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BRANCHING PROCESSES IN AUTOREGRESSIVE RANDOM ENVIRONMENT

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We consider the model of alternating branching processes where two Markov branching processes act alternately at random observation and treatment times. The sequences of cycles (observation, treatment) = (δ_n, τ_n) constitute a random environment for branching mechanisms. We suppose in addition that the lengthes of the cycles $\sigma_n = \delta_n + \tau_n$ are generated by the linear additive first order autoregressive schema EAR(1).

1. Introduction

Stochastic processes in random environment is a vast domain in now-days: Random walks in random environment, Brownian diffusion in random environment, Branching processes in random environment, and so on. They arise as the solution of the stochastic differential equations or as the stochastic control on the space of branching trees, see [1], [6], [9], [5]. The objective of the present communication is to describe the random environment, as the first order autoregressive point process.

It is well known that the classical models of branching processes investigate the dynamics of an isolated population with an reproduction independent of the individuals. However, in the real situation there is always an interaction between

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the individuals by intermediary of the environment. One way to describe the interaction is by the model of controlled branching processes, including immigration and emigration, see [8], [14]. In our previous paper [10] we introduced the idea to control a branching process $\xi(t)$ by means of another branching process $\mu(t)$, where the control function is defined by the fractional thinning operator of Steutel and van Harn [13], [12] as the "discrete multiplication". This way, the control consists of testing every one particle from the $n-$th generation according to a dying branching process $\mu(t)$ during an independent time τ. Relatively to the population defined by the Markov branching process $\xi(t)$ the testing of particles is equivalent to the state dependent emigration.

Now we consider the model of alternating branching process with an autoregressive random environment. Suppose there are two Markov branching processes $\xi(t)$ and $\mu(t)$ acting alternately by the random observation times δ_n and treatment times τ_n, respectively. The sequences of cycles (observation, treatment)-(δ_n, τ_n) constitute a random environment for the branching mechanisms. Let $\sigma_n = \delta_n + \tau_n$, $n = 1, 2, \ldots$, denote the lengthes of the cycles. Suppose they form an exponential first-order autoregressive sequence EAR(1), i.e.

$$\sigma_{n+1} = \rho\sigma_n + \tau_{n+1}, \quad n = 1, 2, \ldots,$$

where $\rho \in (0, 1)$ and $\delta_{n+1} = \rho\sigma_n$. We remark that treatment time and observation time are independent in every one cycle. The lengthes of the cycles form a sequence of exponential identically distributed but dependent random variables. It is the same for the observation times. The treatment times form the innovation sequence of independent identically distributed random variables with degenerated exponential distribution. There are some runs of the cycles σ_{n+1} which are equal to the previous one values, σ_n, times ρ and the treatment times are zero. It is well known that a stationary EAR(1) is strong mixing but not exchangeable process, see [7]. This provides the sufficient condition to study the problem of extinction probability and limit theorem in the supercritical case for the reproduction by n cycles, as $n \to \infty$. The critical parameters and the mean of the reproduction by n cycles can be calculated explicitly. We show that in the critical case the mean of the reproduction by n cycles tends to a finite constant grater than one. The calculus show how strong and important is the correlation even in the EAR(1) random environment.

2. Model – fundamental relations

Let $\xi(t), t \geq 0$, and $\mu(t), t \geq 0$, be two Markov branching processes (MBP) starting with one particle: $\xi(0) = 1$ and $\mu(0) = 1$, defined by the composition semigroups of probability generating functions (p.g.f.) $f(t,s)$, $t \geq 0$, and $g(t,s)$, $t \geq 0$, respectively. Let $u(s) = a(U(s) - s)$ and $v(s) = b(V(s) - s)$ be the infinitesimal generating functions to the composition semi-groups $f(t,s)$ and $g(t,s)$, respectively, see [4], page 106, and [11], page 27.

We suppose that $f(t,s)$ is supercritical, (its critical parameter $u'(1) > 0$) and $g(t,s)$ is subcritical, with critical parameter $v'(1) < 0$. We denote by $\mu(t \mid X)$ and $\xi(t \mid Y)$ the same branching mechanism as $\mu(t)$ and $\xi(t)$ but starting with a random number of particles X or Y, independent of $\mu(t)$ and $\xi(t)$, respectively. Suppose that the given Markov branching processes $\xi(t), t \geq 0$ and $\mu(t), t \geq 0$ act alternately by the random observation times δ_n and treatment times τ_n , respectively. Let $\xi_1(\cdot), \xi_2(\cdot), \dots$ be independent copies of $\xi(t)$, representing the observed processes. And, let $\mu_1(\cdot), \mu_2(\cdot), \dots$, being independent copies of $\mu(t)$, represent the treatment processes. The independence of the evolution of particles is represented by the following equality:

