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OPTIMUM CHEMICAL BALANCE WEIGHING DESIGN UNDER CERTAIN CONDITION

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The problem of estimation of unknown weights of p objects is considered. The experiment is carried out according to the standard Gauss-Markoff model of the chemical balance weighing design. Existence conditions of the optimum design are given. New construction method of the optimum design based on the set of the incidence matrices of the ternary balanced block designs is presented.

1. Introduction

The chemical balance weighing design is determined by the model

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e},$$

where

(1) $\mathbf{y}$ is an $n \times 1$ random observed vector of the recorded results of weights,

(2) the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\ (-1,0,1)$, where $\mathbf{\Phi}_{n\times p,m}\ (-1,0,1)$ denotes the class of $n \times p$ matrices of the chemical balance weighing design with elements equal to $-1,\ 0$ or 1,

(3) $m = \max\{m_1, m_2, \ldots, m_p\}$, m_j is the number of elements equal to -1 and 1 in jth column of $\mathbf{X}$, $j = 1, 2, \ldots, p$,

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(4) $\mathbf{w}$ is a $p \times 1$ vector representing unknown weights of objects,

(5) $\mathbf{e}$ is $n \times 1$ random vector of errors,

(6) $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$ and $\mathrm{Var}(\mathbf{e}) = \sigma^2 \mathbf{I}_n$, where $\mathbf{0}_n$ is an $n \times 1$ null vector, $\mathbf{I}_n$ denotes $n \times n$ identity matrix.

The squares estimator of the vector $\mathbf{w}$ representing unknown weights of objects is equal to $\hat{\mathbf{w}} = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{y}$ assuming that $\mathbf{X}$ is of full column rank. The variance matrix of $\hat{\mathbf{w}}$ is given by $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2 (\mathbf{X}'\mathbf{X})^{-1}$.

The problems concerned on determining of unknown measurements of objects in the model of the chemical balance weighing design were considered in Banerjee (1975), Raghavarao (1971), Shah and Sinha (1989). Hotelling (1944) has shown that for a chemical balance weighing design the minimum attainable variance for each of the estimated weights is σ^2/n. He proved the theorem that each of the variance of the estimated weights attains the lower bound if and only if $\mathbf{X}'\mathbf{X} = n\mathbf{I}_p$. For this case several construction methods of the design matrix of the chemical balance weighing design are available in the literature (See Kageyama, Saha (1983), Swamy (1982), Ceranka and Katulska (1999)). The design for which the variance of each of the estimated weights attains its minimum is called by Hotelling optimal. In the case given by Hotelling elements of the design matrix $\mathbf{X}$ may be equal to -1 and 1, only. In this paper we generalize the optimality given by Hotelling. Sometimes, it is not possible to take into account all possible combinations of objects in each measurement operation. Hence in the design matrix $\mathbf{X}$ there are elements equal to 0. The aim of the paper is to present the lower bound of the variance of estimators of unknown weights of objects assuming that in $\mathbf{X}$ are elements equal to -1, 0 or 1 and to give certain ways of construction of the optimal design matrix.

2. Variance limit of estimated weights

For the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ Ceranka and Graczyk (2001) have obtained the following results.

Theorem 2.1. *In the nonsingular chemical balance weighing design* $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ *the variance of the estimated measurements of objects is*

$$\mathrm{Var}(\hat{w}_j) \geq \frac{\sigma^2}{m}, \qquad j = 1, 2, \ldots, p. \tag{1}$$

Definition 2.1. *Any nonsingular chemical balance weighing design is optimal if the variance of each of the estimators attains the lower bound given in* (1).

Theorem 2.2. *Any nonsingular chemical balance weighing design with the design matrix* $\mathbf{X} \in \mathbf{\Phi}_{n \times p,m}\,(-1,0,1)$ *is optimal if and only if*

$$\mathbf{X}'\mathbf{X} = m\mathbf{I}_p. \tag{2}$$

In next sections we give the construction method of the design matrix of the optimum chemical balance weighing design. It is based on the incidence matrices of the ternary balanced block designs.

