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ON A SECOND ORDER CONDITION FOR MAX-SEMISTABLE LAWS*

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In statistics of extremes the great importance of the Normal approximation of intermediate order statistics is well known when the parent distribution function is in a max-stable domain of attraction and verifies the first and the second order extreme value conditions. The generalization of these conditions to max-semistable contexts is the object of this paper, aiming to be a basis of future developments in statistical inference under max-semistability.

1. Introduction

For a long time the semi-stability concept appeared in the literature only in the partial sum context. The *genesis* of the class of max-semistable (MSS) distributions is due to Pancheva [6] and Grinevich [3], [4]. After these essays many efforts have been made in order to characterize this new class and their domains of attraction. The class MSS includes not only the max-stable (MS) class of distributions but also non-degenerate limit distributions for the maxima of independent and identically distributed (i.i.d.) random variables (r.v.'s) with either discrete or multi-modal continuous distribution functions (d.f.'s) which do not belong to the MS class.

Following Pancheva [6], a real d.f. G is MSS if there are reals $r > 1$, $a > 0$ and b such that $G(x) = G^r(ax + b),\ x \in \mathbb{R}$, or equivalently, if there exist a sequence

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of i.i.d. r.v.'s with d.f. F, a nondecreasing sequence of positive integers $\{k_n\}$ satisfying

$$\lim_{n\to+\infty} \frac{k_{n+1}}{k_n} = r, \text{ with } r \text{ in } [1, +\infty[, \tag{1}$$

and two real sequences $\{a_n > 0\}$ and $\{b_n\}$ for which

$$F^{k_n}(a_n x + b_n) \to G(x),\ n \to +\infty, \tag{2}$$

for each continuity point, x, of G. In this case we will say that F belongs to the domain of attraction of G. A characterization of this class, different from the one of Grinevich [4], as well as necessary and sufficient conditions on F such that (2) holds, are given in Canto e Castro et al. [1].

The numerical expression of the elements of the MSS class is given by $G_{\gamma,\nu}\left(\frac{x-\mu}{\sigma}\right)$, with $\mu \in \mathbb{R}$ and $\sigma > 0$, and

$$G_{\gamma,\nu}(x) = \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right)\nu\left(\ln(1+\gamma x)^{-1/\gamma}\right) & 1+\gamma x > 0,\ \gamma \neq 0 \\ \mathbb{1}_{]-\infty,0[}(\gamma) & 1+\gamma x \leq 0,\ \gamma \neq 0 \\ \exp\left(-e^{-x}\nu(x)\right) & \gamma = 0,\ x \in \mathbb{R} \end{cases},$$

where $\gamma \in \mathbb{R}$ and ν is a bounded and periodic function with period $p = \ln r$. Notice that for $\nu = 1$ we obtain the extreme value distribution $G_\gamma(x)$.

The class MSS includes three disjoint families, for $\gamma = 0$, $\gamma > 0$ and $\gamma < 0$, and each family includes infinitely many types. Indeed in Temido and Canto e Castro [5] it is proved that G and G^θ are in the same type if and only if exists $m \in \mathbb{Z}$ such that $\theta = r^m$.

In this work we extend the first order and the second order conditions established for the max-stable domains of attractions and characterize the class of all possible limits.

2. A first order condition

As usually, for a nondecreasing and right continuous function f its left-continuous inverse is defined by $f^{\leftarrow}(x) = \inf\{y : f(y) \geq x\}$. In this context we recall Lemma 1.1.1 of de Haan and Ferreira [2] where conditions under which the convergence of f_n to f implies the convergence of $f_n^{\leftarrow}$ to $f^{\leftarrow}$ are given.

Let F be a d.f. and define $U = \left(\frac{1}{1-F}\right)^{\leftarrow}$. We recall that in the max-stable context, F belongs to the domain of attraction of G_γ, that is, there are real sequences $\{a_n > 0\}$ and $\{b_n\}$ for which

$$F^n\left(a_n x + b_n\right) \to G_\gamma(x) = \exp(-(1+\gamma x)^{-1/\gamma}),\ n \to +\infty, \tag{3}$$

if and only if there is a real function $a(t)$ such that

$$\frac{U\left(tx\right) - U(t)}{a(t)} \to \frac{x^\gamma - 1}{\gamma},\ t \to +\infty. \tag{4}$$

For $\gamma = 0$ the right hand side of (3) and (4) is interpreted as $\exp(-e^{-x})$ and $\ln x$, respectively.