$$\mu(t \mid X) = \sum_{j=1}^{X} \mu_j(t).$$

We define the alternating branching process as the sequence given by:

$$X_0 = 1, \quad Y_0 = 1,$$

$$X_1 = \xi_1(\delta_1 \mid Y_0) \quad \text{and} \quad Y_1 = \mu_1(\tau_1 \mid X_1) \dots,$$

$$X_n = \xi_n(\delta_n \mid Y_{n-1}) \quad \text{and} \quad Y_n = \mu_n(\tau_n \mid X_n), \dots \ .$$

The sequence $Y_n, n = 1, 2, \dots$, describes the controlled reproduction by n cycles. It represents a branching process with random environment (BPRE). The Markov chain (without explicit immigration) $(Y_n, n = 1, 2 \dots)$ is transient.

The sequence of cycles (observation, treatment) – $(\delta_n, \tau_n), n = 1, 2, \dots$ constitute a random environment for the branching mechanisms. Denote by σ_n the lengthes of the cycles

$$\sigma_n = \delta_n + \tau_n, \quad n = 1, 2 \dots .$$

Suppose that the treatment times τ_n are independent identically distributed (i.i.d.) non-negative random variables. We consider the model when the observation time of the $n-$th cycle is a deterministic part of the length of the previous $(n-1)$-th cycle, i.e.

$$\delta_n = \rho \sigma_{n-1}, \quad 0 < \rho < 1, \quad n = 1, 2 \dots .$$

Then the random environment

$$\sigma = \{(\delta_n, \tau_n), \quad n = 1, 2, \dots\}$$

is created by the additive autoregressive equation

$$\sigma_n = \rho\sigma_{n-1} + \tau_n, \quad n = 1, 2 \dots. \tag{1}$$

The initial condition σ_0 will be chosen by the reason of stationarity and we suppose that σ_0 is independent of $\tau_n, n = 1, 2 \dots$.. The equation (1) has a solution for a given distribution of σ_n as an infinite moving average, see [7], namely

$$\sigma_{n-1} = \rho^{n-1}\sigma_0 + \rho^{n-2}\tau_1 + \cdots + \rho\tau_{n-2} + \tau_{n-1}, \quad n = 2, 3, \dots$$

This way the observation time of the $n-$th cycle is given by

$$\delta_n = \rho^n\sigma_0 + \rho^{n-1}\tau_1 + \cdots + \rho\tau_{n-1}, \quad n = 2, 3, \dots, \tag{2}$$

$$\delta_1 = \rho\sigma_0.$$

Obviously σ_{n-1} is a function only of $(\sigma_0, \tau_1, \dots \tau_{n-1})$ and is therefore independent of τ_n. This way the observation and treatment tines are independent in every one cycle. Naturally we require the length of the cycles σ_n to be positive random variables. Assume that the sequence $\sigma_n, n = 1, 2, \dots$ is marginally stationary. Then the Laplace-Stiltjes transform of the equation (1) indicates that the random variables σ_n are exponentially distributed. Let $A(y) = P(\delta_i \leq y)$ and $B(y) = P(\sigma_i \leq y)$ and $C(y) = P(\tau_i \leq y)$ be the probability distribution functions (p.d.f.) to the observation times , the lengthes of the cycles and treatment times, respectively. Denote the density of the exponential distribution by $B(dx) = \beta e^{-\beta x}dx$.

Proposition 2.1. *The solution of the equation* (1) *is given by the following sequences of random variables:*

The lengthes of the cycles $\sigma_n, n = 1, 2, \dots$ *constitute the sequence of identically distributed but dependent random variables with exponential distribution function* $B(\cdot)$ *and intensity* β*.*

The observation times $\delta_n, n = 1, 2, \dots$ *form the sequence of identically distributed but dependent random variables with exponential distribution function* $A(\cdot)$ *and intensity* $\alpha = \dfrac{\beta}{\rho}$ *.*

The treatment times $\tau_n, n = 1, 2, \dots$ *form the innovation sequence for the* $EAR(1)$ *of the i.i.d. random variables with degenerate exponential distribution* $C(\cdot)$ *having the atom of mass* ρ *at* 0 *and density* $B(dx) = \beta e^{-\beta x}dx$ *with probability* $(1 - \rho)$*.*

