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SENSITIVITY ANALYSIS OF SOME APPLIED PROBABILITY MODELS

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The aim of the paper is two-fold, namely, to give a brief survey of sensitivity analysis methods and to use them for investigation of two input-output models arising in applied probability.

1. Introduction

The paper treats the stability problems for various stochastic models. We provide a brief survey of different approaches and methods. After that we concentrate on the sensitivity analysis of two important models arising in inventory theory and insurance.

First of all we recall the following well-known facts:

1. In order to study a real-life process or a system it is useful to construct its mathematical model.

2. There are a lot of models describing more or less precisely a given system.

3. The same model can describe the processes arising in different research domains.

A crucial question in all investigations pertaining to decision making is how to choose an appropriate mathematical model. The usual procedure is given by the scheme of Fig. 1.

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Key words: Asymptotically optimal policy; Incomplete information; Input-output model; Insurance; Inventory; Risk measures; Sensitivity analysis; Stability

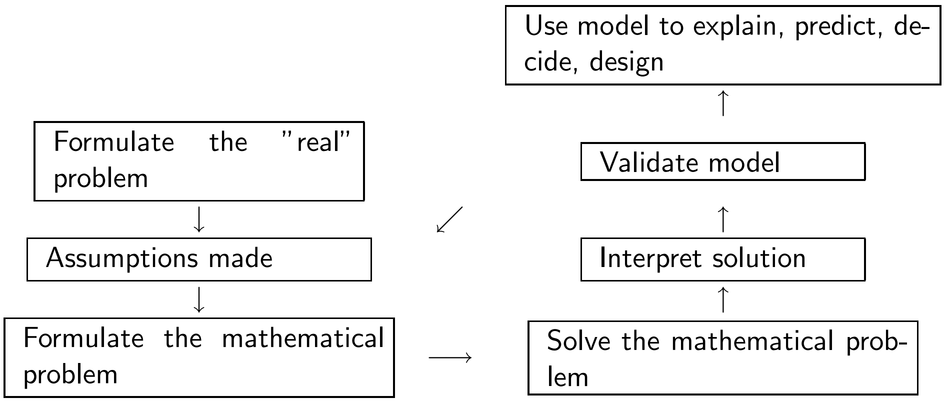

Fig. 1. Decision making

As Bellman emphasized in [**?**], it is necessary to avoid oversimplification and excessive complexification. Although it is not difficult to solve a simple mathematical problem, the solution may not describe precisely the system's performance. Hence, the *poor model fit* is the first source of decision errors.

On the other hand, a complex model depending on a great number of parameters and giving a rather precise system description can also be the source of decision errors. In fact, it is often impossible to obtain an explicit form of the solution, so one has to use a numerical one. The high speed of modern computers lets to get it without problems, if all the parameters are known exactly. Thus, along with computational errors, the *parameters variability* (and necessity of their estimation on the base of previous system observations) is the second source of decision errors, the third one being *perturbations of the underlying processes*. Therefore the model *stability* is a must and before making a decision applying some mathematical model it is desirable to perform its *sensitivity analysis*.

We are interested in input-output models, arising in such applications as insurance, finance, inventory, queueing, storage, reliability theory and many others. They can be described by a five-tuple $(\mathsf{Z}, \mathsf{Y}, \mathsf{U}, \Psi, \mathcal{L})$, a scheme is given by Fig. 2. Here $\mathsf{Z} = \{Z(t), t \geq 0\}$ and $\mathsf{Y} = \{Y(t), t \geq 0\}$ are input and output processes, respectively, $\mathsf{U} = \{U(t), t \geq 0\}$ is a control, Ψ reflects the system's configuration and performance mode, whereas $\mathcal{L}$ is an objective function (valuation criterion, risk measure). One takes also into account the planning horizon $T \leq \infty$ and the system state $\mathsf{X} = \Psi(\mathsf{Z}, \mathsf{Y}, \mathsf{U})$, a function in t as well, for details see, e.g. [**?**].

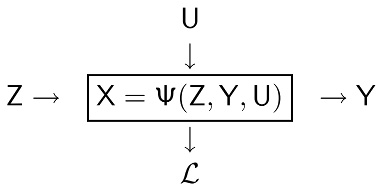

Fig. 2. Input-output model

Table 1. Interpretation of systems elements

Research field	**Input**	**Output**	**System state**
Insurance	Premium	Indemnity	Reserve
Inventory	Supply	Demand	Inventory level
Queueing	Arrival	Departure	Queue length
Storage	Inflow	Outflow	Water level
Finance	Income	Expenses	Capital
Reliability	New units	Defective units	Working units
Population growth	Birth, Immigration	Death, Emigration	Population size

This general description is useful for models classification, making obvious the similarity of different research fields as well. Another interpretation of model "elements" leads to another research field, as shown by Table 1.

Thus, the methods useful in one research field may be of interest in others. For example, distribution of a random walk maximum can be interpreted either as a distribution of waiting time in a one-server queueing system $G|G|1$ or as a ruin probability in the Sparre Andersen insurance model, see e.g. [**?**].

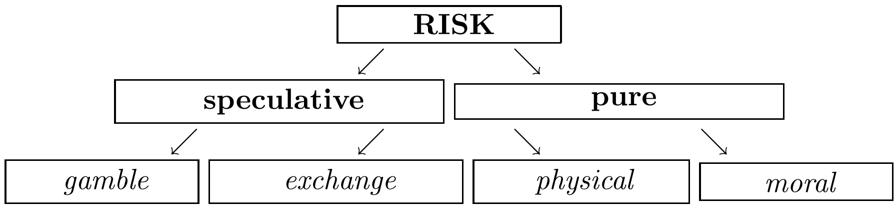

Fig. 3. Risk classification

2. Risk measures

The systems investigation is usually aimed at optimization of their performance, thus eliminating or minimizing the risk.

Risk, as defined by the Concise Oxford English Dictionary, is "a hazard, a chance of bad consequences, loss or exposure to mischance". The basic risk classification is given by Fig. 3.

Pure risk (physical or moral), pertaining mainly to insurance, entails loss only, whereas speculative one, connected in particular with finance, can entail profit as well as loss.

Risk management is a discipline for living with the possibility that future events may cause adverse effects. That means one needs (quantitative) risk measurement, see, e.g. [**?**], [**?**]. Risk is usually associated with uncertainty, hence risk management may be considered as decision making under uncertainty, see, e.g. [**?**].

Risk measures (or objective functions) are based on loss distributions or their functionals, such as mean, variance, quantile, premium principle etc.

The choice of objective function determines the approach used by a researcher. The most well known approaches are cost and reliability ones. The cost approach was used in inventory theory from the beginning, see, e.g. [**?**]. On the other hand, the reliability approach was typical for actuarial sciences, see, e.g. [**?**]. Nowadays, both are used, more or less frequently, in all the research domains mentioned in Table 1, see, e.g. [**?**].

In the framework of the **cost approach** one can choose as an objective function total expected costs during the planning period, total discounted expected costs, see, e.g. [?], [?], long-run average cost per unit time, net present value, see, e.g. [?], [?], internal rate of return, see, e.g. [?], etc. Thus, an objective function can depend on some scalar parameters (such as order and holding costs, penalties for delays, inflation and interest rates etc.) as well as distributions of underlying processes (for example demand and supply) if they are stochastic. The main goal is *cost minimization* (or *profit maximization*).

We remind here some definitions which will be needed in Section 4. Although the objective function $\mathcal{L}_T(\mathsf{Z},\mathsf{Y},\mathsf{U})$ we are going to use a simpler notation $\mathcal{L}_T(U_T)$ denoting by $\mathcal{U}_T$ a class of feasible controls $U_T = \{U(t), t \in [0,T]\}$, for a given planning horizon T. A collection of controls $U = \{U_T, T \geq 0\}$ is called a *policy*. Further we assume the distributions of underlying (stochastic) processes Y and Z and all the parameters to be known.

Definition 1. *An optimal control U_T^*, if exists, is determined by the formula*

$$\mathcal{L}_T(U_T^*) = \inf_{U_T \in \mathcal{U}_T} \mathcal{L}_T(U_T),$$

whereas $U^* = (U_T^*, T \geq 0)$ is an optimal policy.