In order to extend that well known first order condition (4), which holds in the max-stable setup, we are going to apply that lemma of de Haan and Ferreira to relation (2). In fact we can establish the equivalence stated in the following theorem.

Theorem 1. *The distribution function F belongs to the domain of attraction of the max-semistable distribution function G if and only if there exist a nondecreasing positive integer sequence $\{k_n\}$ satisfying* (1), *a positive real sequence $\{a_n\}$ and a real γ such that*

$$\frac{U(k_n x) - U(k_n)}{a_n} \to L(x) := \begin{cases} \frac{h^\gamma(x) - h^\gamma(1)}{\gamma} & \gamma \neq 0 \\ g(x) - g(1) & \gamma = 0 \end{cases}, \tag{5}$$

or each point of continuity x of L, where h and g are such that $h^{\leftarrow}(x) = \dfrac{x}{\nu(-\ln x)}$ and $g^{\leftarrow}(x) = \dfrac{e^x}{\nu(x)}$.

P r o o f. Applying Lemma 1.1.1 of de Haan and Ferreira [2] to the convergence (2), we deduce

$$\frac{U\left(k_n x\right) - b_n}{a_n} \to \left(\frac{1}{-\ln G_{\gamma,\nu}}\right)^{\leftarrow}(x) := L^*(x),\ n \to +\infty, \tag{6}$$

for each point x of continuity of L^*. Since for $\gamma \neq 0$

$$\inf\left\{y : \frac{(1+\gamma\, y)^{\frac{1}{\gamma}}}{\nu(\ln(1+\gamma\, y)^{-\frac{1}{\gamma}})} \geq x\right\} = \inf\left\{\frac{z^\gamma - 1}{\gamma} : \frac{z}{\nu(-\ln z)} \geq x\right\}$$

and $\dfrac{z^\gamma - 1}{\gamma}$ is nondecreasing, we obtain

$$L^*(x) = \begin{cases} \frac{h^\gamma(x)-1}{\gamma} & \gamma \neq 0 \\ g(x) & \gamma = 0 \end{cases}.$$

In other hand, considering $b'_n = U(k_n)$, from (5) we get

$$\frac{U(k_n x) - b'_n}{a_n} \to L^*(x) + C,\ n \to +\infty,$$

where C is a constant. So, there is b_n such that $\dfrac{U(k_n x) - b_n}{a_n} \to L^*(x)$, $n \to +\infty$. Consequently, applying again the same lemma, we have $\dfrac{U^{\leftarrow}(a_n y + b_n)}{k_n} \to (L^*)^{\leftarrow}(y)$, $n \to +\infty$, and then

$$\frac{1}{k_n(1 - F(a_n y + b_n))} \to \frac{1}{-\ln G(y)},\ n \to +\infty,$$

for each point y of continuity of G. □

Remark 1. *Observe that if ν is constant we obtain the max-stable context and the functions h and g have the expected expressions.*

Remark 2. *Note that if $\{k_n\}$ is a real positive and nondecreasing sequence satisfying (1) then $k_n(1-F(u_n)) \to \tau$, $n \to +\infty$, if and only if $[k_n](1-F(u_n)) \to \tau$, $n \to +\infty$. Thus, in what follows, $\{k_n\}$ is not necessarily an integer sequence.*

In the following proposition, a characterization of the functions h and g is given.