Knowing the environment $\sigma = \{(\delta_n, \tau_n), n = 1, 2, \ldots\}$ the reproduction by the $n-$th cycle of (observation, treatment)- (δ_n, τ_n) has random p.g.f.

$$\varphi_n(s, \sigma) := f(\delta_n, g(\tau_n, s)), \quad \text{for} \quad n = 1, 2, \ldots. \tag{3}$$

Indeed, one particle observed by the time δ_n and its offsprings tested by the time τ_n are transformed into ζ_n particles, such as

$$\zeta_n = \sum_{j=1}^{\xi_n(\delta_n)} \mu_{nj}(\tau_n),$$

where $\mu_{nj}(\cdot), j = 1, 2, \ldots$ are independent copies of $\mu_n(\cdot)$. The random p.g.f.

$$\varphi_n(s, \sigma) = E(s^{\zeta_n} \mid \sigma), \quad n = 1, 2, \ldots$$

are all identically distributed but dependent random variables.

The remarkable properties of EAR(1) point process are that it is a stationary ergodic strongly mixing but not exchangeable process. For the model of BPRE this random environment provides the sufficient conditions to study the extinction probability and limit theorem at supercritical cases.

Let T denote the shift operator on the random environment, i.e. translation by one cycle, defined by

$$T\sigma = \{(\delta_i, \tau_i), \quad i = 2, 3, \ldots\}.$$

Proposition 2.2. *If the random environment σ is defined by the EAR(1) point process independent of the branching mechanisms $\xi(\cdot)$ and $\mu(\cdot)$, then the sequence (Y_n) is a Markov chain representing Galton-Watson processes in random environment. The sequence (Y_n) has random reproduction law by one cycle $\varphi_n(s, \sigma)$, defined by (3). Reproduction by the first n cycles is equal in distribution to the reproduction by n successive cycles. Namely, let*

$$\phi_n^{\rightarrow}(s, \sigma) := E(s^{Y_n} \mid \sigma) = \varphi_1 \circ \varphi_2 \circ \ldots \varphi_n(s, \sigma),$$

then for each $0 \leq s \leq 1$

$$\phi_n^{\rightarrow}(s, T\sigma) := \varphi_2 \circ \varphi_3 \circ \ldots \varphi_{n+1}(s) = \phi_n^{\rightarrow}(s, \sigma), \ \textit{in distribution.}$$

3. Extinction probability and critical parameters

Traditionally q denotes the extinction probability, what is the smallest root of the equation $u(s) = 0$ or $v(s) = 0$ for the corresponding infinitesimal generating function. The supercritical process $\xi(t)$ has the extinction probability $q(\xi) < 1$ and the subcritical process $\mu(t)$ has the extinction probability $q(\mu) = 1$. For the model of alternating branching processes the events of extinction by n cycles form the following increasing sequence :

$$(X_n = 0) \subset (Y_n = 0) \subset (X_{n+1} = 0) \subset \cdots ,$$

We define the unconditional probability of extinction by the following constant

$$q(X,Y) := P(X_n = 0 \text{ for some } n = 1,2,\ldots) = P(Y_n = 0 \text{ for some } n = 1,2,\ldots).$$

The conditional probability of extinction knowing the environment σ is defined by the random variable (r.v.) Q as follows

$$Q = P(Y_n = 0 \text{ for some } n = 1,2,\ldots \mid \sigma) = \lim_{n\to\infty} \phi_n^{\rightarrow}(0,\sigma), a.s.$$

(By monotony, the limit exists with probability one).

Definition 3.1. *(Critical parameters) We define the constants m and γ as follows*

$$m := E\exp[u'(1)\delta + v'(1)\tau], \tag{4}$$

$$\gamma := E[u'(1)\delta + v'(1)\tau], \tag{5}$$

where the random variable $\delta = \delta_n$ has probability distribution function $A(\cdot)$, the random variable $\tau = \tau_n$ has degenerated probability distribution function $C(\cdot)$. The random variables $\delta = \delta_n$ and $\tau = \tau_n$ are independent in every one cycle. The reproductions by one cycle $\zeta = \zeta_n$ will be labeled supercritical, critical or subcritical as $\gamma > 0$, $\gamma = 0$ or $\gamma < 0$, respectively. The random variables ζ_n, $n = 1,2,\ldots$ are all identically distributed but dependent random variables.

The Jensen's inequality provides $m \geq e^\gamma$ always. We can calculate and compare explicitly the critical parameters. Naturally the inequality $m \leq 1$ provides the extinction with probability one, i.e. $q(X,Y) = 1$. But the most precise sufficient condition is given by the critical parameter γ.