Definition 2. *A policy $\widehat{U} = (\widehat{U}_T, T \geq 0)$ is asymptotically optimal if it satisfies the following relation*

$$\lim_{T\to\infty} T^{-1}\mathcal{L}_T(\widehat{U}_T) = \lim_{T\to\infty} T^{-1}\mathcal{L}_T(U_T^*).$$

Definition 3. *A policy $\overline{U} = (\overline{U}_T, T \geq 0)$ is stationary if, for all $S, T \geq 0$,*

$$\overline{U}_T(t) = \overline{U}_S(t), \quad t \leq \min(S,T).$$

Although the **reliability approach** will not be used in the paper, we mention, for completeness sake, that, in the framework of such an approach, the most frequently chosen objective functions are ruin probability [?], [?] (probability of system failure [?]), expected time until failure [?], value at risk, expected shortfall (conditional value at risk), upper and lower partial moments [?]. During the last decade, coherent risk measures became popular in finance and insurance, details can be found in [?].

3. Sensitivity analysis

The *objective* of sensitivity analysis (SA) is to ascertain how a model depends on its parameters and distributions of underlying processes and establish its *stability* (or instability) with respect to small deviations of parameters or perturbations of processes.

There are a lot of methods for investigation of systems stability. One can mention the Lyapunov stability in differential equations theory. An interesting research direction in transportation networks is connected with stability of dynamic systems, see, e.g. [?]. The effects of perturbations in inventory networks were considered e.g. in [?], [?].

The study of stochastic models stability using the probability metrics was initiated by V.M. Zolotarev in 1970s, see [?], [?], [?], [?], [?] and references therein. The probability metrics are useful to measure the systems response to perturbations of underlying stochastic processes, see, e.g. [?].

Here, giving a brief survey far from completeness, we concentrate on the so-called local and global SA techniques appropriate for uncertain scalar parameters. Their applications will be considered in Section 4.

Denote by $R = g(a)$ a valuation criterion (objective function, decision made or optimal control), $a = (a_1, a_2, \ldots, a_n)$ being a vector of model parameters. From now on, as usual in SA, R will be called the system output, $g(\cdot)$ the model and a_i the i-th input parameter (or factor).

The *local SA* school studies the local response obtained by varying input factors one at a time, while others are fixed at a nominal (or base-case) value.

The *global SA* school explores the space of input factors within finite (or infinite) region, and variation of output is usually averaged over variation of all the factors.

3.1. Local techniques

Let $R^0 = g(a^0)$, a^0 being the base-case value of parameters. Such values reflect the decision maker (researcher) knowledge of assumptions made. Sensitivity of model to changes in input parameters is usually tested through the so-called "one-way" or a combined SA scheme. That is, the change $\Delta R = R - R^0$ is registered when a parameter (or a combination of parameters) is varied within a rationally chosen range in order to draw conclusions on the consistency and correctness of the valuation model, see [?] for description of the classical "Tornado diagram". However this SA scheme should not be used to infer parameter importance, since parameter changes Δa_i are not taken in consideration, see, e.g. [?]. Several local SA techniques were developed to establish parameters importance, see e.g. [?], [?], [?], [?] and [?].

Differential importance measure (DIM) for the input a_s is defined as follows

$$D_s(a^0, da) = g'_{a_s}(a^0)\, da_s \left(\sum_{j=1}^{n} g'_{a_j}(a^0)\, da_j \right)^{-1} \quad (= dg_s(a^0)/dg(a^0). \tag{1}$$

DIM produces importance of parameters, for small deviations of their value from the base-case value, under the assumptions: g is differentiable at a^0 and $dg(a^0) \neq 0$, see, e.g. [**?**]. Thus DIM technique enables the researcher to distinguish the more influential factors. It is possible to consider either uniform or proportional parameters changes.

(H1) *Uniform parameters changes*: $da_s = h$, for all s, whence

$$D1_s(a^0) = g'_{a_s}(a^0) \Big/ \sum_{j=1}^{n} g'_{a_j}(a^0).$$

(H2) *Proportional parameters changes*: $da_s/a_s^0 = 1/w$, for all s and some $w \neq 0$, whence

$$D2_s(a^0) = g'_{a_s}(a^0)a_s^0 \Big/ \sum_{j=1}^{n} g'_{a_j}(a^0)a_j^0.$$

DIM technique generalizes other local methods such as applying Fussel-Vesely importance measure and local importance measures based on normalized partial derivatives, also known as measures of criticality importance or elasticity.

Elasticity of R^0 with respect to the s-th parameter is given by

$$E_s(a^0) = g'_{a_s}(a^0)a_s^0/R^0,$$

thus

$$D1_s(a^0) = (E_s(a^0)/a_s^0) \left(\sum_{j=1}^{n} E_j(a^0)/a_j^0 \right)^{-1},$$

$$D2_s(a^0) = E_s(a^0) \left(\sum_{j=1}^{n} E_j(a^0) \right)^{-1}.$$

In case (H1) Elasticity and DIM produce, in general, different ranking and no definitive conclusions can be inferred using Elasticity, see [?].

In case (H2) Elasticity and DIM differ by a normalization factor.
If $\sum_{j=1}^{n} E_j(a^0) > 0$, parameters are ranked in the same order,
and if $\sum_{j=1}^{n} E_j(a^0) < 0$, the order is reversed.

DIM was used for analysis of investment projects by Borgonovo and Peccati, see [?]. They considered NPV (net present value) of CF(cash flow)

$$R^0 = \sum_{s=0}^{n} b_s a_s^0 (1+\alpha)^{-s}$$

and IRR (internal rate of return) determined by

$$\sum_{s=0}^{n} b_s a_s (1+IRR)^{-s} = 0,$$

here b_s is the non-extinction probability and α is the discount rate.

Since

$$D1_s(a^0) = b_s(1+\alpha)^{-s} \Big/ \sum_{j=0}^{n} b_j (1+\alpha)^{-j},$$

the importance of CF (determined only by the CF profile) is independent of CF magnitude. Furthermore, $D1_s(a^0) > D1_{s+1}(a^0)$, for all s, iff $b_s > b_{s+1}/(1+\alpha)$ for all s. Therefore if $b_s = 1$ for all s, CFs near in time are more important than CFs at later times.

For a review of local methods see also Turanyi [?].

3.2. Global techniques

The *main goals of global sensitivity analysis* are the following:

1. Understanding of the model dependence on the (uncertain) input parameters, say, to make sure whether the output is determined by individual parameters or their groups.

2. Determination of the global importance of parameters. This information can be used to direct data collection and parameters estimation.

A qualitative type of global SA analysis is offered by some screening methods aimed at identifying the active factors of a model at low computational costs and ranking the input parameters in order of their importance, see [?], [?]. Monte

Carlo regression and correlation methods, scatterplots, standardized regression coefficients, Pearson correlation measures, partial correlation coefficients, Spearman correlation etc. were used for global SA, see the review by Helton [**?**].

Now the most popular ones are two ANOVA (analysis of variance) type methods, namely, the Sobol' decomposition and FAST (Fourier Amplitude Sensitivity Test). They are reminded below.

3.2.1. SOBOL' decomposition

Although the systems parameters are some (often unknown) constants it is useful to treat them as r.v.'s. Assume that $A = (A_1, \dots, A_n)$ is uniformly distributed in $K^n = [0,1]^n$ and the function $g(a)$, $a \in K^n$, is integrable. Put

$$g_0 = \mathsf{E}R = \int\limits_{K^n} g(a)\, da,$$

$$g_i(a_i) = \int\limits_0^1 \dots \int\limits_0^1 g(a) \prod_{k \neq i} da_k - g_0,$$

$$g_{i,j}(a_i, a_j) = \int\limits_0^1 \dots \int\limits_0^1 g(a) \prod_{k \neq i,j} da_k - (g_0 + g_i(a_i) + g_j(a_j)),$$

$$\dots \quad \dots .$$

The Sobol' method is based on the important result proved in [**?**].

Theorem 1 (Sobol') *For any $a \in K^n$, the following decomposition of $g(a)$ is unique:*

$$g(a) = g_0 + \sum_{i=1}^n g_i(a_i) + \sum_{i<j}^n g_{i,j}(a_i, a_j) + \dots + g_{1,\dots,n}(a_1, \dots, a_n).$$

Moreover, for any $1 \le i < \dots < s \le n$,

$$\int\limits_0^1 \dots \int\limits_0^1 g_{i,\dots,s}(a_i, \dots, a_s) \prod_{k=i,\dots,s} da_k = 0$$

and if $(i, j, \dots, m) \neq (k, l, \dots, p)$, then

$$\int\limits_{K^n} g_{i,j,\dots,m} \cdot g_{k,l,\dots,p}\, da = 0.$$

Corollary 1. *The following decomposition of variance holds for a square integrable random variable $R = g(A)$:*

$$V[R] = \sum_{i=1}^{n} V_i + \sum_{i<j} V_{i,j} + \sum_{i<j<k} V_{i,j,k} + \ldots + V_{1,2,\ldots,n}, \tag{2}$$

where $V[R] = \int\limits_{K^n} g^2(a)\,da - g_0^2$ and partial variances are calculated by way of

$$V_{i_1,\ldots,i_s} = \int_0^1 \ldots \int_0^1 g^2_{i_1,\ldots,i_s}(a_{i_1},\ldots,a_{i_s}) \prod_{k=i_1,\ldots,i_s} da_k. \tag{3}$$

Now we can formulate further definitions assuming $V[R] \neq 0$.

