PLISKA
STUDIA MATHEMATICA BULGARICA

ПЛИСКА
БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

Pliska Stud. Math. Bulgar. **18** (2007), 213–224

PLISKA
STUDIA MATHEMATICA
BULGARICA

OPTION PRICING BY BRANCHING PROCESS

Georgi Mitov Kosto Mitov [1]

The randomly indexed Galton-Watson branching process has been used for the model of daily stock prices. Using this stock price process we derive a new formula for the price of European call options.

1. Introduction

The "call" option on the given *underlying stock* is the right to buy a share of the stock at a certain fixed price K (the "strike price") at a certain fixed time T in the future (the "maturity date"). Let us denote by $S(t),\ t \geq 0$ the price of a share of the underlying stock at time t. The buyer is paying today (at $t < T$) some money (present value of the option $C(T;t)$) in return for the *right* to force the seller to sell him a share of the stock, if the buyer wants it, at the strike price K on the maturity date T. Clearly, if $S(T) > K$, then the buyer of the option will exercise his right at time T, buying the stock for K and selling it for $S(T)$, gaining a net profit $S(T) - K$. If $S(T) \leq K$, then it is not profitable to buy the stock at price K, so the option is not exercised, and the gain at time T is 0. In summary, for the call option, the gain at maturity date is $(S(T) - K)^+ = \max(S(T) - K, 0)$.

The main problem in the option pricing is to determine the "fair" present value of the option. Apparently, it should depend at least on the value $S(t)$ of the underlying stock, the time $T - t$ to maturity, and the strike price K.

---

[1] This paper is partially supported by NSF of Bulgaria, Grant No VU-MI-105/2005.

2000 *Mathematics Subject Classification:* 60J80, 62P05

Key words: Branching process; Galton-Watson process; Geometric distribution; Option pricing; Stock-price process

Thus, the problem for finding security prices processes which agree well with the market data is the central problem for both practitioners and scientists. Since 1973 the most widely used model is the Black-Scholes [1] model. It assumes that $S(t)$ is a Geometric Brownian motion, which implies that

i) *trading takes place continuously in time*;

ii) *the price dynamics of the stock have a continuous sample path with probability one*;

iii) *the distribution of log-returns is normal with constant volatility.*

Many empirical investigations during the last thirty years do not agree with these assumptions (see e.g. Mandelbrot[8], Fama [6], Madan and Seneta [7]). Many analytic models have been introduced during this time.

Generally speaking, there are two ways for relaxing the assumption that stock prices follow a geometric Brownian motion.

First, one may specify an alternative stochastic process for the price. (see e.g. Cox and Ross [3], Merton [9] and [10], Cont and Tankov [2], Mittnik and Rachev [11] among others for models of this kind (models with jumps; jump diffusion models; models with stable distributions of the returns)).

The second way is to specify a stochastic model for the stock price volatility. (see e.g. Madan and Seneta [7] for a model with stochastic volatility (variance-gamma model)).

Most of the analytical models assume that the stock prices vary continuously, i.e. they can take any nonnegative value. On the other hand, it is well known that the stock prices are measured in units of \$1/16, i.e. they are discrete variables.

At 1996 T. Epps [5] introduced a randomly indexed branching process for modeling of daily stock prices. The model is constructed by Galton-Watson branching processes subordinated with the Poisson process. It has the following attractive features (see [5]):

i) The extra randomness introduced by the subordination produces in the increments and in the returns the same high proportion of outliers observed in high-frequency stock data; but, unlike the other tick-tailed models of stock returns this one capture the discreteness of prices.

ii) The model predicts an inverse relation between variance of returns and the initial price which is well documented empirically.

iii) The possibility of extinction of the stock price process and the distribution of the extinction time have natural interpretation as bankruptcy of the corresponding firm and can be applied in investigations of bankruptcy risk.

