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BOOTSTRAP FOR CRITICAL BRANCHING PROCESS WITH NON-STATIONARY IMMIGRATION

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In the critical branching process with a stationary immigration the standard parametric bootstrap for an estimator of the offspring mean is invalid. We consider the process with non-stationary immigration, whose mean and variance $\alpha(n)$ and $\beta(n)$ are finite for each $n \geq 1$ and are regularly varying sequences with nonnegative exponents α and β, respectively. It turns out that if $\alpha(n) \to \infty$ and $\beta(n) = o(n\alpha^2(n))$ as $n \to \infty$, then the standard parametric bootstrap procedure leads to a valid approximation for the distribution of the conditional least squares estimator. We state a theorem which justifies the validity of the bootstrap. By Monte-Carlo and bootstrap simulations for the process we confirm the theoretical findings. The simulation study highlights the validity and utility of the bootstrap in this model as it mimics the Monte-Carlo pivots even when generation size is small.

1. Introduction

Let $Z(n), n \geq 0, Z(0) = 0$ be a discrete time branching stochastic process with immigration. It is defined by two families of independent, nonnegative integer

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valued random variables $\{X_{ni}, n, i \geq 1\}$ and $\{\xi_n, n \geq 1\}$ recursively as

$$Z(n) = \sum_{i=1}^{Z(n-1)} X_{ni} + \xi_n, \quad n \geq 1. \tag{1.1}$$

Assume that X_{ni} have a common distribution for all n and i, and families $\{X_{ni}\}$ and $\{\xi_n\}$ are independent. Variables X_{ni} will be interpreted as the number of offspring of the ith individual in the $(n-1)$th generation and ξ_n is the number of immigrating individuals in the nth generation. Then $Z(n)$ can be considered as the size of nth generation of the population.

In this interpretation, $a = EX_{ni}$ the mean number of offspring of a single individual is crucial. Process $Z(n)$ is called *subcritical, critical* or *supercritical* depending on $a < 1$, $a = 1$ or $a > 1$, respectively. The independence assumption of families $\{X_{ni}\}$ and $\{\xi_n\}$ means that reproduction and immigration processes are independent. However, unlike in the classical models, we do not assume that $\xi_n, n \geq 1$, are identically distributed, i. e. the immigration distribution depends on the time of immigration. It is well known that asymptotic behavior of the process with immigration is very sensitive to any changes of the immigration process in time. In the case of Bernoulli offspring distribution, i.e. $P\{X_{ni} = 1\} = 1 - P\{X_{ni} = 0\} = p$, process defined in equation (1.1) can be considered as an integer-valued, first order autoregressive (INAR(1)) time series model with non stationary noise ξ_k.

As it was shown in [9], if a sample $\{Z(k), k = 1, \ldots, n\}$ is available, the weighted conditional least squares estimator (WCLSE) of the offspring mean is

$$\hat{a}_n = \frac{\sum_{k=1}^{n} (Z(k) - \alpha(k))}{\sum_{k=1}^{n} Z(k-1)}, \quad \alpha(k) = E\xi_k. \tag{1.2}$$

The maximum likelihood estimators (MLE) for the offspring and immigration means in the process with a stationary immigration were derived in [2] for the power series offspring and immigration distributions and are based on the sample of pairs $\{(Z(k), \xi_k),\ k = 1, \ldots, n\}$. The MLE for the offspring mean has the same form as $\hat{a}_n$ with ξ_k in place of $\alpha(k)$, and the MLE for the immigration mean is just the arithmetic mean of the number of immigrating individuals.

In [11] the author investigated validity of the bootstrap estimator of the offspring mean based on MLE and demonstrated that in the critical process with a stationary immigration the asymptotic validity of the parametric bootstrap does not hold. Similar invalidity of the parametric bootstrap for the first order autoregressive process with autoregressive parameter ± 1 was earlier proved in [1]. The main cause of the failure is the fact that in the critical case the MLE does not have the desired rate of convergence (faster than n^{-1}).

The results obtained recently in [9] show that in the process with non-stationary immigration the conditional least squares estimator (CLSE) may have a normal limit distribution and the rate of convergence of the CLSE to the parameter under estimation is faster than n^{-1}. Given this, will the standard parametric bootstrap for the weighted CLSE of the offspring mean be valid in this non-classical model? In this paper we address this question. We demonstrate that the validity of the bootstrap depends on the relative rate of the immigration mean and variance. Assuming that the immigration mean and variance vary regularly with nonnegative exponents α and β, respectively, we state results which show that if $\beta < 1 + 2\alpha$, the bootstrap leads to a valid approximation for the CLSE. More precisely, conditions $\alpha(n) \to \infty$ and $\beta(n) = o(n\alpha^2(n))$ as $n \to \infty$, are sufficient for the validity of the standard parametric bootstrap.