Definition 4. *Sensitivity index $S_{i_1,i_2,\ldots,i_s}$ for a group of parameters $(a_{i_1}, a_{i_2}, \ldots, a_{i_s})$, $1 \le i_1 < i_2 < \ldots < i_s \le n$, is given by $V_{i_1,i_2,\ldots,i_s}/V[R]$, whereas the sensitivity index of order s is $\sum_{1\le i_1<\ldots<i_s\le n} S_{i_1,i_2,\ldots,i_s}$.*

Thus S_i is the first order contribution of the i-th parameter to the output variance, while $S_{i_1,i_2,\ldots,i_s}$ represents the parameters interaction.

Definition 5. *Global sensitivity index $GI(a_i)$ of parameter a_i is the sum of all indices $S_{i_1,\ldots,i_s}$, $s \ge 1$, containing i*

$$GI(a_i) = (V_i + \sum_{j\neq i} V_{i,j} + \ldots + V_{1,2,\ldots,n})/V[R].$$

Thus, $GI(a_i)$ represents the total contribution of parameter a_i to variance of output. Now we are able to answer the following questions. Which of the uncertain input factors is so uninfluential that we can safely fix it (them)? If we could eliminate the uncertainty in one of the input factors, which should we choose to reduce most the variance of output? The applicability of these sensitivity indices is related to the possibility of evaluating the multidimensional integrals such as (**??**) using Monte-Carlo methods.

Remark 1. *Variance decomposition* (**??**) *is valid* (*with obvious changes*) *for any distribution of A.*

In the framework of numerical experiments similar decompositions are discussed in [**?**], [**?**], [**?**] and [**?**], for details see also [**?**].

3.2.2. Fourier Amplitude Sensitivity Test (FAST)

This method of SA was introduced in the 1970s by Cukier, Fortuin, Shuler, Petschek and Schailby, see [**?**], and further developed by many researchers, see, e.g. [**?**], [**?**] and references therein. FAST computes the "main effect" contribution of each input factor (parameter) to variance of output, in other words, its "importance measure", see, e.g. [**?**]. It is closely related to design of experiments and ANOVA studies, see [**?**]. The core feature of FAST is that the multidimensional space of input parameters is explored by a suitably chosen search curve.

Let A be a random vector with pdf $p(a_1,\dots,a_n)$ on K^n and $R=g(A)$, then (if exists)

$$\mathsf{E}[R^r]=\int\limits_{K^n} g^r(a_1,\dots,a_n)p(a_1,\dots,a_n)\,da, \quad r=1,2.$$

Using multi-dimensional Fourier transformation of g it would be possible to perform ANOVA-like decomposition of $V[R]$.

A monodimensional Fourier decomposition is done along a curve defined by a set of parametric equations

$$a_i(s)=G_i(\sin\omega_i s), \quad i=\overline{1,n}, \quad s\in(-\infty,+\infty). \tag{4}$$

$G_i(\cdot)$ are transformation functions (which will be described below) and $\{\omega_i\}$, $i=\overline{1,n}$, is the set of different frequencies associated with each parameter.

The curve is space filling, that is, arbitrarily close to any point of K^n iff $\sum_{i=1}^n r_i\omega_i\neq 0$ for any integer r_i (incommensurable frequencies). In this case one can estimate $\mathsf{E}R^r$ using the Weyl ergodic theorem.

Due to the finite precision of computers ω_i cannot be really incommensurable. In practice ω_i are positive integers, so $g(s)=g(a_1(s),\dots,a_n(s))$ is periodic, more precisely, as shown in [**?**], $g(s+2\pi)=g(s)$. Therefore we may expand $g(s)$ in a Fourier series

$$R=g(s)=\sum_{j=-\infty}^{+\infty}(C_j\cos js+B_j\sin js).$$

Variance estimate has the form

$$\widehat{D}=(2\pi)^{-1}\int\limits_{-\pi}^{\pi} g^2(s)\,ds-[(2\pi)^{-1}\int\limits_{-\pi}^{\pi} g(s)\,ds]^2$$

where, by the Parseval theorem,

$$\widehat{D} = \sum_{j\neq 0} \Lambda_j = 2\sum_{j=1}^{\infty} \Lambda_j, \quad \Lambda_j = C_j^2 + B_j^2, \quad j \in \mathbb{Z}.$$

Variance estimate due to the i-th parameter is obtained by evaluating the spectrum for the fundamental frequency ω_i and its higher harmonics $p\omega_i$, namely,

$$\widehat{D}_i = \sum_{p\neq 0} \Lambda_{p\omega_i} = 2\sum_{p=1}^{\infty} \Lambda_{p\omega_i}.$$

Now

$$S_i^{FAST} = \widehat{D}_i/\widehat{D}$$

is the estimate of the main effect of a_i on Y. Its magnitude does not depend, in principle, on the choice of the set of frequencies taken for computation.

We note in passing that each $a_i(s)$ in (**??**) oscillates periodically at the corresponding frequency ω_i whatever G_i is. Output R shows different periodicities combined with different frequencies ω_i whatever the model g is. If the i-th factor has the strong influence on the output, the oscillations of R at frequency ω_i will be of high amplitude.

It was shown in [**?**] that S_i^{FAST}, $i = \overline{1,n}$, are "equivalent" to the Sobol' sensitivity indices of the first order, as well as to "importance measures" considered in [**?**], [**?**], [**?**], [**?**] or "correlation ratio" treated in [**?**], [**?**]. All these measures estimate the same underlying statistical quantity given by $V[E(R|A_i)]/V[R]$.

An important question is how to choose the functions G_i. It was proposed in [**?**] to take

$$a_i(s) = a_i^0 e^{\nu_i \sin \omega_i s}$$

where a_i^0 is the nominal value of the i-th factor, ν_i defines the endpoints of the estimated range of uncertainty for a_i and $s \in (-\pi/2, \pi/2)$. Such transformation is suitable for a factor with long-tailed and positively skewed pdf.

Another choice, see [**?**], is setting

$$a_i(s) = a_i^0(1 + \nu_i \sin \omega_i s),$$

which is appropriate for U-shaped pdf. The transformation

$$a_i(s) = (1/2) + (1/\pi)\arcsin(\sin \omega_i s) \tag{5}$$

considered in [?] fits well the uniform distribution. This is the solution of the following equation introduced in [?]

$$\pi(1 - a_i^2)^{1/2} p_i(G_i)(dG_i(a_i)/da_i) = 1,$$

for the case $p_i(a_i) = const$, here p_i is pdf of A_i.

In FAST the model g must be evaluated at N_s equally spaced sample points along the closed path in the interval $(-\pi, \pi)$. According to the Nyquist principle the minimum sample size $N_s = 2M\omega_{max} + 1$ where M is the interference factor (≥ 4) and ω_{max} is the largest of the frequencies ω_i. It is useful to choose N_s odd to include the point $s = 0$ in the symmetric set of samples, for details see, e.g. [?].

All the transformations introduced above have a drawback, namely, as s varies in $(-\pi/2, \pi/2)$, they always return the same points in K^n. To make the model evaluations used more efficiently the following modification of (**??**) was proposed in [?]

$$a_i(s) = (1/2) + (1/\pi) \arcsin(\sin(\omega_i s + \varphi_i))$$

where φ_i is a random phase-shift uniformly distributed in $[0, 2\pi)$.

By selecting various sets $\{\varphi_1, \ldots, \varphi_n\}$ different curves can be generated. This procedure is called *resampling* by the authors of [?]. Let k_r be the number of curves obtained. The sample size in resampling scheme must be $N_s = k_r(2M\omega_{max} + 1)$.

An extension of FAST, to calculate the total contribution of each parameter to the output variance, is introduced in [?] and it is compared with the Sobol' method.