3. Ternary balanced block designs

In this section we remind the definition of the ternary balanced block design given in Billington (1984).
A ternary balanced block design is defined as the design in which

(1) we place v treatments in b blocks, each of the size k,

(2) each treatment occurs r times altogether and 0, 1 or 2 times in each block,

(3) each of the distinct pairs of elements appears λ times,

(4) each treatment occurs once in ρ_1 blocks and twice in ρ_2 blocks, where ρ_1 and ρ_2 are constants for the design,

(5) $\mathbf{N}$ is the incidence matrix of the ternary balanced block design,

(6) the parameters are connected by the following equalities

$$vr = bk,$$

$$r = \rho_1 + 2\rho_2,$$

$$\lambda(v-1) = \rho_1(k-1) + 2\rho_2(k-2) = r(k-1) - 2\rho_2,$$

$$\mathbf{N}\mathbf{N}' = (\rho_1 + 4\rho_2 - \lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v' = (r + 2\rho_2 - \lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v'.$$

4. The design matrix

Let us consider the set of the incidence matrices $\mathbf{N}_h$ of the ternary balanced block designs with the parameters $v,\ b_h,\ r_h,\ k_h,\ \lambda_h,\ \rho_{1h},\ \rho_{2h},\ h = 1,2,3$. We form the design matrix $\mathbf{X}$ as

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}_1' - \mathbf{1}_{b_1}\mathbf{1}_v' \\ \mathbf{N}_2' - \mathbf{1}_{b_2}\mathbf{1}_v' \\ \mathbf{N}_3' - \mathbf{1}_{b_3}\mathbf{1}_v' \end{bmatrix}. \tag{3}$$

Each column of design matrix $\mathbf{X}$ contains $\sum\limits_{h=1}^{3} \rho_{2h}$ elements equal to 1, $\sum\limits_{h=1}^{3} (b_h - \rho_{1h} - \rho_{2h})$ elements equal to -1 and $\sum\limits_{i=h}^{3} \rho_{1h}$ elements equal to 0. That means each object is weighed $m = \sum\limits_{h=1}^{3} (b_h - \rho_{1h})$ times in $n = \sum\limits_{h=1}^{3} b_h$ measurement operations.

Lemma 4.1. *Any chemical balance weighing design with the design matrix* $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\,(-1,0,1)$ *given in* (3) *is nonsingular if and only if* $v \neq k_h$ *for at least one* $h,\ h = 1,2,3$.

Proof. For the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\,(-1,0,1)$ of the form (3) we have

$$\mathbf{X}'\mathbf{X} = \left(\sum_{h=1}^{3}(r_h - \lambda_h + 2\rho_{2h})\right)\mathbf{I}_v + \left(\sum_{h=1}^{3}(b_h + \lambda_h - 2r_h)\right)\mathbf{1}_v\mathbf{1}_v'. \tag{4}$$

Hence $\det(\mathbf{X}'\mathbf{X}) = \left(\sum\limits_{h=1}^{3}(r_h - \lambda_h + 2\rho_{2h})\right)^{v-1} \cdot \left(\sum\limits_{h=1}^{3} \frac{r_h}{k_h}(v-k_h)^2\right)$. Because $\sum\limits_{h=1}^{3}(r_h - \lambda_h + 2\rho_{2h}) > 0$ then $\det(\mathbf{X}'\mathbf{X}) = 0$ if and only if $v = k_h$ for each $h,\ h = 1,2,3$. Thus the lemma is proved. □

Theorem 4.1. *Nonsingular chemical balance weighing design with the design matrix* $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\,(-1,0,1)$ *given in* (3) *is optimal if and only if*

$$\sum_{h=1}^{3}(b_h + \lambda_h - 2r_h) = 0. \tag{5}$$

P r o o f. Nonsingular chemical balance weighing design with the matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\,(-1,0,1)$ of (3) is optimal if and only if condition (2) is fulfilled. Thus we compare the elements of both matrices. The diagonal elements of $\mathbf{X}'\mathbf{X}$ are $\sum_{i=1}^{3}(b_h - r_h + 2\rho_{2h})$. Taking into account relations between parameters of the ternary balanced block designs we get $\sum_{i=1}^{3}(b_h - \rho_{1h}) = m$. The offdiagonal elements of $\mathbf{X}'\mathbf{X}$ are $\sum_{i=1}^{3}(b_h + \lambda_h - 2r_h)$. Thus the theorem follows.