In the present paper we continue to study the properties of this model under the following basic assumptions:

i) The offspring of a particle in the branching process has the *two parameter geometric distribution.*

ii) The subordinator is a *Poisson process.*

Under these assumptions it is possible to find a formula for the price of the European call option.

The paper is organized as follows: The second section contains the definitions and some properties of the randomly indexed branching process in supercritical case, needed later. In the third section an option pricing formula is derived. Two numeric examples are given in the last section.

2. Randomly indexed branching process

Let on the probability space $(\Omega, \mathcal{A}, \Pr)$ be given:

(i) a Galton-Watson branching process, $Z_n, n = 0, 1, 2, \dots$ with non-random number of ancestors $Z_0 > 0$, and the offspring probability distribution

$$\begin{aligned}&\Pr(Z_{n+1} = 0|Z_n = 1) = (1-a);\\&\Pr(Z_{n+1} = k|Z_n = 1) = ap(1-p)^{k-1}, \quad k = 1, 2, \dots,\\&0 < a < 1, \quad 0 < p < 1;\end{aligned}$$

(ii) an independent of it Poisson process $N(t),\ t \geq 0$ with intensity $\lambda > 0$.

Following Epps, define the randomly indexed branching process

$$S(t) = Z_{N(t)},\ t \geq 0.$$

Denote the probability generating function (pgf) of the offspring distribution by

$$f(s) = 1 - a + \frac{aps}{1-(1-p)s}, \quad s \in [0, 1], \tag{1}$$

and by $f_n(s) = f(f_{n-1}(s)), f_1(s) = f(s), f_0(s) = s$ the functional iterates of the pgf $f(s)$.

It is easily seen that

$$m = f'(1) = \frac{a}{p} \quad \text{and} \quad b = f''(1) = 2\frac{1-p}{p}m. \tag{2}$$

In what follows we assume that the process is supercritical, i.e. $m > 1$.

The following properties of the process Z_n are known:

The offspring variance is

$$\sigma^2 = b + m - m^2 = \frac{a}{p}\left(\frac{(1-p)+(1-a)}{p}\right).$$

Remark 1. *It is not difficult to be seen that the inequalities* $1 < m < 1 + \varepsilon$ *and* $0 < \sigma^2 < \delta$ *can be satisfied for any positive numbers* δ *and* ε *after an appropriate choice of the parameters* a *and* p. *This property of the branching mechanism is important for the price process. Epps [5] have used mixtures of distributions to satisfy this condition.*

Using the relations (1) and (2) one can write $f(s)$ in the form

$$f(s) = 1 - \frac{m(1-s)}{1 + \frac{b}{2m}(1-s)}.$$

It is well known that (see e.g. Sevastyanov [12]),

$$f_n(s) = E[s^{Z_n}|Z_0 = 1] = 1 - \frac{m^n(1-s)}{1 + \frac{b}{2m}\frac{1-m^n}{1-m}(1-s)}. \tag{3}$$

In the next section we need the probabilities

$$\Pr(Z_n = k|Z_0), \quad k = 0, 1, 2, \ldots, \qquad n = 1, 2, \ldots.$$

Let $Z_0 = 1$. Differentiating in (3) we obtain

$$f_n'(s) = \frac{m^n}{\left[1 + \frac{b(1-m^n)}{2m(1-m)}(1-s)\right]^2},$$

and for $k = 2, 3, \ldots,$

$$f_n^{(k)}(s) = \frac{k! m^n \left[\frac{b(1-m^n)}{2m(1-m)}\right]^{k-1}}{\left[1 + \frac{b(1-m^n)}{2m(1-m)}(1-s)\right]^{k+1}}.$$

Since $\Pr(Z_n = k|Z_0 = 1) = \dfrac{f_n^{(k)}(0)}{k!}$, we get for $k = 1, 2, \ldots$

$$\Pr(Z_n = k|Z_0 = 1) = \frac{m^n \left[\frac{b(1-m^n)}{2m(1-m)}\right]^{k-1}}{\left[1 + \frac{b(1-m^n)}{2m(1-m)}\right]^{k+1}}. \tag{4}$$