4. Applications

To illustrate the performance of the above described SA techniques we begin by applying them to a simple inventory model introduced for the first time in 1915 by Harris [?] and rediscovered later independently by many authors, for the last time by Wilson in 1934, see [?]. Many modifications of this model appeared since then and are widely used in inventory management, see, e.g. [?], [?], [?].

We consider this model because it is possible to obtain an explicit form of the global sensitivity indices, using the Sobol' decomposition.

Remark 2. *It is always possible to obtain an explicit form of global sensitivity indices when parameters A_i, $i = \overline{1, n}$, are independent and $g(a) = \prod g_i(a_i)$ with $\mathsf{E} g_i^2(A_i) < \infty$. Moreover, for linear $g(a) = \sum g_i(a_i)$ the decomposition* (**??**) *contains only summands of the first order.*

In the second part of this Section we introduce and study a new discrete-time insurance model. The cost approach is used in both models.

4.1. Wilson-Harris formula for EOQ

Consider the choice of an EOQ (economic order quantity), minimizing the total costs incurred in the following deterministic model.

Let a_1 be a fixed order cost, a_2 a constant demand during a planning period and a_3 a fixed holding cost per unit time, furthermore, there is no delivery lag. Denoting by Q the order quantity we have to minimize $(a_1a_2/Q) + (a_3Q/2)$. Hence

$$Q^* = g(a) = \sqrt{2}a_1^{p_1}a_2^{p_2}a_3^{p_3}, \quad p = (0.5, 0.5, -0.5).$$

According to Definition **??** and Corollary **??** one has to calculate

$$\begin{aligned}
\mathsf{E}[Q^*] &= \sqrt{2}\prod \mathsf{E}[A_i^{p_i}], \\
V[Q^*] &= 2(\prod \mathsf{E}[A_i^{2p_i}] - (\prod \mathsf{E}[A_i^{p_i}])^2), \\
V_i &= 2(\prod_{j\neq i} \mathsf{E}[A_j^{p_j}])^2 V[A_i^{p_i}], \\
V_{s,r} &= 2\mathsf{E}[A_i^{p_i}]^2 V[A_r]V[A_s], \quad i \neq s, r, \\
V_{1,2,3} &= V - \sum_{i=1}^{3} V_i - V_{1,2} - V_{1,3} - V_{2,3}.
\end{aligned}$$

In the paper [**?**] the indices $GI(a_i)$, $i = 1, 2, 3$, were calculated for two particular cases (relative error in parameters estimation being 25% and 99%). More general results are proved by the author's students in [**?**]:

Theorem 2. *Let A_i be uniformly distributed on $[a_i(1-k), a_i(1+k)]$, $k \in (0, 1)$. Then*

1. *$GI(a_i)$, $i = 1, 2, 3$, depend only on k,*
2. *$GI(a_1) = GI(a_2) \le GI(a_3)$.*

Corollary 2. *The same results are true if A_i has a Gamma-distribution with $\mathsf{E}A_i = a_i$, $V[A_i] = (a_ik)^2$.*

Along with the Sobol' method, FAST was used for a_i of the form (**??**) and with $\omega = (5, 7, 13)$. Parameter a_3 was found the most influential, since $Q^*(s) = \infty$ for $s = (3\pi + 4\pi k)/2\omega_3$, $k \neq 0$. For details see [**?**].

4.2. Insurance company model (discrete time)

Assume that by the end of a year the company can make one of the following decisions: I – to sell some assets (immediately), II – to borrow some money, the loan being available by the end of the next year, III – both above decisions simultaneously.

We take into account the following parameters: c_1 is the loss incurred by selling assets unit, c_2 is the interest rate while borrowing, r is the penalty for payment delay, p is the inflation rate, x is the initial capital (if $x < 0$ its absolute value is the company debt) and ξ_n is the excess of claims Y_n over premiums Z_n in the year n, in other words, $\xi_n = (Y_n - Z_n)^+$.

Suppose that $\{\xi_n\}_{n\geq 1}$ is a sequence of i.i.d. nonnegative r.v's with a finite mean $\gamma = \mathsf{E}\xi_n < \infty$ and a density $\varphi(s) > 0$ for s belonging to some finite or infinite interval of $\mathbb{R}_+$. The corresponding distribution function

$$F(t) = P(\xi_n \leq t) = \int_0^t \varphi(s)\,ds, \quad \overline{F}(t) = 1 - F(t). \tag{6}$$

4.2.1. Known distribution

We denote by $f_n(x)$ the minimal average n years costs. According to the Bellman optimality principle

$$f_n(x) = \min_{z_1\geq 0, z_2\geq 0}[c_1 z_1 + c_2 z_2 + L(x+z_1) + \mathsf{E} f_{n-1}(x+z_1+z_2-\xi_1)]$$

where $f_0(x) \equiv 0$ and

$$L(v) = p\int_0^v (v-s)\varphi(s)\,ds + r\int_v^\infty (s-v)\varphi(s)\,ds.$$

Putting $v = x + z_1$, $u = v + z_2$ and

$$G_n(u,v) = (c_1 - c_2)v + c_2 u + L(v) + \int_0^\infty f_{n-1}(u-s)\varphi(s)\,ds$$

one gets

$$f_n(x) = -c_1 x + \min_{u\geq v\geq x} G_n(u,v). \tag{7}$$

To establish the optimal control we introduce the following notation

$$K_n(v) = \frac{\partial G_n}{\partial v}(u,v) = c_1 - c_2 + L'(v) \quad (= K(v)),$$

$$S_n(u) = \frac{\partial G_n}{\partial u}(u,v) = c_2 + \int_0^\infty f'_{n-1}(u-s)\varphi(s)\,ds$$

and

$$T_n(u) = K_n(u) + S_n(u) = c_1 + L'(u) + \int_0^\infty f'_{n-1}(u-s)\varphi(s)\,ds.$$

Let $\overline{v}$ satisfy $K(\overline{v}) = 0$, that is $F(\overline{v}) = (r + c_2 - c_1)/(r + p)$, moreover,

$$S_n(u_n) = 0, \quad T_n(t_n) = 0, \quad H(\overline{u}) = 0, \tag{8}$$

where

$$H(u) = c_2 - c_1 + \int_0^{u-\overline{v}} K(u-s)\varphi(s)\,ds \tag{9}$$

and

$$F(\overline{t}) = r/(r+p), \quad F^{2*}(\widehat{u}) = r/(r+p). \tag{10}$$

Theorem 3. *Optimal behaviour at the first step of n-step process has the form:*

(A) If $c_1 \le c_2$, $(l-1)r < c_1 \le lr$, then $u = v = x$ for $n < l$ and $u = v = \max(t_n, x)$ for $n \ge l$. The sequence $\{t_n\}$ is bounded, increasing, and $\lim_{n\to\infty} t_n = \overline{t}$.

(B) If $c_2 m/(m-1) \le c_1 \le \min(c_2 + r, c_2(m-1)/(m-2))$, $m \ge 2$, $(l-1)r < c_1 \le lr$, $l \ge 1$ (hence $m \ge l$ and $u_n \ge \overline{v}$ for $n \ge m$), then $u = v = x$ for $n < l$ and $v = \max(\overline{v}, x)$, $u = \max(u_n, x)$ for $n \ge m$. The sequence $\{u_n\}$ is bounded, increasing, and $\lim_{n\to\infty} u_n = \overline{u}$.

If $l \le n < m$, then the optimal decision may be determined either by parameters $(u_n, \overline{v})$, or t_n, moreover, if t_{n_0} is optimal for some n_0, then t_n is also optimal for $l \le n < n_0$.

(C) If $c_1 > c_2 + r$, $(k-1)r < c_2 \le kr$, $k \ge 1$, then $u = v = x$ for $n \le k$ and $v = x$, $u = \max(u_n, x)$ for $n > k$. The sequence $\{u_n\}$ is bounded, increasing, and $\lim_{n\to\infty} u_n = \widehat{u}$.

The **proof** is carried out by induction. According to (**??**), in order to obtain $f_n(x)$ we have to minimize $G_n(u,v)$ over the set $\{(u,v) : u \geq v \geq x\}$, see Fig. 4. The minimum can be attained either inside this set or on its boundary. In the first case the corresponding point is the solution of the system $S_n(u) = 0$, $K(v) = 0$ (since $K_n(v)$ does not depend on n). In the second case there are two possibilities. Either $u = v$, then one has to take the solution of $T_n(u) = 0$, or $v = x$, then it is necessary to use the solution of $S_n(u) = 0$.