Investigation of the problems related to the bootstrap methods and their applications has been an active area of research since its introduction by Efron [6]. As a result, a large number of papers and monographs have been published. We note monographs [5] and [7] and the most recent review articles [4] and [8] as important sources of the literature on bootstrap methods. As it was mentioned before, invalidity of the bootstrap for the critical process with a stationary immigration was shown in [11]. In [3] a modification of the standard bootstrap procedure was proposed, which eliminated the invalidity in the critical case. The second-order correctness of the bootstrap for a studentized version of MLE in subcritical case is proved in [12].

In Section 2 of the paper, we present the theoretical development of the parametric bootstrap and state main result on validity of the bootstrap. We also demonstrate that in the important particular case of the Poisson immigration process the bootstrap is valid. The proof of this result is based on certain approximation theorems for the sequence of nearly critical processes and will be published elsewhere. In present paper we concentrate our attention on a simulation study of the problem. So, Section 3 contains Monte-Carlo and bootstrap simulations and empirical investigation of the process with non-stationary immigration. The Conclusion section, Section 4, highlights the theoretical and the empirical findings for the behavior of the estimator of this stochastic process with non-stationary immigration and discusses other future research avenues of interest.

2. The theoretical developments

The process with time-dependent immigration is given by the offspring distribution of $X_{ki}, k, i \geq 1$, and by the family of distributions of the number of immigrating individuals $\xi_k, k \geq 1$. We assume that the offspring distribution has the probability mass function

$$p_j(\theta) = P\{X_{ki} = j\},\ j = 0, 1, \ldots, \tag{2.1}$$

depending on parameter θ, where $\theta \in \Theta \subseteq \mathbb{R}$. Then $a = E_\theta X_{ki} = f(\theta)$ for some function f. We assume throughout the paper that f is one-to-one mapping of Θ to $[0, \infty)$ and is a homeomorphism between its domain and range. It is known that these assumptions are satisfied, for example, by the distributions of the power series family [3].

What concerns the distributions of the number of immigrating individuals, we assume that ξ_k follows a known distribution with the probability mass function

$$q_j(k) = P\{\xi_k = j\},\ j = 0, 1, \ldots, \tag{2.2}$$

for any $k \geq 1$.

Throughout the paper " $\xrightarrow{d}$ " and " $\xrightarrow{P}$ " will denote convergence of random variables in distribution and in probability, respectively, and also $X \stackrel{d}{=} Y$ denotes equality of distributions. We assume that $b = VarX_{ni} < \infty$ and $\alpha(k) = E\xi_k,\ \beta(k) = Var\xi_k$ are finite for any $k \geq 1$ and are regularly varying sequences of nonnegative exponents α and β, respectively. Then $A(n) = EZ(n)$ and $B^2(n) = VarZ(n)$ are finite for each $n \geq 0$ and $a = 1$. To provide the asymptotic distribution of $\hat{a}_n$ defined in (1.2), we assume that there exists $c \in [0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\beta(n)}{n\alpha(n)} = c \tag{2.3}$$

and denote for any $\varepsilon > 0$

$$\delta_n(\varepsilon) = \frac{1}{B^2(n)} \sum_{k=1}^{n} E[(\xi_k - \alpha(k))^2; |\xi_k - \alpha(k)| > \varepsilon B(n)].$$

As it was proved in [9], if $a = 1,\ b \in (0, \infty),\ \alpha(n) \to \infty,\ \beta(n) = o(n\alpha^2(n))$, condition (2.3) is satisfied and $\delta_n(\varepsilon) \to 0$ as $n \to \infty$ for each $\varepsilon > 0$, then as $n \to \infty$

$$\frac{nA(n)}{B(n)}(\hat{a}_n - a) \xrightarrow{d} (2 + \alpha)\mathcal{N}(0, 1). \tag{2.4}$$

Furthermore, under the above conditions, $A(n)/B(n) \to \infty$ as $n \to \infty$ and when $c = 0$ the condition $\delta_n(\varepsilon) \to 0$ is satisfied automatically. More detailed discussion and examples can be seen in [9].