Then the probability for extinction in the nth generation is

$$(5)\quad \Pr(Z_n=0|Z_0=1)=1-\sum_{k=1}^{\infty}\frac{m^n\left[\dfrac{b(1-m^n)}{2m(1-m)}\right]^{k-1}}{\left[1+\dfrac{b(1-m^n)}{2m(1-m)}\right]^{k+1}}=1-\frac{m^n}{1+\dfrac{b(1-m^n)}{2m(1-m)}}.$$

Denote by $q=\lim_{n\to\infty}1-\dfrac{m^n}{1+\dfrac{b(1-m^n)}{2m(1-m)}}=1-\dfrac{2m(m-1)}{b}$, the probability for extinction if the process starts with $Z_0=1$. Then for $Z_0>1$ the probability for extinction of the process Z_n is $q^{Z_0}=\left(1-\dfrac{2m(m-1)}{b}\right)^{Z_0}<1$.

The process $W_n=Z_nm^{-n}$ is a nonnegative martingale and $\lim_{n\to\infty}W_n=W$, a.s., where W has the distribution of the sum of Z_0 iid r.v. with distribution function

$$F(x)=\begin{cases} q+(1-q)(1-e^{-qx}), & x\geq 0;\\ 0, & x<0.\end{cases}$$

Some properties of the process $S(t)$ are collected in the following proposition.

Proposition 1. *Under the conditions assumed above:*
(i) The probability generating function of the process $S(t), t\geq 0$ is

$$\Phi(t;s)=E(s^{S(t)}|S(0))=\left(E\left(1-\frac{m^{N(t)}(1-s)}{1+\dfrac{b}{2m}\dfrac{1-m^{N(t)}}{1-m}(1-s)}\right)\right)^{S(0)}$$

$$=\left(\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}e^{-\lambda t}\left(1-\frac{m^k(1-s)}{1+\dfrac{b}{2m}\dfrac{1-m^k}{1-m}(1-s)}\right)\right)^{S(0)}.$$

(ii) The process $S(t)m^{-N(t)}$, $t\geq 0$ is a nonnegative martingale.
The $\lim_{t\to\infty}S(t)m^{-N(t)}=W^*$ exists almost surely and it has the same distribution as the r.v. W.
(iii) The process $S(t)e^{-\lambda t(m-1)}$, $t\geq 0$ is a nonnegative martingale.
The $\lim_{t\to\infty}S(t)e^{-\lambda t(m-1)}=W^{**}$ exists almost surely.

The proof follows immediately from the definition and the properties of the process Z_n, $n=0,1,2\ldots$

3. Pricing of European call option

Let us consider the *arbitrage-free* market which consists of

- a risk-free asset (government bond) whose price process is $B(t) = B_0 e^{rt}, \quad t \in [0, T]$, where the interest rate $r > 0$ is not random and fixed;

- a risky asset whose price follows the stochastic process $S(t), \quad t \in [0, T]$, on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \Pr)$, where $\mathcal{F}_t$ is the natural filtration of $S(t)$;

- an European call option with maturity T and strike price K, written on the risky asset. Denote by $C(t;T)$ the price of the option at the instant $t \in [0, T]$.

The assumption that the market is arbitrage-free is equivalent to the existence of at least one probability measure Q on the given probability space, which is absolutely continuous with respect to the measure Pr and such that the discounted process $S(t)e^{-rt}, \quad t \in [0, T]$ is a martingale.

If the market is *complete* the martingale measure is unique. Otherwise, i.e. if the market is not complete, there are infinitely many different martingale measures and the problem "How to choose the right one?" might be answered in different ways.