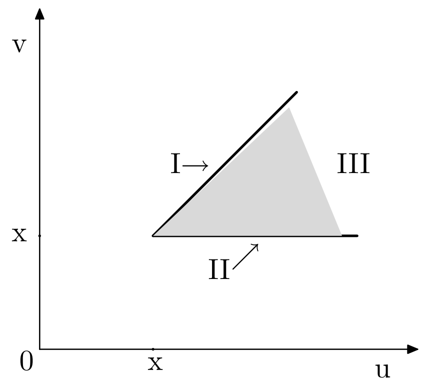

Fig. 4. Optimization set

Now put $n = 1$. Since $S_1(u) = c_2 > 0$, we have to consider $T_1(u) = c_1 + L'(u)$. In other words, the optimal behaviour is determined by t_1 satisfying the relation $F(t_1) = (r - c_1)/(r + p)$, if such t_1 exists, that is, for $c_1 \leq r$. Then one sells assets to get $(t_1 - x)^+$ and

$$f_1(x) = \begin{cases} -c_1 x + G_1(t_1, t_1), & x \leq t_1, \\ L(x), & x \geq t_1, \end{cases} \qquad f_1'(x) = \begin{cases} -c_1, & x \leq t_1, \\ L'(x), & x \geq t_1. \end{cases}$$

If $c_1 > r$ then $f_1(x) = L(x)$ for all x.

Note that the solution of $K(v) = 0$ is given by $F(\overline{v}) = (r + c_2 - c_1)/(r + p)$. Hence it exists only for $c_1 \leq c_2 + r$, otherwise $K(v) > 0$. That means, for $c_1 > c_2 + r$ one always takes $v = x$. In other words, in case (C) it is optimal never to sell assets and one has only to decide how much to borrow. For $n = 2$ it is easily seen that

$$S_2(u) = c_2 + \int_0^\infty L'(u-s)\varphi(s)\,ds = c_2 - r + (r+p)F^{2*}(u).$$

Thus, if $c_2 \le r$, there exists u_2 satisfying the relation $F^{2*}(u_2) = (r-c_2)/(r+p)$ and

$$f_2(x) = \begin{cases} -c_2x + L(x) + G_2(u_2, x), & x \le u_2, \\ L(x) + \int_0^\infty L(x-s)\varphi(s)\,ds, & x \ge u_2, \end{cases}$$

$$f_2'(x) = \begin{cases} -c_2 + L'(x), \\ L'(x) + \int_0^\infty L'(x-s)\varphi(s)\,ds. \end{cases}$$

If $c_2 > r$ then $f_2(x) = L(x) + \int_0^\infty L(x-s)\varphi(s)\,ds$ for all x.

Now we finish induction procedure for the case (C), namely, $c_1 > c_2+r$, under additional assumption $c_2 \le r$. Then

$$S_3(u) = S_2(u) + \int_0^\infty (f_2'(u-s) - f_1'(u-s))\varphi(s)\,ds$$

and $f_2'(x) - f_1'(x) = -c_2$ for $x \le u_2$, whence $S_3(u_2) < 0$ easily follows, so $u_3 > u_2$. Moreover,

$$(11)\quad S_n(u) = \int_0^\infty L'(x-s)\varphi(s)\,ds + \int_0^{u-u_{n-1}} S_{n-1}(u-s)\varphi(s)\,ds \ge -r + (r+p)F^{2*}(u),$$

leading to conclusion $u_n \le \widehat{u}$ with $F^{2*}(\widehat{u}) = r/(r+p)$. The sequence $\{u_n\}$ being bounded and nondecreasing there exists $\lim_{n\to\infty} u_n$ and its coincidence with $\widehat{u}$ is readily deduced from (**??**). The other subcases of (C), $(k-1)r \le c_2 \le kr$, $k > 1$, are treated similarly.

Next, assuming $c_1 \le c_2$, case (A), and $c_1 \le r$, one has, for $n = 2$,

$$S_2(u) = c_2 - c_1 + \int_0^{u-t_1} T_1(u-s)\varphi(s)\,ds > 0,$$

hence $u = v$ (there is no need to borrow) and it remains to consider

$$T_2(u) = L'(u) + \int_0^{u-t_1} T_1(u-s)\varphi(s)\,ds.$$

From this point one proceeds in the same way as in case (C), establishing that the sequence $\{t_n\}$ is bounded, nondecreasing and $\lim_{n\to\infty} t_n = \bar{t}$ with $L'(\bar{t}) = 0$.

Finally, in case (B), that is, $c_2 < c_1 \le c_2 + r$, there exists $\overline{v}$ satisfying $K(\overline{v}) = 0$. So, for $n \ge 2$, one has to choose between decision I with $u = v = \max(t_n, x)$ and decision III with $v = \max(\overline{v}, x)$ and $u = \max(u_n, x)$.

Assume additionally, $l = 1$, $m = 2$, for $n = 2$, that is, $2c_2 \le c_1 \le r$. Since

$$S_2(u) = c_2 - c_1 + \int_0^{u-t_1} [c_1 + L'(u-s)]\varphi(s)\,ds,$$

it follows easily that $S_2(\overline{v}) < 2c_2 - c_1 \le 0$. Therefore there exists $u_2 > \overline{v}$ satisfying $S_2(u_2) = 0$ and decision III is optimal for $n = 2$. The first step of induction is verified. Assuming the same is true for all steps up to n one gets

$$f_n'(x) = \begin{cases} -c_1, & x < \overline{v}, \\ -c_2 + L'(x), & \overline{v} \le x < u_n, \\ -c_2 + L'(x) + S_n(x), & x \ge u_n. \end{cases}$$

Then $S_{n+1}(u) = S_n(u) + (f_n' - f_{n-1}') * F(u)$ where

$$f_n'(x) - f_{n-1}'(x) = \begin{cases} 0, & x < u_{n-1}, \\ -S_{n-1}(x), & u_{n-1} \le x < u_n, \\ (f_{n-1}' - f_{n-2}') * F(x), & x \ge u_n. \end{cases}$$

It is obvious that $S_{n+1}(u_n) < 0$, that is, $u_{n+1} > u_n$. Moreover, for any $n > 2$,

$$S_n(u) \ge H(u) = c_2 - c_1 + \int_0^{u-\overline{v}} K(u-s)\varphi(s)\,ds,$$

whence it follows $u_n \le \overline{u}$ and $H(\overline{u}) = 0$. It is also not difficult to establish that $\overline{u} = \lim_{n\to\infty} u_n$.
The other combinations of l and m are treated along the same lines. □

Now recall *advantages* and *shortcomings* of dynamic programming. Equation (**??**) lets to use n-fold minimization of a function of two variables instead of minimization of a function of $2n$ variables. However the planning horizon n must be known beforehand and it cannot be changed later. Critical levels t_n and u_n are calculated recurrently.

Fortunately, there exists a stationary asymptotically optimal policy Π. Its form depends on the parameters set one considers. Let x_{k-1} be the capital at the end of the $(k-1)$-th year, $k \geq 1$, and $x_0 = x$. Then Π prescribes:

In case (A) sell the assets to get the money amount $(\overline{t} - x_{k-1})^+$.

In case (B) sell the assets to get $(\overline{v} - x_{k-1})^+$ and borrow $\min[(\overline{u} - \overline{v}), (\overline{u} - x_{k-1})^+]$.

In case (C) borrow $(\widehat{u} - x_{k-1})^+$.

Due to the lack of space we treat below only cases (A) and (B), denoting by $\widehat{f}_n(x)$ the n years expected costs incurred under policy Π.

Theorem 4. *Stationary policy* Π *is asymptotically optimal, that is, for any* $x \in \mathbb{R}$

$$\lim_{n\to\infty} n^{-1}\widehat{f}_n(x) = \lim_{n\to\infty} n^{-1} f_n(x).$$

The **proof** consists of two lemmas.

Lemma 1. *For any* $x \in \mathbb{R}$*, there exists*

$$\lim_{n\to\infty} n^{-1}\widehat{f}_n(x) = d_0,$$

where in case (A)

$$d_0 = \int_0^\infty [c_1 s + L(\overline{t} - s)]\varphi(s)\, ds$$

and in case (B)

$$d_0 = \int_0^{\overline{u}-\overline{v}} [c_2 s + L(\overline{u} - s)]\varphi(s)\, ds + \int_{\overline{u}-\overline{v}}^\infty [c_1 s + (c_2 - c_1)(\overline{u} - \overline{v}) + L(\overline{v})]\varphi(s)\, ds.$$

P r o o f. We consider below the more complicated case (B), the changes in case (A) being obvious. If $x \leq \overline{u}$ then it is easily seen that $x_k = \overline{u} - \xi_k$, for all $k \geq 1$. Therefore $\widehat{f}_n(x) = \widehat{f}_1(x) + (n-1)d_0$ with

$$\widehat{f}_1(x) = \begin{cases} c_1(\overline{v} - x) + c_2(\overline{u} - \overline{v}) + L(\overline{v}), & x \leq \overline{v}, \\ c_2(\overline{u} - x) + L(x), & \overline{v} < x \leq \overline{u}. \end{cases}$$

Since $\widehat{f}_1(x)$ is finite for a fixed $x \le \overline{u}$, the statement of Lemma is obvious for such x.