Proposition 1. *Let ν be a bounded and periodic function with period $p = \ln r$, with $r > 1$. The functions $h(x) = \left(\dfrac{x}{\nu(-\ln x)}\right)^{\leftarrow}$ and $g(x) = \left(\dfrac{e^x}{\nu(x)}\right)^{\leftarrow}$ verify*

$$\text{i) } h(z\,y) = z\,h(y) \quad \Leftrightarrow \quad z \in \{r^m,\ m \in \mathbb{Z}\};$$

$$\text{ii) } g(z\,y) = g(y) + \ln z \quad \Leftrightarrow \quad z \in \{r^m,\ m \in \mathbb{Z}\}.$$

P r o o f. For positive reals y and z we have

$$\begin{aligned} h(y\,z) &= \inf\left\{x : \frac{x/z}{\nu(-\ln x)} \geq y\right\} \\ &= z\inf\left\{w : \frac{w}{\nu(-\ln(w\,z))} \geq y\right\}. \end{aligned}$$

Now, since the period of ν is $\ln r$, we get $\nu(-\ln w - \ln z) = \nu(-\ln w)$ if and only if there is an integer m such that $\ln z = m \ln r$. Then

$$h(z\,y) = z\inf\left\{w : \frac{w}{\nu(-\ln w)} \geq y\right\} = r^m h(y).$$

The result ii) follows similarly from

$$\begin{aligned} g(y\,z) &= \inf\left\{x : \tfrac{e^x}{\nu(x)} \geq z\,y\right\} = \inf\left\{x : \frac{e^{x-\ln z}}{\nu(x)} \geq y\right\} \\ &= \inf\left\{w : \frac{e^w}{\nu(w+\ln z)} \geq y\right\} + \ln z = \inf\left\{w : \frac{e^w}{\nu(w)} \geq y\right\} + m\,\ln r \\ &= g(y) + m\,\ln r. \end{aligned}$$

□

Example 1. Consider the Von Misès d.f. defined by $F(x) = 1 - e^{-x-1/2\sin x}$, for $x > 0$, which is strictly increasing. This d.f. does not belong to any max--stable domain of attraction. The inverse function U verifies

$$U(x) = \ln x - \frac{1}{2}\sin U(x),\ x > 0.$$

Taking $a_n = 1$ we will check (5). In fact, taking into account that

$$U(k_n x) - U(k_n) = \ln k_n + \ln x - \frac{1}{2}\sin U(k_n x) - \ln k_n + \frac{1}{2}\sin U(k_n),$$

considering k_n such that $\sin U(k_n) = 1$, for all n, that is, $k_n = U^{\leftarrow}(2n\pi + \pi/2 - 1/2) = e^{2n\pi+\pi/2-1/2}$, we get

$$U(k_n x) - U(k_n) = \ln x - \frac{1}{2}\sin U(k_n x) + \frac{1}{2}, \forall n \in \mathbb{N}. \tag{7}$$

With $a_n = 1$ and this choice for k_n, a convergence exists like in (5), if and only if there is a function $\ell(x)$, not depending on n such that $U(k_n x) = \ell(x) + 2m\pi$, with $m \in \mathbb{Z}$.

Since

$$U(k_n x) = \ell(x) + 2m\pi \quad \Leftrightarrow \quad k_n x = e^{\ell(x)+2m\pi+\frac{1}{2}\sin\ell(x)}$$
$$\Leftrightarrow \quad e^{\pi/2-1/2}x = e^{\ell(x)+\frac{1}{2}\sin\ell(x)}$$

holds if $m = n \in \mathbb{N}$, we obtain $\ell(x) = U(e^{\pi/2-1/2}x)$, because F is strictly increasing. Thus, from (7) we conclude that

$$U(k_n x) - U(k_n) = \ln x - \frac{1}{2}\sin\ell(x) + \frac{1}{2}$$

and that $g(x) = \ln x - \dfrac{1}{2}\sin\ell(x)$. Moreover, since

$$g\left(\frac{e^x}{\nu(x)}\right) = x \Leftrightarrow -\ln\nu(x) = \frac{1}{2}\sin U\left(e^{\pi/2-1/2}\frac{e^x}{\nu(x)}\right) \tag{8}$$

and

$$U\left(e^{\pi/2-1/2}\frac{e^x}{\nu(x)}\right) = z \Rightarrow \frac{1}{2}\sin z = -z + \frac{\pi}{2} - \frac{1}{2} + x - \ln\nu(x),$$

from (8), we deduce $z = x + \dfrac{\pi}{2} - \dfrac{1}{2}$ and thus

$$\nu(x) = e^{-\frac{1}{2}\sin(x+\pi/2-1/2)} = e^{-\frac{1}{2}\cos(x-\frac{1}{2})},\ x \in \mathbb{R}. \qquad \square$$

In the next result, for a special class of twice differentiable distribution functions, we prove that the first order condition (5) can be established choosing $a_n = k_n U'(k_n)$.