We now describe the bootstrap procedure to approximate the sampling distribution of the pivot

$$V_n = \frac{nA(n)}{B(n)}(\hat{a}_n - a).$$

Given a sample $\mathcal{X}_n = \{Z(k), k = 1, \dots, n\}$ of population sizes, we estimate the offspring mean a by the weighted CLSE $\hat{a}_n$ and obtain the estimate of the parameter θ as $\hat{\theta}_n = f^{-1}(\hat{a}_n)$ from equation $a = f(\theta)$. Further, we replace θ in the probability distribution (2.1) by its estimate. Given $\mathcal{X}_n$, let $\{X_{ki}^{*(n)}, k, i \geq 1\}$ be a family of i.i.d. random variables with the probability mass function $\{p_j(\hat{\theta}_n), j = 0, 1, \dots\}$. Now we obtain the bootstrap sample $\mathcal{X}_n^* = \{Z^{*(n)}(k), k = 1, \dots, n\}$ recursively from

$$Z^{*(n)}(k) = \sum_{i=1}^{Z^{*(n)}(k-1)} X_{ki}^{*(n)} + \xi_k, \quad n, k \geq 1, \tag{2.5}$$

with $Z^{*(n)}(0) = 0$, where $\xi_k, k \geq 1$, are independent random variables with the probability mass functions $\{q_j(k), j = 0, 1, \dots\}$. Then, we define the bootstrap analogue of the pivot V_n by

$$V_n^* = \frac{nA(n)}{B(n)}(\hat{a}_n^* - \hat{a}_n), \tag{2.6}$$

where $\hat{a}_n^*$ is the weighted CLSE based on $\mathcal{X}_n^*$, i.e.

$$\hat{a}_n^* = \frac{\sum\limits_{k=1}^{n} (Z^{*(n)}(k) - \alpha(k))}{\sum\limits_{k=1}^{n} Z^{*(n)}(k-1)}. \tag{2.7}$$

To state our main result, we need the following conditions be satisfied.

A1. $a = 1$ and moments $E_\theta[(X_{ki})^2]$ and $E_\theta[(X_{ki})^{2+l}]$ are continuous functions of θ for some $l > 0$.

A2. $\delta_n(\varepsilon) \to 0$ as $n \to \infty$ for each $\varepsilon > 0$.

A3. $\alpha(n) \to \infty$, $\beta(n) = o(n\alpha^2(n))$ as $n \to \infty$.

Theorem 2.1. *If conditions* A1–A3 *and* (2.3) *are satisfied, then*

$$\sup_x |P\{V_n^* \leq x|\mathcal{X}_n\} - P\{V_n \leq x\}| \xrightarrow{P} 0 \tag{2.8}$$

as $n \to \infty$.

Remark 2.1. As it was mentioned before, in the case where $c = 0$ condition A2 is automatically satisfied. In the case where $c > 0$ the condition is equivalent to the Lindeberg condition for the family $\{\xi_n, n \geq 1\}$ of the number of immigrating individuals.

Example 2.1. Let $\xi_k, k \geq 1$, be Poisson with the mean $\alpha(k)$ such that $\alpha(k) \to \infty, k \to \infty$, and regularly varies with exponent α. In this case $\beta(n) = o(n\alpha(n))$ as $n \to \infty$ and condition A3 is satisfied. Moreover, we realize that $c = 0$ in (2.3), which implies that condition A2 is also fulfilled. Thus we have the following result.

Corollary 2.1. *If* ξ_k, $k \geq 1$, *are Poisson with mean* $\alpha(k) \to \infty$, $k \to \infty$, *and* $(\alpha(k))_{k=1}^{\infty}$ *is regularly varying with exponent* α *and condition* A1 *is satisfied, then* (2.8) *holds.*

Due to (2.4), we immediately obtain the following result from Theorem 2.1.

Corollary 2.2. *If conditions of Theorem* 2.1 *are satisfied, then*

$$\sup_x |P\{V_n^* \leq x|\mathcal{X}_n\} - \Phi(2+\alpha, x)| \xrightarrow{P} 0$$

as $n \to \infty$, *where* $\Phi(\sigma, x)$ *is the normal distribution with mean zero and variance* σ^2.