Now, we give the series of the parameters of the ternary balanced block designs. Based on the incidence matrices of these designs we, as the next step, form the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\,(-1,0,1)$ in (3) of the optimum chemical balance weighing design. □

Corollary 4.1. *If the parameters of the ternary balanced block design are equal to*

(i) $v = 5,\ b_1 = 20,\ r_1 = 12,\ k_1 = 3,\ \lambda_1 = \rho_{11} = \rho_{21} = 4$ *and* $v = 5,\ b_2 = 10,\ r_2 = 12,\ k_2 = 6,\ \lambda_2 = 13,\ \rho_{12} = \rho_{22} = 4$ *and* $v = 5,\ b_3 = 15,\ r_3 = 21,\ k_3 = 7,\ \lambda_3 = 28,\ \rho_{13} = \rho_{23} = 7,$

(ii) $v = k_1 = 9,\ b_1 = r_1 = u + 8,\ \lambda_1 = u + 7,\ \rho_{11} = u,\ \rho_{21} = 4$ *and* $v = 9,\ b_2 = 6s,\ r_2 = 4s,\ k_2 = 6,\ \lambda_2 = 2s + 1,\ \rho_{12} = 8,\ \rho_{22} = 2(s-2)$ *and* $v = 9,\ b_3 = 9s,\ r_3 = 11s,\ k_3 = 11,\ \lambda_3 = 13s,\ \rho_{13} = 5s,\ \rho_{23} = 3s,\quad s = 3,4,\ldots,\ u = 1,2,\ldots,$

(iii) $v = k_1 = 9,\ b_1 = r_1 = u + 17,\ \lambda_1 = u + 15,\ \rho_{11} = u + 1,\ \rho_{21} = 8$ *and* $v = 9,\ b_2 = 3(s+4),\ r_2 = 2(s+4),\ k_2 = 6,\ \lambda_2 = s + 5,\ \rho_{12} = 8,\ \rho_{22} = s$ *and* $v = 9,\ b_3 = 3(s+4),\ r_3 = 4(s+4),\ k_3 = 12,\ \lambda_3 = 5s + 21,\ \rho_{13} = 8,\ \rho_{23} = 2(s+2),\quad u,\ s = 1,2,\ldots,$

(iv) $v = k_1 = 11,\ b_1 = r_1 = u + 21,\ \lambda_1 = u + 19,\ \rho_{11} = u + 1,\ \rho_{21} = 10$ *and* $v = b_2 = 11,\ r_2 = k_2 = 7,\ \lambda_2 = 4,\ \rho_{12} = 5,\ \rho_{22} = 1$ *and* $v = b_3 = 11,\ r_3 = k_3 = 15,\ \lambda_3 = 20,\ \rho_{13} = \rho_{23} = 5,\quad u = 1,2,\ldots,$

(v) $v = b_1 = r_1 = k_1 = 11,\ \lambda_1 = 10,\ \rho_{11} = 1,\ \rho_{21} = 5$ *and* $v = b_2 = 11,\ r_2 = k_2 = 7,\ \lambda_2 = 4,\ \rho_{12} = 5,\ \rho_{22} = 1$ *and* $v = 11,\ b_3 = 44,\ r_3 = 48,\ k_3 = 12,\ \lambda_3 = 52,\ \rho_{13} = 40,\ \rho_{23} = 4,$

(vi) $v = k_1 = 12,\ b_1 = r_1 = u + 22,\ \lambda_1 = u + 20,\ \rho_{11} = u,\ \rho_{21} = 11$ *and* $v = 12,\ b_2 = 3(2s+5),\ r_2 = 2(2s+5),\ k_2 = 8,\ \lambda_2 = 2(s+3),\ \rho_{12} = 6-2s,\ \rho_{22} = 3s + 2$ *and* $v = 12,\ b_3 = 3(2s+5),\ r_3 = 4(2s+5),\ k_3 = 16,\ \lambda_3 = 2(5s+13),\ \rho_{13} = 6 - 2s,\ \rho_{23} = 5s + 7,\quad s = 0,1,2,\ u = 1,2,\ldots,$