If $x > \overline{u}$ nothing is done until the capital becomes less than $\overline{u}$, therefore

$$x_k = \begin{cases} x - \sum_{i=1}^{k} \xi_i, & k \le \nu_x, \\ \overline{u} - \xi_k, & k > \nu_x, \end{cases}$$

where

$$\nu_x = \max\left\{k : \sum_{i=1}^{k-1} \xi_i < x - \overline{u}\right\}.$$

It follows immediately that

$$\widehat{f}_n(x) = \sum_{m=1}^{n-1} \mathsf{P}(\nu_x = m)\varkappa(x, m, n) + \mathsf{P}(\nu_x \ge n)\varkappa_1(x, n).$$

The functions $\varkappa(x, m, n)$ and $\varkappa_1(x, n)$, being the expected conditional n years costs under assumptions $\nu_x = m$ and $\nu_x \ge n$ respectively, have the form

$$\varkappa_1(x, n) = L(x) + \sum_{k=1}^{n-1} \int_0^{x-\overline{u}} L(x - s)\varphi^{k*}(s)\, ds,$$

$$\varkappa(x, m, n) = \varkappa_1(x, m) + \varkappa_2(x, m) + (n - m - 1)d_0$$

with $\varkappa_2(x, m)$ given by

$$\int_{x-\overline{u}}^{x-\overline{v}} [c_2(\overline{u} - x + s) + L(x - s)]\varphi^{m*}(s)\, ds + \int_{x-\overline{v}}^{\infty} [c_1(\overline{v} - x + s) + c_2(\overline{u} - \overline{v}) + L(\overline{v})]\varphi^{m*}(s)\, ds.$$

After some transformations one gets $\widehat{f}_n(x) = nd_0 \sum_{m=1}^{n-1} \mathsf{P}(\nu_x = m) + \delta_n(x)$ where

$$-d_0(2 + \mathsf{E}\nu_x) \le \delta_n(x) \le c_2(\overline{u} - \overline{v}) + [c_1\gamma + l(x)](1 + \mathsf{E}\nu_x).$$

Since $l(x) = \max_{z \in [\overline{v}, x]} L(z)$ and $\mathsf{E}\nu_x$ are finite for any $x > \overline{u}$, the proof is completed. □

Lemma 2. *For any $x \in \mathbb{R}$, there exists*

$$\lim_{n\to\infty} n^{-1}(\widehat{f}_n(x) - f_n(x)) = 0.$$

P r o o f. Let $f_n^l(x)$ denote the expected n-step costs under assumption that $(\overline{u}, \overline{v})$-policy is used during the first l steps and after that we apply the optimal policy. It is obvious that $f_n(x) = f_n^0(x)$ and $\widehat{f}_n(x) = f_n^n(x)$. Since $u_n \to \overline{u}$, as $n \to \infty$, for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that $\overline{u} \geq u_n \geq \overline{u} - \varepsilon$, for $n > N$. Furthermore,

$$|f_{N+r}^r(x) - f_{N+r}(x)| \leq \sum_{i=0}^{r-1} |f_{N+r}^{i+1}(x) - f_{N+r}^i(x)|, \tag{12}$$

and

$$\max_x |f_{N+r}^{i+1}(x) - f_{N+r}^i(x)| \leq \max_x |f_{N+r-i}^1(x) - f_{N+r-i}^0(x)|. \tag{13}$$

Hence, taking into account (**??**) and (**??**) along with $\max_x |f_n^1(x) - f_n^0(x)| \leq d\varepsilon$, for $n > N$, and $d = c_1 + 2c_2 + mp$, if $mc_2 \leq (m-1)c_1$, we obtain

$$|f_{N+r}^r(x) - f_{N+r}(x)| \leq rd\varepsilon.$$

To finish the proof, note that $|\widehat{f}_{N+r}(x) - f_{N+r}^r(x)| \leq B(x)$ where

$$B(x) = \max_{\overline{u} \leq y \leq \max(x, \overline{u})} \int_0^\infty (\widehat{f}_N(y-s) + f_N(y-s))\varphi(s)\, ds$$

is finite for any x. □

For the investigation of the unknown demand case, under assumptions (A) and (B), we need also another result, namely, the *stability of the policy* Π with respect to small perturbations of distribution F.

Denote by v^k (resp. u^k) the values of $\overline{v}$ (resp. $\overline{u}$) obtained taking $F_k(t)$ instead of $F(t)$ (t^k being value of $\overline{t}$). Moreover, set

$$\mu(F_k, F) = \sup_t |F_k(t) - F(t)|,$$

that is, μ is the Kolmogorov (or uniform) metric.

Lemma 3. *Let d.f's $F_k(t)$ be continuous and strictly increasing for any $k \geq 1$. Then $v^k \to \overline{v}$, $u^k \to \overline{u}$ (and $t^k \to \overline{t}$), provided that $\mu(F_k, F) \to 0$ as $k \to \infty$.*

P r o o f. According to assumptions $F_k(v^k) = F(\overline{v})$ and $|F_k(v^k) - F(v^k)| \leq \mu(F, F_k)$. Hence $|F(\overline{v}) - F(v^k)| \leq \mu(F, F_k)$. That means $v^k \to \overline{v}$, as $k \to \infty$. Similarly $t_k \to \overline{t}$.

In order to establish the result concerning u^k and $\overline{u}$ it is sufficient to verify that $\mu(H_k, H) \to 0$, as $k \to \infty$. It is not difficult to see that $\mu(H_k, H) \le (r + c_2 - c_1)\mu_{k1} + (r+p)\mu_{k2}$ with $\mu_{k1} = \mu(F_k, F) + w(|\overline{v} - v^k|)$, $\mu_{k2} = 2\mu(F_k, F) + w(|\overline{v} - v^k|)$ and $w(t) = \sup_{|s_1 - s_2| < t} |F(s_1) - F(s_2)|$. Since a continuous distribution function is uniformly continuous, $w(t) \to 0$, as $t \to 0$, ensuring $\mu(H_k, H) \to 0$ as $k \to \infty$. $\square$

4.2.2. Unknown distribution

In the most practical cases there is no a priori information concerning the distribution F.

Let ξ_i, $i = \overline{1,k}$, be loss observations during k years. The ordered sample $\eta_1 = \min_{1 \le i \le k} \xi_i \le \eta_2 \le \ldots \le \eta_k \max_{1 \le i \le k} \xi_i$ is used for calculation of empirical distribution function and its continuous analogue:

$$\tilde{F}_k(t) = \nu_k(t)/k, \qquad \widehat{F}_k(t) = k^{-1} \sum_{i=1}^{k} \phi_{kl}(t),$$

where $\nu_k(t) = \max\{i : \eta_i \le t\}$, $\phi_{kl}(t)$ is a d.f. of a r.v. uniformly distributed on (η_{l-1}, η_l), $l = \overline{1,k}$, $\eta_0 = 0$.

Lemma 4. *Let $\overline{t}_k$, $\overline{u}_k$ and $\overline{v}_k$ be the values of t^k, u^k and v^k corresponding to $F_k(t) = \widehat{F}_k(t)$. Then almost surely $\mu(F_k, F) \to 0$, $\overline{t}_k \to \overline{t}$, $\overline{u}_k \to \overline{u}$ and $\overline{v}_k \to \overline{v}$, as $k \to \infty$.*

As previously, let x_{k-1} be the capital at the beginning of the k-th step.

The empirical asymptotically optimal policy (EAOP) $\widehat{\Pi}$ is described as follows: at the first step do nothing, for $k \ge 2$ in case (A) sell assets to get $(\overline{t}_{k-1} - x_{k-1})^+$ and in case (B) sell assets to get $(\overline{v}_{k-1} - x_{k-1})^+$ and borrow $\min[(\overline{u}_{k-1} - \overline{v}_{k-1}), (\overline{u}_{k-1} - x_{k-1})^+]$.