In order to prove Theorem 2.1, one needs to obtain a series of results on convergence in Skorokhod topology of an array of the branching processes. The scheme of the proof is as following. Since the bootstrap sample $\mathcal{X}_n^*$ is based on the sequence of branching processes (2.5), we first investigate the array of processes under suitable assumptions of near criticality. Second, we derive limit distributions for the CLSE of the offspring mean in the sequence of nearly critical processes. In the third step, we show that conditions of the limit theorems for the CLSE are fulfilled by the bootstrap pivot V_n^*. A detailed proof of Theorem 2.1 can be seen in [10], where it is proven in a slightly more general situation.

Results, related to the threshold of the validity of the bootstrap will be published elsewhere.

3. Simulation study

For the purpose of empirical investigation of bootstrapping for the branching stochastic process with non-stationary immigration, we conducted Monte-Carlo and bootstrap simulations of the process and compared them in this section. All simulations in this study were conducted with computer programs written in MATLAB 7.0. These programs are currently being prepared for user-friendliness and are going to be published online. However, their specifications are described in brief in this section.

To investigate the limiting behavior of the offspring mean estimator, the Monte-Carlo simulation was conducted with the following specifications:

(1) Offspring distribution is Poisson with parameter $a = 1$. This signifies a critical process.

(2) Immigration distribution is Poisson with parameter $\alpha(k) = k$, $k \geq 2$ and $\alpha(1) = 2$.

(3) Totally $n = 99$ generations and $m = 2500$ sample paths.

The graphs in Figure 1 below show the results of this simulation. In the Figure, graph (a) represents supremum $|CDF(Vk) - CDF\mathcal{N}(0, 2+\alpha)|$, $\alpha = 1$, versus generation k, graph (b) displays cumulative distribution function (CDF) of Monte-Carlo pivot V_k versus $\mathcal{N}(0, 2+\alpha)$ at the last generation, graph (c) depicts kurtosis (a sample analog of $\gamma_k = E[(V_k - E[V_k])^4/\sigma_{V_k}^4])$ of pivots $V_k$versus generation k, and graph (d) shows standard deviation of pivots V_k versus generation k.

Figure 1 (a) shows that the supremum of $|CDF(V_k) - CDF(N(0, 2+\alpha))|$ decreases as generation value grows larger. Since the graph of the supremum difference in CDFs generally approaches zero as $k \longrightarrow \infty$, apart from sampling errors, the Monte-Carlo simulation produced pivots that can be considered to approach normality as reported earlier in our theoretical results. This is further supported by the close empirical and theoretical cumulative distribution functions at the 99th generation as shown in Figure 1 (b). In particular, Figure 1 (c) shows the kurtosis (blue line) of V_k hovering around the usual normal distribution kurtosis of 3 (red dotted line) from about the 20th generation onwards. The standard deviation of V_k in Figure 1(d) also decreases with increasing generation number. However, unlike the kurtosis, the standard deviation of V_k reaches our