Theorem 2. *Let $\{k_n\}$ be a nondecreasing real sequence satisfying (1) with $r > 1$. Suppose that the inverse function U is twice differentiable. If there is a function h satisfying $(h^\gamma)'(1) = \gamma$ for some real γ, i) of Proposition 1 and*

$$\lim_{n\to+\infty}\frac{k_n U''(k_n x)}{U'(k_n x)} = \frac{(h^\gamma)''(x)}{(h^\gamma)'(x)}$$

or there is a function g satisfying $g'(1) = 1$, ii) of Proposition 1, and

$$\lim_{n\to+\infty}\frac{k_n U''(k_n x)}{U'(k_n x)} = \frac{g''(x)}{g'(x)},$$

then (5) holds with $a_n = k_n U'(k_n)$.

P r o o f. For $x > 0$ and n sufficiently large such that $k_n x > 1$, we have

$$\ln U'(k_n x) - \ln U'(k_n) = \int_1^x \frac{k_n U''(k_n s)}{U'(k_n s)} ds.$$

Then, for all reals a and b, we have

$$\lim_{n\to+\infty} \sup_{x\in[a,b]} \left| \ln \frac{U'(k_n x)}{U'(k_n)} - \ln \frac{(h^\gamma)'(x)}{\gamma} \right| = 0$$

and consequently, since the Lagrange theorem gives us $|e^s - e^t| < C|s-t|$ for $s,t \in [a,b]$ and $C > 0$, we also deduce

$$\lim_{n\to+\infty} \sup_{x\in[a,b]} \left| \frac{U'(k_n x)}{U'(k_n)} - \frac{(h^\gamma)'(x)}{\gamma} \right| = 0. \tag{9}$$

In this case we conclude that

$$\frac{U(k_n\, x) - U(k_n)}{k_n U'(k_n)} - \frac{h^\gamma(x) - h^\gamma(1)}{\gamma} = \int_1^x \left(\frac{U'(k_n s)}{U'(k_n)} - \frac{(h^\gamma)'(s)}{\gamma} \right) ds$$

converges to zero, when n goes to infinity.

Mutatis mutandis the result holds as well as for the limit g. □

From the proof of the last theorem we can establish the following result.

Proposition 2. *Let $\{k_n\}$ be a nondecreasing real sequence satisfying* (1) *with $r > 1$. Suppose that the inverse function U is differentiable. If there is a function h satisfying i) of Proposition* 1 *and* (9) *for some real γ, or there is a function g satisfying ii) of Proposition* 1*, and*

$$\lim_{n\to+\infty} \sup_{x\in[a,b]} \left| \frac{U'(k_n x)}{U'(k_n)} - g'(x) \right| = 0,$$

then (5) *holds with $a_n = k_n U'(k_n)$.*

Example 2. Consider the d.f. of Example 1. Since

$$U'(x) = \frac{1}{x\left(1 + \frac{1}{2}\cos U(x)\right)},$$

for $x > 0$, with $\{k_n\}$ such that $\cos U(k_n) = 0$, we obtain

$$\frac{U'(k_n x)}{U'(k_n)} = \frac{1}{x\left(1 + \frac{1}{2}\cos(U(k_n x))\right)} = \frac{1}{x\left(1 + \frac{1}{2}\cos(\ell(x))\right)}.$$

In other hand

$$g'(x) = \frac{1}{x} - \frac{1}{2}\cos(\ell(x))\ell'(x) = \frac{1}{x\left(1 + \frac{1}{2}\cos(\ell(x))\right)}.$$ □

3. A second order condition

In this section, we establish a second order condition related to the first order condition (5). Indeed we will suppose that there exists a real sequence $\{A_n\}$ such that $\lim_{n\to+\infty} A_n/a_n = 0$ and

$$\lim_{n\to+\infty} \frac{U(k_n\,x) - U(k_n) - a_n L(x)}{A_n} = H(x), \tag{10}$$

for all x positive. In what follows we determine analytically the function H.