(vii) $v = 12,\ b_1 = 18,\ r_1 = 15,\ k_1 = 10,\ \lambda_1 = 11,\ \rho_{11} = 1,\ \rho_{21} = 7$ *and* $v = 12,\ b_2 = 15,\ r_2 = 10,\ k_2 = 8,\ \lambda_2 = 6,\ \rho_{12} = 6,\ \rho_{22} = 2$ *and* $v = 12,\ b_3 = 48,\ r_3 = 56,\ k_3 = 14,\ \lambda_3 = 52,\ \rho_{13} = 48,\ \rho_{23} = 4$,

(viii) $v = k_1 = 12,\ b_1 = r_1 = 46,\ \lambda_1 = 44,\ \rho_{11} = 24,\ \rho_{21} = 11$ *and* $v = 12,\ b_2 = 3(2s+5),\ r_2 = 2(2s+5),\ k_2 = 8,\ \lambda_2 = 2(s+3),\ \rho_{12} = 6-2s,\ \rho_{22} = 3s+2$ *and* $v = 12,\ b_3 = 3(2s+5),\ r_3 = 4(2s+5),\ k_3 = 16,\ \lambda_3 = 2(5s+13),\ \rho_{13} = 6-2s,\ \rho_{23} = 5s+7, \quad s = 0,1,2$,

(ix) $v = k_1 = 15,\ b_1 = r_1 = u+29,\ \lambda_1 = u+27,\ \rho_{11} = u+1,\ \rho_{21} = 14$ *and* $v = 15,\ b_2 = 3(s+4),\ r_2 = 2(s+4),\ k_2 = 10,\ \lambda_2 = s+5,\ \rho_{12} = 6-2s,\ \rho_{22} = 2s+1$ *and* $v = 15,\ b_3 = 3(s+4),\ r_3 = 4(s+4),\ k_3 = 20,\ \lambda_3 = 5s+21,\ \rho_{13} = 6-2s,\ \rho_{23} = 3s+5, \quad s = 1,2,\ u = 1,2,\ldots$,

then $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ *in (3) is the design matrix of the optimum chemical balance weighing design.*

Corollary 4.2. *If the parameters of the first ternary balanced block design are equal to*

(i) $v = k_1 = 5,\ b_1 = r_1 = 10,\ \lambda_1 = 8,\ \rho_{11} = 2,\ \rho_{21} = 4$,

(ii) $v = k_1 = 5,\ b_1 = r_1 = 14,\ \lambda_1 = 12,\ \rho_{11} = 6,\ \rho_{21} = 4$ *or*

(iii) $v = k_1 = 5,\ b_1 = r_1 = 22,\ \lambda_1 = 20,\ \rho_{11} = 14,\ \rho_{21} = 4$

and the parameters of second and third ternary balanced block designs are equal to $v = 5,\ b_2 = 5(s+2),\ r_2 = 3(s+2),\ k_2 = 3,\ \lambda_2 = s+3,\ \rho_{12} = s+6,\ \rho_{22} = s$ *and* $v = 5,\ b_3 = 5(s+2),\ r_3 = 7(s+2),\ k_3 = 7,\ \lambda_3 = 3s+7,\ \rho_{13} = s+6,\ \rho_{23} = 3s+4, \quad s = 1,2,\ldots,$ *respectively, then* $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1,0,1)$ *in the form* (3) *is the design matrix of the optimum chemical balance weighing design.*

Corollary 4.3. *If the parameters of the first ternary balanced block design are equal to* $v = k_1 = 5,\ b_1 = r_1 = u+4,\ \lambda_1 = u+3,\ \rho_{11} = u,\ \rho_{21} = 2$ *and the parameters of the second and third ternary balanced block designs are equal to*