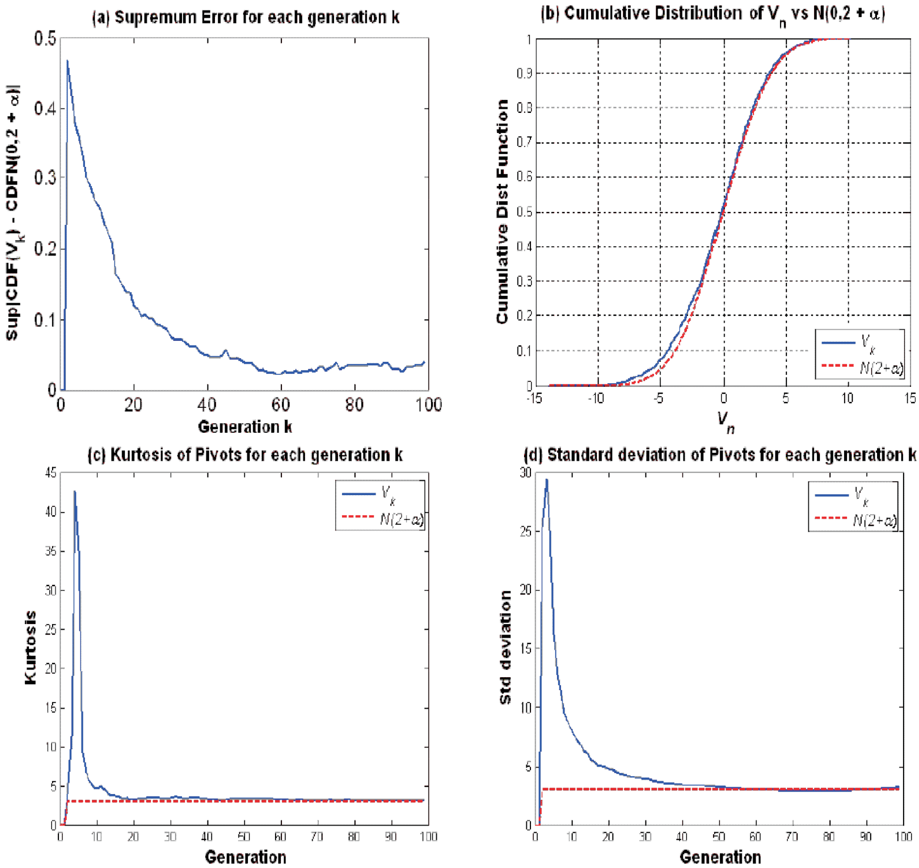

Figure 1: Pivot Behavior for $m = 2500$ Monte-Carlo Simulations and $n = 99$

theoretical standard deviation of $2 + \alpha$ (red dotted line) at a slower rate since this occurs only at the 60th generation onwards.

Figure 2 below shows the behavior of the weighted conditional least square estimates (WCLSE) of the offspring means over a long term where the process reaches 99 generations. Although initially the WCLSE went up to more than 2, in the long run, these estimates of offspring means stably reaches 1 (for a critical case) starting from generation of 4 onwards.

For the purpose of empirical investigation of the bootstrap, we also conducted simulations with the following specifications:

(1) An initial random sample from the process was generated with $n = 15$

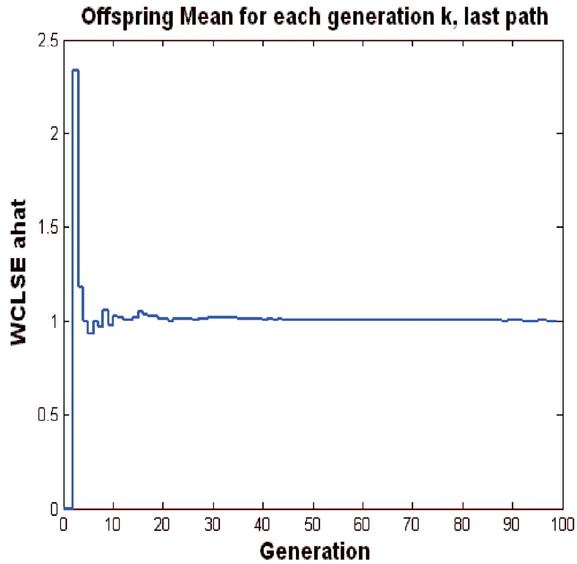

Figure 2: Offspring Means through WCLSE from a single path, $n = 99$

generations and the estimated offspring mean $\hat{a}_n$ was obtained.

(2) The initial random sample was treated as a pseudo-population with offspring mean $\hat{a}_n$ treated as a known parameter. Resampled offspring paths were obtained from this pseudo-population, where the offspring distribution is Poisson with parameter $a = \hat{a}_n$

(3) Immigration distribution is Poisson with parameter (a) $\alpha(k) = \log(k)$, $k \geq 2$ and $\alpha(1) = \log(2)$, (b) $\alpha(k) = k$, $k \geq 2$ and $\alpha(1) = 2$, or (c) $\alpha(k) = k^3$, $k \geq 2$ and $\alpha(1) = 2^3 = 8$.

(4) Totally, $n = 15$ generations and $m = 10,000$ bootstrap resampled paths.

(5) Monte-Carlo simulations were also conducted to match each bootstrap specification above for comparison purposes.

Figure 3 shows the behavior of the WCLSE of offpspring means for a single path from the process when immigration rates are (a) $\alpha(k) = \log(k)$, (b) $\alpha(k) = k$, or (c) $\alpha(k) = k^3$. Although the WCLSE can be negative for lower generations (see equation 1.2), from Figure 3, the estimator hovers around 1 (for the critical case) for larger generations according to the rate of immigration. For bootstrapping