We recall again that, in the max-stable setup, for a d.f. F satisfying the first order condition, we assume that there exists a function A(t) such that $\lim_{t\to+\infty} A(t) = 0$ and

$$\lim_{t\to+\infty} \frac{\frac{U(t\,x)-U(t)}{a(t)} - \frac{x^\gamma-1}{\gamma}}{A(t)} = H(x).$$

It is proved (see de Haan and Ferreira [2]) that, if $H(x)$ is not a multiple of $\frac{x^\gamma-1}{\gamma}$, there is a constant $\rho \leq 0$, such that $H(x) = \frac{1}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma-1}{\gamma}\right)$.

We will need the following lemma.

Lemma 1. *Let $\{k_n\}$ and U be under the conditions of Theorem 1. Then*

$$\lim_{n\to+\infty} \frac{U(k_n\,r^m\,x) - U(k_n\,r^m)}{a_n} = \lim_{n\to+\infty} \frac{U(k_{n+m}\,x) - U(k_{n+m})}{a_n}$$

and

$$\lim_{n\to+\infty} \frac{U(k_n\,r^m\,x) - U(k_n\,r^m)}{U(k_{n+m}\,x) - U(k_{n+m})} = 1.$$

Proof. In fact, using Theorem 1 we get, for $\gamma \neq 0$,

$$\begin{aligned}
&\frac{U(k_n\,r^m\,x) - U(k_n)}{a_n} + \frac{U(k_n) - U(k_n\,r^m)}{a_n}\\
&\longrightarrow \frac{h^\gamma(xr^m) - h^\gamma(1)}{\gamma} - \frac{h^\gamma(r^m) - h^\gamma(1)}{\gamma}\\
&= r^{m\gamma}\frac{h^\gamma(x) - h^\gamma(1)}{\gamma},\ n\to+\infty,
\end{aligned}$$

and

$$\frac{U(k_{n+m}\,x)-U(k_{n+m})}{a_n} = \frac{U(k_{n+m}\,x)-U(k_{n+m})}{a_{n+m}}\frac{a_{n+m}}{a_n}$$
$$\longrightarrow \frac{h^\gamma(x)-h^\gamma(1)}{\gamma}a^m = \frac{h^\gamma(x)-h^\gamma(1)}{\gamma}r^{m\gamma},\ n\to+\infty.$$

The result follows as well as for the function g. □

In the following theorem we establish the class of possible limits in (10). For the proof we need the following well known result.

Lemma 2.

1. *Suppose that $\{V_m\}$ is a real positive sequence satisfying $V_{m+p}=V_mV_p$, for all positive integers p and m. Then either $V_m=1$ or exists $\zeta>0$ such that $V_m=\zeta^m$, for all positive integer m.*
2. *Suppose that $\{Q_m\}$ is a real positive sequence such that, for all positive integers p and m, $Q_{m+p}=Q_p\theta^m+Q_m = Q_m\theta^p+Q_p$, for some $\theta>0$. Then there is a real C such that*
   (a) *if $\theta\neq 1$, then $Q_m=C\,(1-\theta^m)$;*
   (b) *if $\theta=1$, then $Q_m=Cm$.*