(i) $v = 5,\ b_2 = 5(s+2),\ r_2 = 3(s+2),\ k_2 = 3,\ \lambda_2 = s+3,\ \rho_{12} = s+6,\ \rho_{22} = s$ *and* $v = 5,\ b_3 = 5(s+2),\ r_3 = 7(s+2),\ k_3 = 6,\ \lambda_3 = 9(s+2),\ \rho_{13} = s+2,\ \rho_{23} = 3(s+2)$,

(ii) $v = 5,\ b_2 = 20,\ r_2 = 12,\ k_2 = 3,\ \lambda_2 = 6,\ \rho_{12} = 8,\ \rho_{22} = 2$ *and* $v = 5,\ b_3 = 20,\ r_3 = 28,\ k_3 = 7,\ \lambda_3 = 36,\ \rho_{13} = 4,\ \rho_{23} = 12,$

where $u,\ s = 1, 2, \ldots,$ *then* $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\ (-1, 0, 1)$ *in the form (3) is the design matrix of the optimum chemical balance weighing design.*

Corollary 4.4. *If the parameters of the first ternary balanced block design are equal to* $v = k_1 = 5,\ b_1 = r_1 = u + 9,\ \lambda_1 = u + 7,\ \rho_{11} = u + 1,\ \rho_{21} = 4$ *and the parameters of the second and third ternary balanced block designs are equal to*

(i) $v = 5,\ b_2 = 5s,\ r_2 = 3s,\ k_2 = 3,\ \lambda_2 = s+2,\ \rho_{12} = s+8,\ \rho_{22} = s-4$ *and* $v = 5,\ b_3 = 5s,\ r_3 = 7s,\ k_3 = 7,\ \lambda_3 = 9s,\ \rho_{13} = s,\ \rho_{23} = 3s,\quad s = 5, 6, \ldots,$

(ii) $v = 5,\ b_2 = 5(s + 2),\ r_2 = 3(s + 2),\ k_2 = 3,\ \lambda_2 = s + 3,\ \rho_{12} = s + 6,\ \rho_{22} = s$ *and* $v = 5,\ b_3 = 5(s + 2),\ r_3 = 7(s + 2),\ k_3 = 7,\ \lambda_3 = 9s + 19,\ \rho_{13} = s + 6,\ \rho_{23} = 3s + 4,\quad s = 1, 2, \ldots,$ *or*

(iii) $v = 5,\ b_2 = 5(s+4),\ r_2 = 3(s+4),\ k_2 = 3,\ \lambda_2 = s+6,\ \rho_{12} = s+12,\ \rho_{22} = s$ *and* $v = 5,\ b_3 = 5(s + 4),\ r_3 = 7(s + 4),\ k_3 = 7,\ \lambda_3 = 9(s + 4),\ \rho_{13} = s + 4,\ \rho_{23} = 3(s + 4),\quad s = 1, 2, \ldots,$

where $u = 1, 2, \ldots,$ *then* $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\ (-1, 0, 1)$ *in the form* (3) *is the design matrix of the optimum chemical balance weighing design.*

Corollary 4.5. *If the parameters of the first ternary balanced block design are equal to* $v = k_1 = 6,\ b_1 = r_1 = u + 10,\ \lambda_1 = u + 8,\ \rho_{11} = u,\ \rho_{21} = 5$ *and the parameters of the second and third ternary balanced block designs are equal to*

(i) $v = 6,\ b_2 = 4s,\ r_2 = 2s,\ k_2 = 3,\ \lambda_2 = 2,\ \rho_{12} = 2(5 - s),\ \rho_{22} = 2s - 5$ *and* $v = 6,\ b_3 = 6s,\ r_3 = 8s,\ k_3 = 8,\ \lambda_3 = 10s,\ \rho_{13} = 2s,\ \rho_{23} = 3s,\quad s = 3, 4,$

(ii) $v = 6,\ b_2 = 16,\ r_2 = 8,\ k_2 = 3,\ \lambda_2 = 2,\ \rho_{12} = 2,\ \rho_{22} = 3$ *and* $v = 6,\ b_3 = 24,\ r_3 = 32,\ k_3 = 8,\ \lambda_3 = 40,\ \rho_{13} = 8,\ \rho_{23} = 12,$

where $u = 1, 2, \ldots,$ *then* $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\ (-1, 0, 1)$ *in the form (3) is the design matrix of the optimum chemical balance weighing design.*