The existence of a martingale measure yields to the following general formula for the price of an European call option (see e.g. [3]). Since the price of the option at the maturity date is

$$C(T;T) = \max(0, S(T) - K)$$

then under the martingale measure, the price of the option at time $0 \le t < T$ is

$$C(T;t) = e^{-r(T-t)} E^Q[\max(0, S(T) - K)]. \tag{6}$$

Let us look first at the standard Black-Scholes model which needs the following assumptions:

1. The stock price follows a geometric Brownian motion, i.e.

$$S(t) = S(0)e^{\mu t + \sigma W_t}, \quad t \in [0, T],$$

where μ and σ are constants.

2. The short selling of securities with the full use of proceeds is permitted.

3. There are no transactions costs or taxes.

4. All securities are perfectly divisible.

5. There are no dividends during the life of the derivative.

6. There are no riskless arbitrage opportunities (the no-arbitrage principle).

7. Security trading is continuous.

8. The riskfree rate of interest r is constant and the same for all maturities.

The martingale measure is unique and under this measure the price process takes the form

$$S(t) = e^{rt+\sigma W_t}, \quad t \in [0, T].$$

The last equation and (6) are the basis for the famous Black-Scholes formula for the European call option

$$C(T; t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\ln \frac{S(t)}{K} + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$

$$d_2 = \frac{\ln \frac{S(t)}{K} + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$

and $N(x)$ is the standard normal cumulative probability distribution function. As specified previously, $S(t)$ is the stock price at the present time $t \in [0, T]$, K is the strike price, r is the continuously compounded riskfree interest rate, σ is the stock price volatility, and T is the time to maturity of the option.

Assume the following condition 1' instead of the condition 1.

1'. The stock price follows a randomly indexed branching process $S(t)$ from the previous section.

Proposition 1, (iii) shows that the discounted stock price process $S(t)e^{-rt}$, $t \in [0, T]$ will be a martingale if the parameters of the distribution of $S(t)$ are such that

$$\lambda(m - 1) = r \Leftrightarrow \lambda\frac{a - p}{p} = r. \tag{7}$$

Clearly, any choice of the parameters a, p, and λ, which satisfies this equation, leads to a martingale measure that is absolutely continuous with respect to the real measure and under which the stock price process is a martingale.

Remark 2. *We have no any criterion for choosing the martingale measure by now. In the numeric examples presented in the next section we keep the estimated values of p and λ and change the estimated value of a to a new one, so that the equation (7) to be satisfied.*

Assume that the equation (7) is satisfied for some $a \in (0,1)$, $p \in (0,1)$ such that $m = a/p > 1$ and $\lambda > 0$. Using the general formula (6) we can write

$$\begin{aligned} C(T;t) &= e^{-r(T-t)} E[\max\{0, S(T) - K\}] \\ &= e^{-r(T-t)} \sum_{k=0}^{\infty} \Pr(S(T) = k) \max\{0, k - K\}. \end{aligned}$$

Let $t = 0$. Then for $C(T;0)$ we have

$$\begin{aligned} & C(T;0) = e^{-rT} E\left[\max\{S(T) - K, 0\} | S(0)\right] \\ = \ & e^{-rT} \left[\sum_{k=K+1}^{\infty} k \Pr(S(T) = k | S(0)) - K \Pr(S(T) > K | S(0)) \right] \\ = \ & e^{-rT} \left[E[S(T)|S(0)] - \sum_{k=1}^{K} k \Pr(S(T) = k|S(0)) - K(1 - \Pr(S(T) \le K|S(0))) \right] \\ = \ & e^{-rT} \left[E[S(T)|S(0)] - K \right] \\ + \ & e^{-rT} \left[K \Pr(S(T) \le K|S(0)) - \sum_{k=1}^{K} k \Pr(S(T) = k|S(0)) \right]. \end{aligned}$$

Using the relations

$$\Pr(S(T) = k|S(0)) = \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} e^{-\lambda T} P(Z_n = k|Z_0), k = 0, 1, 2, \dots,$$

$$E(S(T)|S(0)) = S(0) e^{\lambda T(m-1)},$$

and (7) we obtain the following exact formula for the price of the European call option

$$\begin{aligned} & C(T;0) = S(0) - e^{-rT} K \\ + \ & e^{-(r+\lambda)T} \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} \sum_{k=0}^{K} (K - k) \Pr(Z_n = k|Z_0). \end{aligned} \tag{8}$$