Theorem 3. *Let $\{k_n\}$ be a nondecreasing real sequence satisfying (1) with $r>1$. If the second order condition (10) holds, where H is not a multiple of $L(x)$, then there are real constants C, C_1, C_2 and C_3 and periodic functions ξ, ξ_1, ξ_2 and ξ_3, with period $\ln r$, and a parameter $\zeta>0$ such that the function H is given by*

$$H(x)=\begin{cases} x^{\frac{\ln\zeta}{\ln r}}\xi(\ln x)+\dfrac{C}{\gamma}\left(h^\gamma(x)-h^\gamma(1)\right) & \gamma\neq 0,\zeta\neq 1\\[2ex] \xi_1(\ln x)+\dfrac{C_1}{\gamma}\left(h^\gamma(x)-h^\gamma(1)\right) & \gamma\neq 0,\zeta=1\\[2ex] x^{\frac{\ln\zeta}{\ln r}}\xi_2(\ln x)+C_2\left(g(x)-g(1)\right) & \gamma=0,\zeta\neq 1\\[2ex] \xi_3(\ln x)+C_3\left(g^2(x)-g^2(1)\right)/\ln r^2 & \gamma=0,\zeta=1\end{cases},$$

with $x>0$. Moreover $\zeta=\lim\limits_{n\to+\infty}\dfrac{A_{n+m}}{A_n}$ and $\xi(0)=\xi_1(0)=\xi_2(0)=\xi_3(0)=0$.

P r o o f. We divide the proof in cases A and B, for $\gamma \neq 0$ and $\gamma = 0$, respectively.

Case A : $\gamma \neq 0$. Due to (5) and Lemma 2 we have

$$\lim_{n\to+\infty} \frac{U(k_n r^m x) - U(k_n) - a_n(h^\gamma(r^m x) - h^\gamma(1))/\gamma}{A_n} - \frac{U(k_n r^m) - U(k_n) - a_n(h^\gamma(r^m) - h^\gamma(1))/\gamma}{A_n} =$$

$$\lim_{n\to+\infty} \frac{U(k_{n+m}x) - U(k_{n+m}) - a_{n+m}(h^\gamma(x) - h^\gamma(1))/\gamma}{A_{n+m}} \frac{A_{n+m}}{A_n} + \frac{(a_{n+m} - a_n r^{m\gamma})(h^\gamma(x) - h^\gamma(1))/\gamma}{A_n}$$

and then

$$(11)\quad H(r^m x) - H(r^m) = H(x) \lim_{n\to+\infty} \frac{A_{n+m}}{A_n} + \frac{h^\gamma(x) - h^\gamma(1)}{\gamma} \lim_{n\to+\infty} \frac{a_{n+m} - r^{m\gamma}a_n}{A_n},$$

only if these two limits exist. Indeed, because of the fact that H is not a multiple of $L^*(x) = \dfrac{h^\gamma(x) - h^\gamma(1)}{\gamma}$, we conclude that the relation

$$\frac{H(x_1)}{\frac{h^\gamma(x_1)-h^\gamma(1)}{\gamma}} = \frac{H(x_2)}{\frac{h^\gamma(x_2)-h^\gamma(1)}{\gamma}}$$

does not hold for all positive reals x_1 and x_2. Thus, there are two different positive reals, x_1 and x_2, such that $H(x_1) - \dfrac{h^\gamma(x_1) - h^\gamma(1)}{h^\gamma(x_2) - h^\gamma(1)} H(x_2) \neq 0$. Now, using (11) twice, one of them multiplying by $\theta := \dfrac{h^\gamma(x_1) - h^\gamma(1)}{h^\gamma(x_2) - h^\gamma(1)}$, we obtain

$$(H(x_1) - \theta H(x_2)) \lim_{n\to+\infty} \frac{A_{n+m}}{A_n} = H(r^m x_1) - \theta H(r^m x_2) - (1-\theta)H(r^m)$$

what enables us to conclude that $\lim_{n\to+\infty} \dfrac{A_{n+m}}{A_n}$ exists. Hence, once again from (11) we deduce that $\lim_{n\to+\infty} \dfrac{a_{n+m} - r^{m\gamma}a_n}{A_n}$ also exists.

Consider $V_m = \lim_{n\to+\infty} \dfrac{A_{n+m}}{A_n}$ and observe that $V_{m+p} = V_p V_m$, $\forall\, m, p \in \mathbb{Z}$. By Lemma 2, $V_m = 1$ or, otherwise, exists a real $\zeta > 0$ such that $V_m = \zeta^m$, $\forall\, m \in \mathbb{Z}$.