Corollary 4.6. *If the parameters of the first ternary balanced block design are equal to*

(i) $v = k_1 = 9,\ b_1 = r_1 = 18,\ \lambda_1 = 16,\ \rho_{11} = 2,\ \rho_{21} = 8,$

(ii) $v = k_1 = 9,\ b_1 = r_1 = 30,\ \lambda_1 = 28,\ \rho_{11} = 14,\ \rho_{21} = 8$ *or*

(iii) $v = k_1 = 9,\ b_1 = r_1 = 38,\ \lambda_1 = 36,\ \rho_{11} = 22,\ \rho_{21} = 8$

and the parameters of second and third ternary balanced block designs are equal to $v = 9,\ b_2 = 3(s+4),\ r_2 = 2(s+4),\ k_2 = 6,\ \lambda_2 = s+5,\ \rho_{12} = 8,\ \rho_{22} = s$ *and* $v = 9,\ b_3 = 3(s+4),\ r_3 = 2(s+4),\ k_3 = 12,\ \lambda_3 = s+5,\ \rho_{13} = 8,\ \rho_{23} = s,\quad s = 1, 2, \ldots,$ *then* $\mathbf{X} \in \mathbf{\Phi}_{n \times p,m}(-1, 0, 1)$ *in the form* (3) *is the design matrix of the optimum chemical balance weighing design.*

Corollary 4.7. *If the parameters of the first ternary balanced block design are equal to*

(i) $v = k_1 = 11,\ b_1 = r_1 = 22,\ \lambda_1 = 20,\ \rho_{11} = 2,\ \rho_{21} = 10,$

(ii) $v = k_1 = 11,\ b_1 = r_1 = 38,\ \lambda_1 = 36,\ \rho_{11} = 18,\ \rho_{21} = 10$ *or*

(iii) $v = k_1 = 11,\ b_1 = r_1 = 46,\ \lambda_1 = 44,\ \rho_{11} = 26,\ \rho_{21} = 10$

and the parameters of second and third ternary balanced block designs are equal to $v = 11,\ b_2 = 11,\ r_2 = 7,\ k_2 = 7,\ \lambda_2 = 4,\ \rho_{12} = 5,\ \rho_{22} = 1$ *and* $v = 11,\ b_3 = 11,\ r_3 = 15,\ k_3 = 15,\ \lambda_3 = 20,\ \rho_{13} = \rho_{23} = 5,$ *then* $\mathbf{X} \in \mathbf{\Phi}_{n \times p,m}(-1, 0, 1)$ *in the form (3) is the design matrix of the optimum chemical balance weighing design.*

Corollary 4.8. *If the parameters of the first and second ternary balanced block designs are equal to* $v = k_1 = 11,\ b_1 = r_1 = u + 10,\ \lambda_1 = u + 9,\ \rho_{11} = u,\ \rho_{21} = 5$ *and* $v = b_2 = 11,\ r_2 = k_2 = 7,\ \lambda_2 = 4,\ \rho_{12} = 5,\ \rho_{22} = 1$ *and the parameters of third ternary balanced block design are equal to*

(i) $v = 11,\ b_3 = 22,\ r_3 = 26,\ k_3 = 13,\ \lambda_3 = 30\ \rho_{13} = 14,\ \rho_{23} = 6,$ *or*

(ii) $v = 11,\ b_3 = 44,\ r_3 = 48,\ k_3 = 12,\ \lambda_3 = 52\ \rho_{13} = 40,\ \rho_{23} = 4$

then $\mathbf{X} \in \mathbf{\Phi}_{n \times p,m}(-1, 0, 1)$ *in the form* (3) *is the design matrix of the optimum chemical balance weighing design.*

Corollary 4.9. *If the parameters of the first ternary balanced block design are equal to*

(i) $v = k_1 = 15,\ b_1 = r_1 = 30,\ \lambda_1 = 28,\ \rho_{11} = 2,\ \rho_{21} = 14,$