For practical purposes an approximation of the infinite series by a partial sum has to be used. Let $\varepsilon > 0$. Then the number N can be determined in such way that

$$e^{-(r+\lambda)T} \frac{K(K+1)}{2} \sum_{n=N+1}^{\infty} \frac{(\lambda T)^n}{n!} < \varepsilon,$$

provided that the values of λ, T, K, and Z_0 are known. Therefore, we can use the approximation

$$C(T;0) \approx \left[S(0) - Ke^{-rT}\right]$$

$$+e^{-(r+\lambda)T} \sum_{n=0}^{N} \frac{(\lambda T)^n}{n!} \sum_{k=0}^{K} (K-k) \Pr(Z_n = k|Z_0), \tag{9}$$

with an error less than ε.

In the last formula we need the probabilities $\Pr(Z_n = k|Z_0 = S(0))$ for $k = 0, 1, 2, \ldots K$ and $n = 1, 2, \ldots N$. They are calculated by the following iterative procedure. Using (4) and (5), we calculate for $n = 1, 2, \ldots, N$:

$$\begin{aligned}
&\Pr(Z_n = k|Z_0 = 2) \\
= &\sum_{j=0}^{k} \Pr(Z_n = j|Z_0 = 1) \Pr(Z_n = k-j|Z_0 = 1), \\
&\Pr(Z_n = k|Z_0 = 3) \\
= &\sum_{j=0}^{k} P(Z_n = j|Z_0 = 2) P(Z_n = k-j|Z_0 = 1), \\
&\ldots\ldots\ldots \\
&\Pr(Z_n = k|Z_0 = S(0)) \\
= &\sum_{j=0}^{k} P(Z_n = j|Z_0 = S_0 - 1) P(Z_n = k-j|Z_0 = 1), \\
&\text{for } k = 0, 1, 2, \ldots, K.
\end{aligned}$$

These formulas are exact. The probabilities calculated in the last equation (10) have to be substituted in the formula (9) for the approximated present value of the option.

4. Numeric examples

The formula obtained in the previous section will be applied to European call options on the stocks of MSFT (Microsoft Corp.)

- Five options on the stock with initial price $S(0) = \$25.43$ on 17-MAY-2005 with the strike prices $K = \$5.00,\ \$10.00,\ \$17.50,\ \$25.00,\ \$27.50$, and maturity date 17-JUN-2005 (time to maturity is $T = 0.04$ years or $= 24$ days).

- Five options on the stock with initial price $S(0) = \$25.43$ on 17-MAY-2005 with the strike prices $K = \$5.00,\ \$10.00,\ \$17.50,\ \$25.00,\ \$27.50$, and maturity date 17-JUL-2005 (time to maturity is $T = 0.079$ years or $T = 48$ days).

The market prices for these options are taken from the web site

`http://quotes.freerealtime.com`

The riskfree interest rate is assumed to be a constant $r = 1\%$ for the periods under question.

The parameters of the process are estimated by the iterative procedure proposed by Epps [5] on the base of the stock prices of MSFT for the one year period from 17-MAY-2004 to 17-MAY-2005. The estimations are

$$\lambda = 2.1621, \quad a = 0.9877, \quad p = 0.9878.$$

Remark 3. *In the paper of Dion and Epps [4] it is pointed out that the process $S(t)$ considered for the positive integer values of $S(j), \quad j = 0, 1, 2, \ldots$ is a particular case of branching process in random environment. The authors have compared, by simulations, several estimates of the parameters of the process and have concluded that only the iterative estimates proposed by Epps [5] are unbiased.*

Keeping the values of λ and p we change the value of a to $a = 0.98782$ obtained from the equation (7), as follows

$$0.01/252 = 2.1621(a/0.9878 - 1).$$

The tables 1 and 2 show the market values, the prices obtained by the Black-Scholes formula and the prices obtained by the formula (9) given in the previous section. The values are calculated for the date 17-MAY-2005.