Now we subdivide case A into A_1 for $\zeta \neq 1$ and A_2 for $\zeta = 1$.

Case $\mathbf{A_1}$: Consider

$$Q_m = \lim_{n\to+\infty} \frac{r^{-(m+n)\gamma}a_{n+m} - r^{-n\gamma}a_n}{r^{-n\gamma}A_n}$$

that exists by (11), and put $a_n^{'} = r^{-n\,r}a_n$ and $A_n^{'} = r^{-n\,\gamma}A_n$. Taking into account that

$$\begin{aligned} Q_{m+p} \quad &:= \quad \lim \frac{a_{n+m+p}^{'} - a_{n+m}^{'} + a_{n+m}^{'} - a_n^{'}}{A_n^{'}} \\ &:= \quad \lim_{n\to+\infty} \frac{a_{n+m+p}^{'} - a_{n+m}^{'}}{A_{n+m}^{'}} \lim_{n\to+\infty} \frac{A_{n+m}^{'}}{A_n^{'}} + \lim_{n\to+\infty} \frac{a_{n+m}^{'} - a_n^{'}}{A_n^{'}} \end{aligned}$$

we deduce

$$Q_{m+p} = Q_p\zeta^m r^{-m\gamma} + Q_m,\ \forall\, m,p \in \mathbb{Z}.$$

In a similar way we can obtain

$$Q_{m+p} = Q_m\zeta^p r^{-p\gamma} + Q_p,\ \forall\, m,p \in \mathbb{Z}.$$

By Lemma 2 there is a constant C such that $Q_m = C\left(1 - \zeta^m r^{-m\gamma}\right)$.

Now, taking again (11) into consideration we obtain

$$H(r^m x) - H(r^m) = H(x)\zeta^m + \frac{h^\gamma(x) - h^\gamma(1)}{\gamma} r^{m\gamma} C\left(1 - \zeta^m r^{-m\gamma}\right).$$

In what follows we solve the linear equation

$$H(rx) - H(r) = H(x)\zeta + \frac{h^\gamma(x) - h^\gamma(1)}{\gamma} C\left(r^\gamma - \zeta\right) \tag{12}$$

beginning with the homogeneous part

$$H(rx) = \zeta H(x). \tag{13}$$

With $x = e^z$ it holds $H\left(e^{z+\ln r}\right) = \zeta H\left(e^z\right)$ and multiplying by $e^{-z\frac{\ln\zeta}{\ln r}}$ we get

$$e^{-(z+\ln\, r)\frac{\ln\zeta}{\ln r}} H\left(e^{z+\ln\, r}\right) = e^{-z\frac{\ln\zeta}{\ln r}} H\left(e^z\right).$$

¿From this last equation we conclude that $\xi(z) := e^{-z\frac{\ln\zeta}{\ln r}} H\left(e^z\right)$ is a periodic function with period $\ln r$. Then

$$H(x) = x^{\frac{\ln\zeta}{\ln r}}\,\xi(\ln x)$$

is the general solution of the homogeneous equation (13).

Since $H_p(x) = \dfrac{C}{\gamma}\,(h^\gamma(x) - h^\gamma(1))$ is a particular solution of the complete equation (12), we conclude that

$$H(x) = x^{\frac{\ln\zeta}{\ln r}}\xi(\log x) + \frac{C}{\gamma}\,(h^\gamma(x) - h^\gamma(1))$$

is the general solution of (12).

Note that $H(1) = 0$ implies $\xi(0) = 0$. Consequently $\xi(0) = \xi(\ln r) = \xi(2\ln r) = \cdots = \xi(m\ln r) = \cdots = 0$.