Theorem 3.1. *If $\gamma \leq 0$ then r.v. $Q = 1$ with probability (w.p.) 1.*

If $\gamma > 0$ and if additionally $E[-\log(1 - f(\delta, g(\tau, 0)))] < \infty$ then $q \leqslant Q < 1$ w.p.1.

Theorem 3.2. *Suppose that $\beta > \rho u'(1)$, for each $0 < \rho < 1$. If the random environment σ is defined by the first order autoregressive point process then the critical parameters are explicitly calculated:*

$$m = \frac{\beta}{\beta - \rho u'(1)}\{\rho + (1-\rho)\frac{\beta}{\beta - v'(1)}\},$$

$$\gamma = \frac{\rho u'(1) + (1-\rho)v'(1)}{\beta}.$$

Obviously the means of the observation and treatment times are given by:

$$E(\delta_n) = \frac{\rho}{\beta}, \quad E(\tau_n) = \frac{1-\rho}{\beta}.$$

The random variables δ_n and τ_n are independent. The Laplace -Stiltjes transform of the random variables $\delta = \delta_n$ and $\tau = \tau_n$ represents the mean of the reproduction by the random time δ and the mean of the removed or emigrated particles by the random time τ, respectively, i.e.:

$$E\exp[u'(1)\delta] = \frac{\beta}{\beta - \rho u'(1)},$$

$$E\exp[v'(1)\tau] = \left\{\rho + (1-\rho)\frac{\beta}{\beta - v'(1)}\right\}.$$

The critical parameter $u'(1)$ of the supercritical MBP $\xi(t)$ represent the rate of the reproduction, what we intend to control. The condition $\beta > \rho u'(1)$ signifies that the intensity of the observation time $\frac{\beta}{\rho}$ must be grater of the rate of reproduction. Otherwise, we can loose any control. Obviously, we have the following relation between the critical parameters:

$$m = 1 + \frac{\beta^2\gamma - \rho u'(1)v'(1)}{(\beta - \rho u'(1))(\beta - v'(1))},$$

$$\rho = \frac{\beta\gamma - v'(1)}{u'(1) - v'(1))}.$$

The main normalizing quantity is the conditional mean of the reproduction by n cycles given by

$$M_n := \prod_{i=1}^{n} \varphi_i'(1) = \exp\left\{\sum_{i=1}^{n} u'(1)\delta_i + v'(1)\tau_i\right\}. \tag{6}$$

We shall study its mean and its asymptotic behavior. Denote by $\alpha_k(n)$ the following deterministic constants:

$$\alpha_k(n) = u'(1)\sum_{j=1}^{n-k} \rho^j + v'(1), \quad k = 1, 2, \ldots, (n-1),$$

$$\alpha_0(n) = u'(1)\sum_{j=1}^{n} \rho^j, \quad \alpha_n(n) = v'(1).$$

Theorem 3.3. *For the random environment σ generated by the EAR(1) we have:*

$$\log M_n = \sigma_0\alpha_0(n) + \sum_{k=1}^{n} \tau_k\alpha_k(n), \tag{7}$$

where σ_0 has p.d.f. $B(\cdot)$ and $\tau_k, k = 1, 2, \ldots$ have degenerate p.d.f. $C(\cdot)$.

Suppose that $\beta - \alpha_0(n) > 0$ and $\beta - \alpha_k(n) > 0, k = 1, 2, \ldots$ for any $\rho \in (0, 1)$, then

$$E(M_n) = \frac{\beta}{\beta - \alpha_0(n)} \prod_{k=1}^{n} \{\rho + \frac{\beta(1-\rho)}{\beta - \alpha_k(n)}\}. \tag{8}$$

Proof. By the expression (2) the sum in (6) is transformed into (7). The treatment times τ_n are i.i.d. random variables and their Laplace-Stiltjes transform proves the expression (8). □

Theorem 3.4. (critical case) *If $\gamma = 0$ and if $\beta > \max(u'(1), -v'(1))$ then the following limit exists*

$$\lim_{n\to\infty} EM_n = \frac{\beta^2}{\beta^2 - (v'(1))^2}. \tag{9}$$

Proof. In the critical case we have the following relations:

$$\rho u'(1) + v'(1) = \rho v'(1), \tag{10}$$

$$\alpha_0(n) + (1-\rho)\sum_{k=1}^{n} \alpha_k(n) = 0,$$

$$\alpha_k(n) = \rho^{n-k} v'(1) < 0, \quad k = 1, 2, \ldots, (n-1).$$

The mean of the conditional mean can be simplified as to be

$$E(M_n) = \frac{\beta}{\beta - \alpha_0(n)} \prod_{k=1}^{n} \left\{ \frac{\beta - \rho\alpha_k(n)}{\beta - \alpha_k(n)} \right\}.$$