Remark 4. *Let us mention that time in the Black - Scholes formula is measured in years, but in the formula (9) time is measured in days assuming that the year has 252 working days. For this reason the daily interest rate is taken as $r = 0.01/252$ in the equation for the parameter a.*

5. Conclusion remarks

The formula derived in the third section is based on the martingale property (Proposition 1, (iii)) of the process $S(t)$. This formula is exact but one has to truncate the infinite series to use it for calculations. Another way is to use the

Market price	20.40	15.40	7.95	0.45	0.05
Black-Scholes price	20.4302	15.4304	7.9307	0.4310	0
Branching process price	20.4302	15.4304	7.9307	0.4464	0.0500
Strike price	5.00	10.00	17.50	25.00	27.50
Time to maturity (in years)	0.004	0.004	0.004	0.004	0.004

Table 1: The prices of the options with T=1 month

Market price	20.45	15.45	8.00	0.75	0.075
Black-Schoels price	20.4304	15.4308	7.9314	0.4320	0
Branching process price	20.4304	15.4308	7.9314	0.4715	0.0550
Strike price	5.00	10.00	17.50	25.00	27.50
Time to maturity (in year)	0.0079	0.0079	0.0079	0.0079	0.0079

Table 2: The prices of the options with T=2 months

martingale property (Proposition 1, (ii)), which leads to the following approximation

$$S(t) \approx W^* m^{N(t)}, \quad t \geq 0.$$

If we use $W^* m^{N(t)}, t \geq 0$ as a price process, a closed formula can be written down. If we proceed this way, we make an error at the beginning instead of truncating the tail of the series. The estimate of the error in the above approximation and the corresponding closed formula is under consideration and the results will be published later.

REFERENCES

[1] Black, F., and Scholes, M. The pricing of options and corporate liabilities, *Journal of Political Economy* **81** (1973), 637–659.

[2] Cont, R., and Tankov, P. Financial modeling with jump processes, Chapman & Hall/CRC, Boca Raton, 2004.

[3] Cox, J.C., and Ross, S. The valuation of options for alternative stochastic processes, *Journal of Financial Economics* **3** (1976), 145–166.

[4] Dion, J.-P. and Epps, T. Stock prises as branching processes in random environement: Estimation, *Communications in statistics: Simulation and computation* **28(4)** (1999), 957–975.

[5] Epps, T. Stock prices as branching processes, *Communications in statistics: Stochastic Models* **12** (1996), 529–558.

[6] Fama, E. The behaviour of stock market prices, *Journal of Business* **38** January (1965), 34–105.

[7] Madan, D.B., and Seneta,E. The variance gamma (V.G.) model for share market returns, *Journal of Business* **63(4)** (1990), 511–524.

[8] Mandelbrot, B.B. New methods in statistical economics, *Journal of Political Economy* **71** (1963), 421–440.

[9] Merton, R.C. Theory of rational option pricing, *Bell Journal of Economics and Management Science* **4** (1973), 141–183.

[10] Merton, R.C. Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics* **3** (1976), 125–144.

[11] Mittnik, S. and Rachev, S. Modeling asset returns with alternative stable distributions, *Econometric Reviews* **12(3)** (1993), 261–330.

[12] Sevastyanov, B.A. Branching Processes, Nauka, Moscow, 1971 (in Russian).

Georgi Kostov Mitov
bul. Georgi Kochev 8, vh A, apt 16
5800 Pleven, Bulgaria
e-mail: `georgi.mitov@finanalytica.com`

Kosto Valov Mitov
bul. Georgi Kochev 8, vh A, apt 16
5800 Pleven, Bulgaria
e-mail: `kmitov@af-acad.bg`