Case $\mathbf{A_2}$: $\gamma \neq 0$ and $V_m = 1$ for all integer m. In a similar way we have

$$Q_{m+p} = Q_p r^{-m\gamma} + Q_m \;=\; Q_m r^{-p\gamma} + Q_p$$

and so, by Lemma 2, there is a constant C_1 such that $Q_m = C_1\,(1 - r^{-m\gamma})$, $\forall\, m, p \in \mathbb{Z}$. Hence (11) gives us

$$H(r^m x) - H(r^m) = H(x) + \frac{h^\gamma(x) - h^\gamma(1)}{\gamma}\, r^{-m\gamma} C_1 \left(1 - r^{-m\gamma}\right).$$

We will solve

$$H(r\,x) - H(r) = H(x) + \frac{h^\gamma(x) - h^\gamma(1)}{\gamma}\, C_1\,(r^\gamma - 1)\,. \tag{14}$$

The general solution of the equation $H(r\,x) = H(x)$ is $\xi_1(\ln x)$, where ξ_1 is a periodic function with period $\ln r$, and $H_p(x) = \dfrac{C_1}{\gamma}\,(h^\gamma(x) - h^\gamma(1))$ is a particular solution of (14). Thus, when $\gamma \neq 1$ and $\zeta = 1$, we can conclude that

$$H(x) = \xi_1(\ln x) + \frac{C_1}{\gamma}\,(h^\gamma(x) - h^\gamma(1))\,.$$

Case B : $\gamma = 0$. From (5) and using again Lemma 2 we deduce

$$\lim_{n\to+\infty} \frac{U\,(k_n r^m x) - U(k_n) - a_n\,(g\,(x r^m) - g(1))}{A_n} - \\ - \frac{U\,(k_n r^m) - U\,(k_n) - a_n\,(g\,(r^m) - g(1))}{A_n} = \\ \lim_{n\to+\infty} \frac{U\,(k_{n+m} x)) - U\,(k_{n+m}) - a_{n+m}\,(g(x) - g(1))}{A_{n+m}}\frac{A_{n+m}}{A_n} + \\ \frac{(a_{n+m} - a_n)\,(g(x) - g(1))}{A_n}$$

and then

$$H(r^m x) - H(r^m) = H(x) \lim_{n\to+\infty} \frac{A_{n+m}}{A_n} + \lim_{n\to+\infty} \frac{a_{n+m}-a_n}{A_n} \left(g(x) - g(1)\right).$$

We have again $Q_{m+p} = Q_p\zeta^m + Q_m = Q_m\zeta^p + Q_p$ and so, from Lemma 2, there are reals C_2 and C_3 such that $Q_m = C_2(1-\zeta^m)$ if $\zeta \neq 1$ or $Q_m = C_3 m$ if $\zeta = 1$.

Case $\mathbf{B_1}$: $\gamma = 0$ and $\zeta \neq 1$. It holds

$$H(r^m x) - H(r^m) = \zeta^m H(x) + C_2(1-\zeta^m)\left(g(x) - g(1)\right).$$

Since $H_p(x) = C_2\left(g(x) - g(1)\right)$ is a particular solution of

$$H(r\,x) - H(r) = \zeta H(x) + C_2(1-\zeta)\left(g(x) - g(1)\right)$$

we conclude that the general solution of the last functional equation is given by

$$H(x) = x^{\frac{\ln\zeta}{\ln r}}\xi_2(\ln x) + C_2\left(g(x) - g(1)\right).$$

Case $\mathbf{B_2}$: $\gamma = 0$ and $\zeta = 1$. Using again the arguments above we obtain

$$H(r^m x) - H(r^m) = H(x) + C_3\, m\, \left(g(x) - g(1)\right), \; \forall m \in \mathbb{Z}.$$

Due to the fact that $H_p(x) = \dfrac{C_3}{2\ln r}\left(g^2(x) - g^2(1)\right)$ is a particular solution of the functional equation

$$H(r\,x) - H(r) = H(x) + C_3\,\left(g(x) - g(1)\right), \tag{15}$$

we conclude that the general solution of this equation is

$$H(x) = \xi_3(\ln x) + \frac{C_3}{2\ln r}\left(g^2(x) - g^2(1)\right),$$

where ξ_3 is a periodic function with period $\ln r$.

Moreover $\xi(0) = \xi_1(0) = \xi_2(0) = \xi_3(0) = 0$ due to $H(1) = 0$. $\square$

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