Consequently

$$E(M_n) = \left\{ \frac{\beta}{\beta - \alpha_0(n)} \right\} \left\{ \frac{\beta - v'(1)\rho^n}{\beta - v'(1)} \right\}.$$

The constant $\alpha_0(n)$ in the critical case is transformed in to the following:

$$\alpha_0(n) = u'(1)\frac{\rho - \rho^{n+1}}{1-\rho}.$$

Obviously using the relation (10) and elementary limit we obtain (9). □

4. Limit theorems

We consider the asymptotic behavior of the BPRE process (Y_n) generated by reproduction ζ to the cycles (observation, treatment). The results are particular cases of those by Athreya and Karlin [2], [3]. We just outline the specific feature of random environment and probability generating functions. The main normalizing quantity is the conditional mean of the reproduction by n cycles

$$M_n := \prod_{i=1}^{n} \varphi_i'(1) = \exp\left\{ \sum_{i=1}^{n} u'(1)\delta_i + v'(1)\tau_i \right\}.$$

Also, we involve the extra moment conditions

$$E[-\log(1 - f(\delta, g(\tau, 0)))] < \infty$$

preventing "catastrophes" in which almost the entire population dies out in a single cycle.

Theorem 4.1. (supercritical case,$\gamma > 0$) *We assume that* $E[-\log(1 - f(\delta, g(\tau, 0)))] < \infty$. *Let* $W_n = \dfrac{Y_n}{M_n}$. *Denote by* $\mathbb{F}_n(\sigma)$ *the filtration generated by the random variables* $(Y_1, Y_2, \cdots, Y_n)$ *and random environment* σ. *Then, the family*

$$\{W_n, \mathbb{F}_n(\sigma), n = 1, 2, \dots\}$$

constitutes a nonnegative martingale and hence

$$\lim_{n\to\infty} W_n = W \text{ exists a.s.}.$$

Suppose, in addition that the conditional mean

$$E(\frac{\zeta \log \zeta}{\varphi'(1)} \mid \sigma) < \infty \quad a.s.. \tag{11}$$

Then, the following limit exists a.s.

$$\lim_{n\to\infty} E(e^{-\lambda W_n} \mid \sigma) := w(\lambda, \sigma)$$

and is the unique solution of the functional equation

$$w(\lambda, \sigma) = f(\delta_1, g(\tau_1, w(\frac{\lambda}{\varphi_1'(1)}, T\sigma))) \quad a.s., \tag{12}$$

among those satisfying

$$\lim_{\lambda\downarrow 0} \frac{1 - w(\lambda, \sigma)}{\lambda} = 1 \quad a.s.. \tag{13}$$

Moreover $E(W \mid \sigma) = 1$ *and* $P(W = 0 \mid \sigma) = Q$ *a.s.*

Proof. Martingale property follows from the definition. The Laplace transform of the random variable W_n knowing the environment σ is

$$\begin{aligned} w_{n+1}(\lambda, \sigma) &:= E(e^{-\lambda W_{n+1}} \mid \sigma) = \phi_{n+1}^{\rightarrow}(e^{-\lambda/M_{n+1}}, \sigma) \\ &= \varphi_1 \circ \phi_n^{\rightarrow}(e^{-\frac{\lambda}{M_{n+1}}}, T\sigma) = \varphi_1 \circ w_n\left(\frac{\lambda}{\varphi_1'(1)}, T\sigma\right). \end{aligned}$$

Convexity of exponential functions provides the monotony of the sequence $w_n, n = 1, 2, \dots,$ namely:

$$w_{n+1}(\lambda, \sigma) \geq w_n(\lambda, \sigma), a.s.$$

Functional equations (12) is a consequence of the stationary ergodic property of σ. Convergence and uniqueness in (12) and (13) follow from the condition (11), see Theorem 1, [3]. Also, as the random environment σ consists on identically distributed (but dependent random variables), we can identify σ and $T\sigma$ in distribution. Then equation (12) can be rewritten in the form:

$$w(\lambda) = f\left(\delta, g\left(\tau, w\left(\frac{\lambda}{\varphi'(1)}\right)\right)\right) \quad \text{in distribution.} \qquad \square$$

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