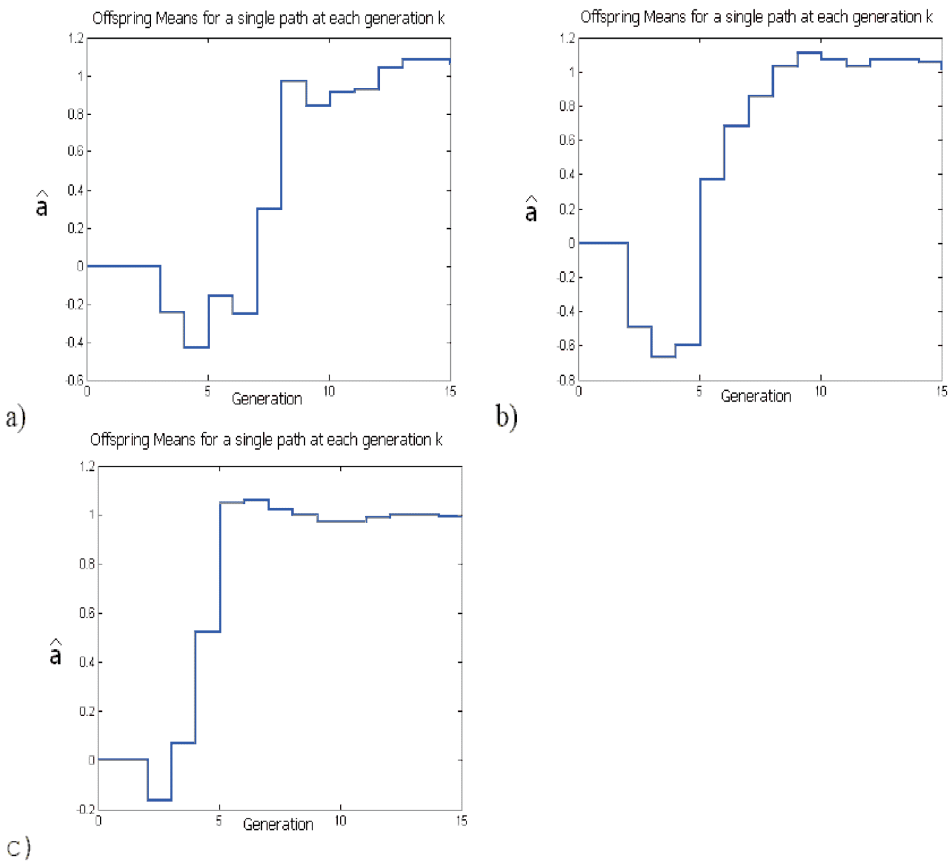

Figure 3: WCLSE offspring means for a single path, $n = 15$

purposes, the WCLSE estimates at generation 15 ($a = \hat{a}_n$) is used as the offspring mean parameter for the process in place of the unknown true parameter.

The graphs in Figure 4 below show the distribution of pivots for the Monte-Carlo (on the left column of Figure 4) and the Bootstrap (right column of Figure 4) simulations. For instance, Figure 4 parts (a) and (b) represent the pivot distribution when the branching process has an immigration mean $\alpha(k) = \log(k)$. The second row shows the same when $\alpha(k) = k$ and the last row shows the distributions when $\alpha(k) = k^3$. In addition, for comparison, the theoretical normal limiting distribution $\mathcal{N}(0, 2 + \alpha)$ is also superimposed on each of the pivot histograms in Figure 4.