The costs associated with $\widehat{\Pi}$ are

$$\widehat{\psi}_1 = p(x - \xi_1)^+ + r(\xi_1 - x)^+, \quad x_0 = x,$$

and, for $n > 1$, in case (A)

$$\widehat{\psi}_n = p(z_n - \xi_n)^+ + r(\xi_n - z_n)^+ + c_1(\overline{t}_{n-1} - x)^+,$$

with $z_n = \max(x, \overline{t}_{n-1})$, whereas in case (B), $\widehat{\psi}_n =$

$$p(y_n - \xi_n)^+ + r(\xi_n - y_n)^+ + c_1(\overline{v}_{n-1} - x_{n-1})^+ + c_2 \min[(\overline{u}_{n-1} - \overline{v}_{n-1}), (\overline{u}_{n-1} - x_{n-1})^+],$$

$y_n = \max(x_{n-1}, \overline{v}_{n-1})$ being the capital after assets selling. Then $G_k(x) = \sum_{n=1}^{k} \mathsf{E}\widehat{\psi}_n$ represents the k steps expected costs under policy $\widehat{\Pi}$.

Theorem 5. *The policy* $\widehat{\Pi}$ *is asymptotically optimal.*

In view of Lemma **??** it is enough to verify

$$\lim_{n\to\infty} n^{-1} G_n(x) = d_0.$$

In case (B) the **proof** is based on the following two results.

Lemma 5. *For any* $\varepsilon > 0$ *there exist a subset* Ω_ε *and positive* $N(\varepsilon)$ *such that* $\mathsf{P}(\Omega_\varepsilon) > 1-\varepsilon$ *and* $|\overline{u} - \overline{u}_n| < \varepsilon$, $|\overline{v} - \overline{v}_n| < \varepsilon$, $x_n < \overline{u} + \varepsilon$ *on* Ω_ε, *for all* $n \geq N(\varepsilon)$.

Lemma 6. *A sequence* $\mathsf{E}\widehat{\psi}_n$ *converges to* d_0 *as* $n \to \infty$.

The hard part of the proof is to establish $x_n < \overline{u} + \varepsilon$, for all $n \geq N(\varepsilon)$, on Ω_ε. Analogous statement is proved in [**?**].

Thus we have proposed the following algorithm for construction of EAOP which can be used in other applications as well:

1. The *first step* is to establish the optimal policy for the known distributions of underlying processes.
2. The *second step* is to construct a stationary asymptotically optimal policy under the same assumptions about distributions of underlying processes.
3. The *third step* is to find an empirical asymptotically optimal policy (EAOP).

4.2.3. Sensitivity analysis of the model

The results of Theorem **??** can be written in the compact form as follows:

$$\mathsf{(A)} \Longleftrightarrow \mathsf{I}, \quad \mathsf{(B)} \Longleftrightarrow \mathsf{III}, \quad \mathsf{(C)} \Longleftrightarrow \mathsf{II},$$

where decisions I, II and III were introduced at the beginning of Section 4.2. Fig. 5 shows clearly the stability domains, that is, the sets of parameters c_1, c_2 with the same type of optimal policy.

Now we study the system response to parameters variations. Namely, we are interested in the behaviour of critical levels $\overline{t}$, $\overline{u}$, $\overline{v}$ and $\widehat{u}$ determining the asymptotically optimal policy for the cases (A), (B) and (C) respectively.

Note that in the case (B), for any fixed r and p, critical levels $\overline{u}, \overline{v}$ depend on $b = c_1 - c_2$ only. From (**??**)-(**??**) it is not difficult to obtain the following result.

Corollary 3. *If* b *increases from* 0 *to* r *the function* $\overline{v}(b)$ *decreases from* $\overline{t}$ *to* 0, *and the function* $\overline{u}(b)$ *increases from* $\overline{t}$ *to* $\widehat{u}$.

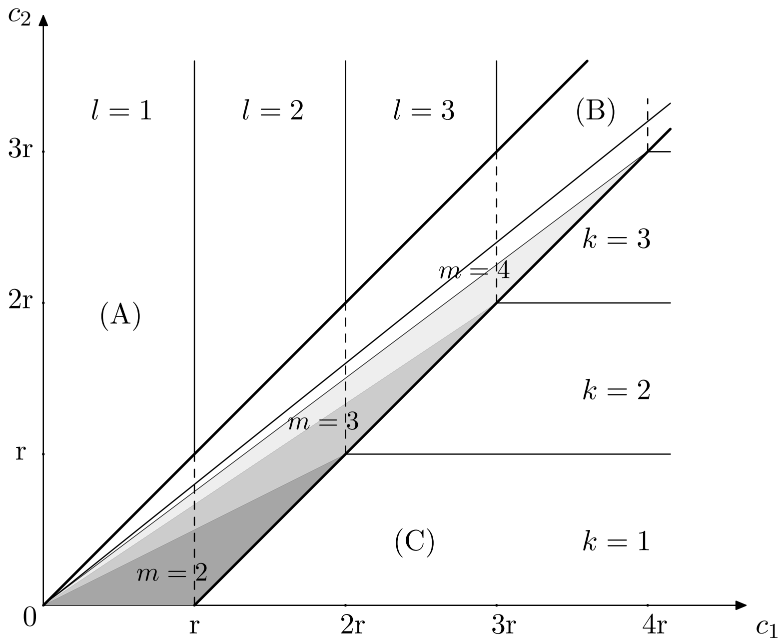

Fig. 5. Stability domains

Hence, although on the boundary $c_2 = c_1$, between (A) and (B), one smoothly switches from policy of type I to that of type III ($\overline{v}(0) = \overline{u}(0) = \overline{t}$), on the other boundary $c_2 = c_1 - r$, between (B) and (C), there is a "jump", since switching from type III to type II means, in fact, choosing $\overline{v} = -\infty$.

Next we use the local SA calculating the differential importance measure (DIM) defined by (**??**). Recall that, according to (**??**), $\overline{t}$ (resp. $\widehat{u}$) are implicit functions $g(a)$ of parameters $a_1 = r$ and $a_2 = p$, whereas $\overline{u}$ and $\overline{v}$ are functions of three parameters, that is, a_1, a_2 and $a_3 = b$.

Theorem 6. *Under assumptions (A), (B) and (C) the differential importance measures for all parameters do not depend on distribution F introduced by* (**??**).

P r o o f. Since $F(\overline{t}) = h(a_1, a_2)$ with $h(a_1, a_2) = a_1/(a_1 + a_2)$, one gets

$$\frac{\partial g}{\partial a_i} = \varphi(\overline{t})\frac{\partial h}{\partial a_i}, \quad i = 1, 2, \tag{14}$$

whence the statement of Theorem for $\overline{t}$ is obvious. The same is true for $\widehat{u}$, since instead of $\varphi(\overline{t})$ one has to put $\varphi^{2*}(\widehat{u})$ in (**??**).

Moreover, $F(\overline{v}) = h(a_1, a_2, a_3)$ with $h(a_1, a_2, a_3) = (a_1 - a_3)/(a_1 + a_2)$. Therefore expressions (**??**) can be written for $\overline{v}$ as well, $i = 1, 2, 3$.

On the other hand, from (**??**) and (**??**) one concludes that

$$\int_0^{\overline{u}-\overline{v}} K(\overline{u} - s)\varphi(s)\, ds = a_3.$$

Thus, for $i = 1, 2$,

$$\left[\frac{\partial \overline{u}}{\partial a_i} - \frac{\partial \overline{v}}{\partial a_i}\right] K(\overline{v})\varphi(\overline{u} - \overline{v}) + \frac{\partial \overline{u}}{\partial a_i}\int_0^{\overline{u}-\overline{v}} K'(\overline{u} - s)\varphi(s)\, ds = 0.$$

Taking into account that $K(\overline{v}) = 0$ and φ is strictly positive one gets $\partial\overline{u}/\partial a_i = 0$, $i = 1, 2$. In a similar way

$$\frac{\partial \overline{u}}{\partial a_3} = \left[(a_1 + a_2)\int_0^{\overline{u}-\overline{v}} \varphi(\overline{u} - s)\varphi(s)\, ds\right]^{-1}.$$

Therefore DIMs of $\overline{u}$ with respect to parameters a_1 and a_2 are equal 0, while for parameter a_3 it is equal to 1. Hence, the parameter a_3 is the most influential and DIM's do not depend on the distribution F. □

Now we calculate DIMs when g represents $\overline{t}$, for graph of $D1_1(a^0)$ see Fig. 6.