(ii) $v = k_1 = 15,\ b_1 = r_1 = 54,\ \lambda_1 = 52,\ \rho_{11} = 26,\ \rho_{21} = 14$ *or*

(iii) $v = k_1 = 15,\ b_1 = r_1 = 62,\ \lambda_1 = 60,\ \rho_{11} = 34,\ \rho_{21} = 14$

and the parameters of second and third ternary balanced block designs are equal to $v = 15,\ b_2 = 3(s+4),\ r_2 = 2(s+4),\ k_2 = 10,\ \lambda_2 = s+5,\ \rho_{12} = 6-2s,\ \rho_{22} = 2s+1$ *and* $v = 15,\ b_3 = 3(s+4),\ r_3 = 4(s+4),\ k_3 = 20,\ \lambda_3 = 2s+9,\ \rho_{13} = 6-2s,\ \rho_{23} = 3s+5,\quad s = 1, 2,$ *then* $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\,(-1,0,1)$ *in the form* (3) *is the design matrix of the optimum chemical balance weighing design.*

Corollary 4.10. *If the parameters of the first and second ternary balanced block designs are equal to* $v = k_1 = 15,\ b_1 = r_1 = u+14,\ \lambda_1 = u+13,\ \rho_{11} = u,\ \rho_{21} = 7$ *and* $v = 15,\ b_2 = 18,\ r_2 = 12,\ k_2 = 10,\ \lambda_2 = 7,\ \rho_{12} = 2,\ \rho_{22} = 5$ *and the parameters of third ternary balanced block design are equal to*

(i) $v = 15,\ b_3 = 30,\ r_3 = 36,\ k_3 = 18,\ \lambda_3 = 42,\ \rho_{13} = \rho_{23} = 12$ *or*

(ii) $v = 15,\ b_3 = 45,\ r_3 = 51,\ k_3 = 17,\ \lambda_3 = 57,\ \rho_{13} = 33,\ \rho_{23} = 9$

where $u = 1, 2, \ldots,$ *then* $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\,(-1,0,1)$ *in the form* (3) *is the design matrix of the optimum chemical balance weighing design.*

5. Example

In the experiment we want to determine unknown measurements of $p = 5$ objects in $n = 40$ measurement operations. We assume that in each weighing each object occurs at most $m = 24$ times. We consider the ternary balanced block designs (See Theorem 4.3 (i)) with the parameters $v = k_1 = 5,\ b_1 = r_1 = 10,\ \lambda_1 = 8,\ \rho_{11} = 2,\ \rho_{21} = 4,\ v = 5,\ b_2 = 15,\ r_2 = 9,\ k_2 = 3,\ \lambda_2 = 4,\ \rho_{12} = 7,\ \rho_{22} = 1,\ v = 5,\ b_3 = 15,\ r_3 = 21,\ k_3 = 7,\ \lambda_3 = 10,\ \rho_{13} = \rho_{23} = 7$ and with the incidence matrices

$$\mathbf{N}_1 = \begin{bmatrix} 2 & 0 & 0 & 2 & 1 & 2 & 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 & 2 & 1 & 2 & 0 & 0 & 2 \\ 2 & 1 & 2 & 0 & 0 & 2 & 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 2 & 0 & 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 2 & 0 & 0 & 2 & 1 & 2 \end{bmatrix},$$

$$\mathbf{N}_2 = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathbf{N}_3 = \begin{bmatrix} 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 0 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 0 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 0 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Based on these matrices we form the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\ (-1,0,1)$ in (3) as $\mathbf{X} = [\mathbf{A}_1 \quad \mathbf{A}_2 \quad \mathbf{A}_3]'$, where

$$\mathbf{A}_1' = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 & 0 & 1 & -1 & -1 \\ -1 & 1 & 0 & 1 & -1 & -1 & 1 & 0 & 1 & -1 \\ -1 & -1 & 1 & 0 & 1 & -1 & -1 & 1 & 0 & 1 \end{bmatrix},$$

$\mathbf{A}_2' =$

$$\begin{bmatrix} 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_3' = \begin{bmatrix} -1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For unknown measurements of objects $\operatorname{Var}(\hat{w}_j) = \dfrac{\sigma^2}{24}$.

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