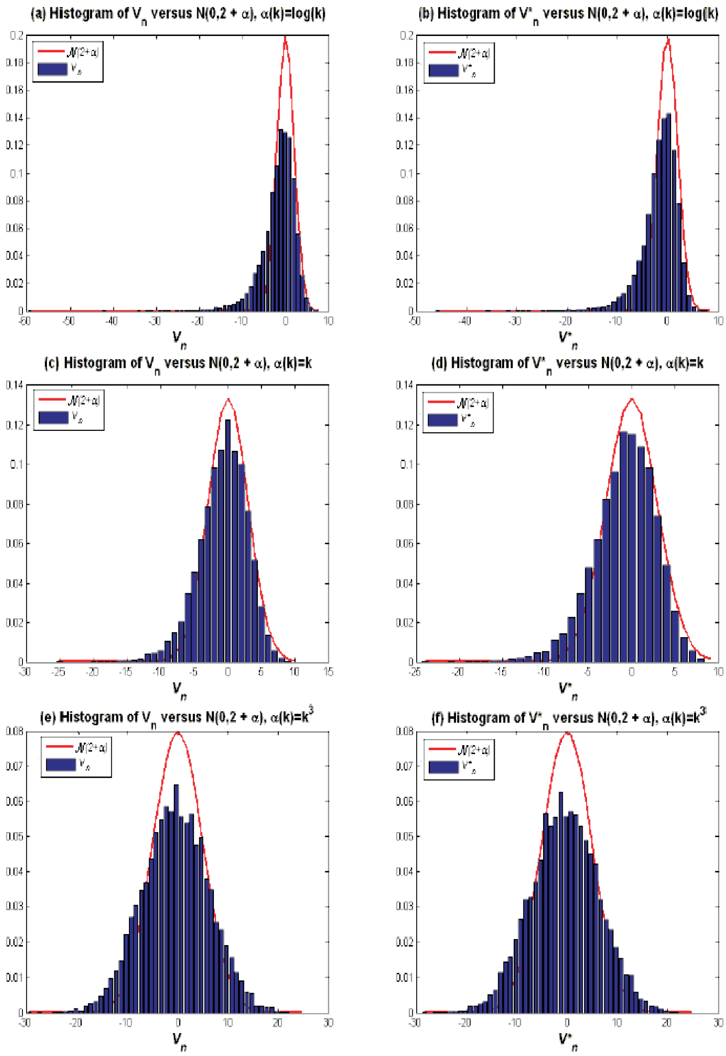

Figure 4: Empirical vs Theoretical Distribution of Pivots at $n = 15$, $m = 10000$

When the immigration rate is slower (i.e. when immigration mean $\alpha(k) = \log(k)$), the pivot distribution seems negatively skewed when compared to the normal distribution. However as immigration rates grow more rapidly, the pivot distributions are more symmetric and are tending towards the normal distribution. Note also that the bootstrap pivot distributions on the right column of

Figure 4 mirror the pivot distributions from the Monte-Carlo simulations very closely. Also, when $\alpha(k) = k^3$, the Monte-Carlo and bootstrap pivot distribution each converges to normality faster than when $\alpha(k) = k$ and when $\alpha(k) = \log(k)$, which confirms our theoretical findings earlier. Thus, the convergence rate is partially governed by the increasing rate of immigration.

For the purpose of studying deviations of Monte-Carlo pivots from normality and the bootstrap pivots, we also defined the supremum differences $\sup|CDF(V_k)$ $-\mathcal{N}(0, \alpha+2)|$ and $\sup|CDF(V_k) - CDF(V_k^*)|$. We compared these supremums for the pivots at each generation from 2 to 15. The results of these comparisons are given in Table 1 for different mean immigration rates, with $a = Sup|CDF(V_k) - CDF\mathcal{N}(0, 2+\alpha)|$, $b = Sup|CDF(V_k) - CDF(V_k^*)|$, and $c = a/b$.

As can be seen in the table, bootstrap approximation is consistently better than normal approximation of the branching process with non-stationary immigration. The ratio of the supremum differences in CDF are generally at least 4 to 5 times smaller for the bootstrap pivots compared to the approximation by normal. This is further verified by examining the CDF in Figure 5 below for this comparison.

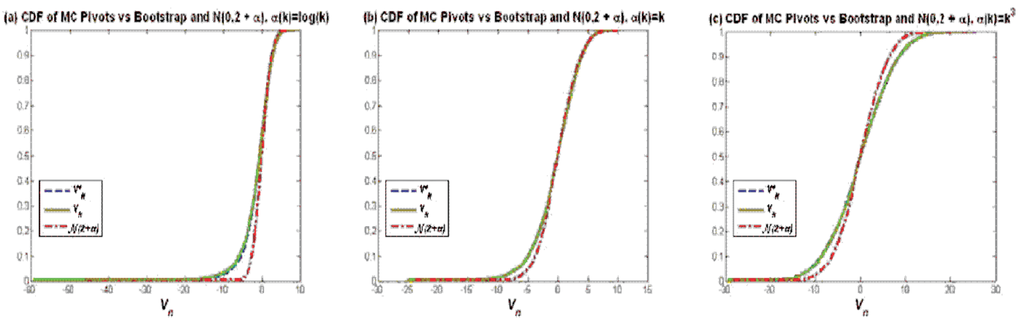

Figure 5: Validity of the Bootstrap for Various Immigration Means

The benefits of bootstrapping is clear. The distribution of the bootstrap pivots mimics the distribution of the Monte-Carlo pivots very closely even at lower generations. This is a very useful feature of the bootstrap for this type of stochastic processes particularly when the process is still at the initial stages and the normality limit theorem does not yet apply. Because of this feature, the initial sample can be used as a pseudo-population for the bootstrapping procedure to provide an empirical basis for an early stage inference on the process.