It follows from (**??**) that, under uniform parameters changes, DIMs for $\overline{t}$ and $\widehat{u}$ are given by

$$D1_1(a^0) = \frac{a_2^0}{a_2^0 - a_1^0}, \quad D1_2(a^0) = -\frac{a_1^0}{a_2^0 - a_1^0} = 1 - D1_1(a^0), \tag{15}$$

that is, they are well defined for $a_1^0 \neq a_2^0$. Moreover, $D1_1(a^0) > 1$, $D1_2(a^0) < 0$ for $a_2^0 > a_1^0$ and $D1_1(a^0) < 0$, $D1_2(a^0) > 1$ for $a_2^0 < a_1^0$. On the other hand, for

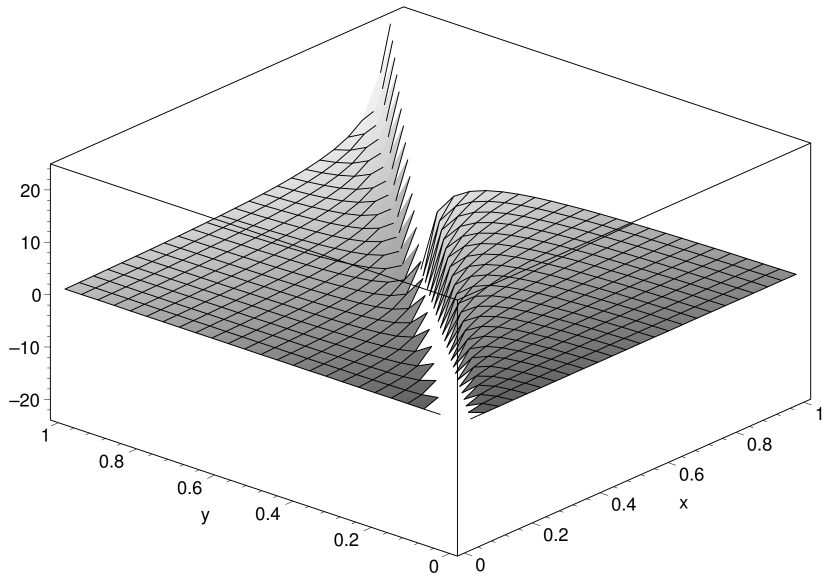

Fig. 6. $D1_1(a^0)$ for $\overline{t}$

proportional parameters changes DIMs do not exist for all $(a_1^0, a_2^0) \in K^2$.

Furthermore, for $\overline{v}$, under uniform parameters changes,

$$D1_1(a^0) = \frac{a_2^0 + a_3^0}{2(a_3^0 - a_1^0)}, \quad D1_2(a^0) = \frac{1}{2}, \quad D1_3(a^0) = -\frac{a_2^0 + a_1^0}{2(a_3^0 - a_1^0)},$$

that is, they are well defined for $a_3^0 \neq a_1^0$ (in other words, except the boundary between (B) and (C)). However, for proportional parameters changes, DIMs do not exist.

Now we turn to global SA. Using Corollary **??** and Definition **??** one can write

$$IG(a_1) = 1 - \frac{V_2}{V}, \quad IG(a_2) = 1 - \frac{V_1}{V}.$$

Suppose the i-th parameter to be uniformly distributed on $(a_i^0(1-k), a_i^0(1+k))$, $0 < k < 1$, $i = 1, 2$. The aim is to obtain the global sensitivity indices as functions of k. Unfortunately, even for simplest distributions F, such as uniform or

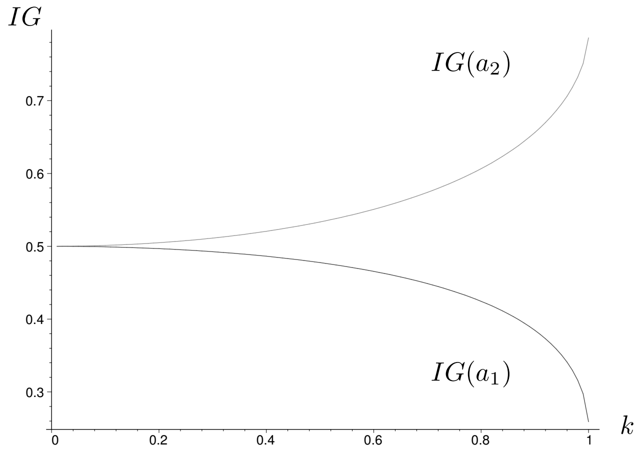

Fig. 7 Global indices

exponential, it turned out impossible to get the explicit form of $IG(a_i)$. However, using for example Maple 8, it is not difficult to draw the graphs desired.

Moreover, in contrast to local SA, it is possible to study the case $a_1^0 = a_2^0$. Thus, Fig. 7 gives the form of global indices as functions of k (for $g(a) = \bar{t}$ and $a_1^0 = a_2^0 = 0.5$). It is easily seen that $IG(a_1)$ decreases, whereas $IG(a_2)$ increases, as k increases, the parameter a_2 being more influential. For $k = 0.25$ one has $IG(a_1) = 0.4949060035$ and $IG(a_2) = 0.5077327062$, that means, even if the relative error in parameters estimation is 25%, the model behaves "almost additively", V_{12} giving only 0.2% of total variance. Moreover, for $k = 0.5$ one has $IG(a_1) = 0.4775592923$ and $IG(a_2) = 0.5334429254$ and V_{12} gives 1.1% of total variance. Hence, parameters interaction is higher for larger errors in their estimation.

Using transformation (**??**) with $\omega = (11, 5)$ expression of $\bar{t}$ was plotted as a function of s for exponential distribution function $F(t) = 1 - e^{-t}$, $t \geq 0$, see Fig. 8 which shows also that parameter a_2 is more influential than a_1.

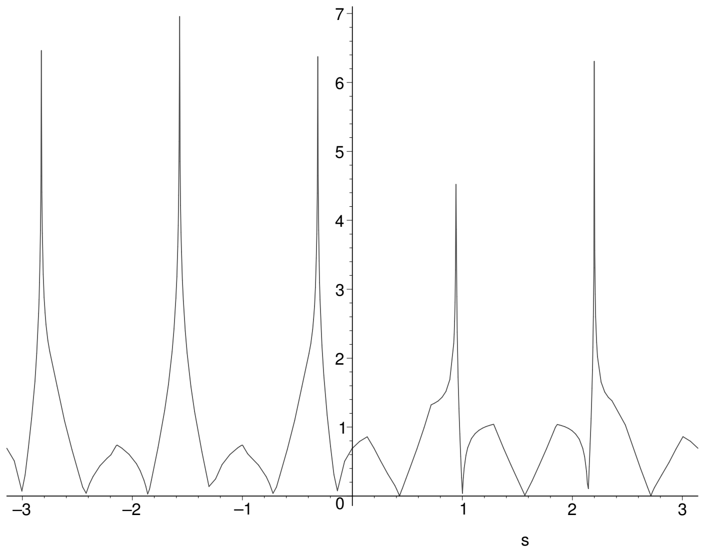

Fig. 8. $\bar{t}(s)$ for exponential F

5. Conclusions

We have recalled the Differential Importance Measure and two global sensitivity analysis techniques (the Sobol' decomposition and FAST) and applied them to the classical Harris-Wilson EOQ formula and a discrete-time insurance model.

It turned out that for the first model it is possible to get the explicit form of global sensitivity indices. Moreover, it was established that for uniform and Gamma distributions of parameters these indices do not depend on the base values.

The second model is a new one. In contrast to usual practice in insurance theory where the time is assumed to be continuous and the objective function is a ruin probability we consider a discrete-time model in the framework of cost approach. Since the reserves are formed by the end of the year, as well as reinsurance treaties have the one year length, the decisions being also made by the end of the year, it is reasonable to study the company behaviour by the end of the year. It is interesting to mention that in the context of dividends payment some authors also began to study the company behaviour "after the ruin". That means that shareholders can use their money to raise the company capital to some positive level, see, e.g. [**?**]. Here we suppose that, having doubts whether the cash available will suffice to pay the indemnity, the company may decide to sell some assets or to borrow money.

At first we studied the optimal behaviour using the dynamic programming. The existence of stationary asymptotically optimal policy was established under the assumption of known loss distribution. The last step was construction of empirical asymptotically optimal policy in case of incomplete information or when nothing is known about the distribution of loss. Thus, we proposed an algorithm of obtaining EAOP which may be useful in other research fields.

At last we studied the model dependence on cost parameters using the above mentioned techniques. It is interesting to note that DIMs for all critical levels do not depend on distribution. Since it is impossible to obtain the explicit formula for global indices, the numerical calculations were performed.

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