k	$\alpha(k) = \log(k)$			$\alpha(k) = k$			$\alpha(k) = k^3$		
	a	b	c	a	b	c	a	b	c
2	0.5577	0.0709	7.9	0.3781	0.0036	105.0	0.2252	0.0102	22.1
3	0.4614	0.0670	6.9	0.2881	0.0066	43.7	0.2255	0.0084	26.9
4	0.3645	0.0821	4.4	0.2281	0.0093	24.5	0.1735	0.0158	11.0
5	0.3201	0.0385	8.3	0.1889	0.0077	24.5	0.1295	0.0131	9.9
6	0.2868	0.0320	9.0	0.1600	0.0093	17.2	0.0973	0.0090	10.8
7	0.2620	0.0255	10.3	0.1368	0.0108	12.7	0.0765	0.0056	13.7
8	0.2372	0.0210	11.3	0.1116	0.0141	7.9	0.0628	0.0060	10.5
9	0.2225	0.0193	11.5	0.1018	0.0155	6.6	0.0481	0.0060	8.0
10	0.2097	0.0224	9.4	0.0941	0.0166	5.7	0.0449	0.0054	8.3
11	0.1971	0.0179	11.0	0.0796	0.0121	6.6	0.0363	0.0053	6.9
12	0.1921	0.0221	8.7	0.0798	0.0105	7.6	0.0364	0.0064	5.7
13	0.1857	0.0256	7.3	0.0734	0.0109	6.7	0.0426	0.0066	6.5
14	0.1819	0.0300	6.1	0.0743	0.0111	6.7	0.0553	0.0054	10.2
15	0.1799	0.0212	8.5	0.0584	0.0090	6.5	0.0734	0.0069	10.6

Table 1: Comparison between Cumulative Distributions of Monte-Carlo(MC) Pivots V_k with $\mathcal{N}(0, 2+\alpha)$ and with Bootstrap Pivots V_k^* for each Generation and Immigration Rate $\alpha(k)$

4. Concluding remarks

The main result of this paper shows that when condition A3 is satisfied the sampling distribution of the CLSE can be approximated by the distribution of the bootstrap version of the estimator. Since the bootstrap estimator can be constructed by a large number of re-sampling, it can be used even when the number of observed generations in the original sample is small, where the limit theorem does not yet apply. This is particularly useful for real application of the stochastic process model, where data may be scarce and early inference on the process is of paramount importance. Moreover, the simulation results of Section 4 show that bootstrap is a better approximation comparatively than the normal approximation (given by the limit theorem) of the CLSE.

It is known [9] that when condition A3 is not satisfied i.e. $n\alpha^2(n) = o(\beta(n))$ as $n \to \infty$ pivot $n(\hat{a}_n - a)$ as $n \to \infty$ converges in distribution to a random variable which is not normal and can be expressed in terms of certain functionals of a time-changed Wiener process. Since the rate of convergence is not faster than n^{-1}, it is heuristically clear that the parametric bootstrap is invalid in this case. To prove this formally, one needs to obtain the approximation results for the array of branching processes in more general set up than it is done in present paper.

Assume now that a sample of pairs $\{(Z(k), \xi_k), k = 1, \ldots, n\}$ is available. In this case a natural estimator of the offspring mean is

$$\tilde{a_n} = \frac{\sum_{k=1}^{n} (Z(k) - \xi_k)}{\sum_{k=1}^{n} Z(k-1)}.$$

The following questions related to this estimator is of interest. How much improvement in the sense of the rate of convergence we will get because of additional observations on the number of immigrating individuals? Will the standard parametric bootstrap procedure be valid for $\tilde{a_n}$ when condition A3 is not satisfied? Since

$$\tilde{a_n} - a = \frac{\sum_{k=1}^{n} \sum_{j=1}^{Z(k-1)} (X_{kj} - a)}{\sum_{k=1}^{n} Z(k-1)},$$

one can easily derive asymptotic distributions for the pivot, corresponding to $\tilde{a_n}$ from a martingale central limit theorem. By the arguments as in the proof of Proposition 4.1 in [9], it is possible to prove that $\tilde{a_n}$ is a strongly consistent estimator of a.

The estimation problems and a justification of the validity of the bootstrap for subcritical and supercritical processes with non-stationary immigration are also open.

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