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TAIL INFERENCE FOR A LAW IN A MAX-SEMISTABLE DOMAIN OF ATTRACTION*

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The class of max-semistable distributions appeared in the literature of extremes, in a work of Pancheva (1992), as the limit distribution of samples with size growing geometrically with ratio $r > 1$. In Canto e Castro *et al.* (2002) it is proved that any max-semistable distribution function has a log-periodic component and can be characterized by the period therein, by a tail index parameter and by a real function y representing a repetitive pattern.

Statistical inference in the max-semistable setup can be performed through convenient sequences of generalized Pickands' statistics, depending on a tuning parameter s. More precisely, in order to obtain estimators for the period and for the tail index, we can use the fact that the mentioned sequences converge in probability only when $s = r$ (or any of its integer powers), having an oscillatory behavior otherwise. This work presents a procedure to estimate the function y as well as high quantiles. The suggested methodologies are applied to real data consisting in seismic moments of major earthquakes in the Pacific Region.

1. Introduction

A distribution function (d.f.) F belongs to the domain of attraction of a max-stable d.f. G if and only if there exist real sequences $\{a_n\}$ and $\{b_n\}$, with $a_n > 0$,

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such that

$$\lim_{n\to+\infty} F^n(a_n x + b_n) = G(x),\ x \in \mathbb{R}.$$

The most common continuous d.f.'s belong to this class, although the same doesn't happen to a wide variety of multimodal continuous d.f.'s as well as to discrete d.f.'s. However, some of those d.f.'s belong to a larger class, the one of max-semistable d.f.'s.

The pionieres in the study of max-semistable laws were Pancheva [7] and Grinevich [5]. Following these authors a d.f. G is max-semistable if and only if there exist $r > 1$, $a > 0$ and $b \in \mathbb{R}$ such that G is solution of the functional equation

$$G(x) = G^r(ax + b),\ x \in \mathbb{R}, \tag{1}$$

or equivalently, if there exist a d.f. F and real sequences $\{a_n\}$ and $\{b_n\}$, with $a_n > 0$, such that

$$\lim_{n\to+\infty} F^{k_n}(a_n x + b_n) = G(x),\ x \in C_G, \tag{2}$$

where C_G is the set of continuity points of the function G and $\{k_n\}$ is a non decreasing sequence that verifies the following geometric growing condition

$$\lim_{n\to+\infty} \frac{k_{n+1}}{k_n} = r \geq 1\ (< \infty). \tag{3}$$

Notice that if $r = 1$ we are in the particular case of the max-stable class. When (2) holds we say that the d.f. F belongs to the domain of attraction of the max-semistable d.f. G.

In [5] are obtained explicit expressions for the max-semistable d.f.'s G, proving that if a d.f. G is max-semistable then it is of the same type as some element of the following family of d.f.'s:

$$G_{\gamma,\nu}(x) = \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right)\nu\left(\ln(1+\gamma x)^{-1/\gamma}\right) & 1+\gamma x > 0,\ \gamma \neq 0 \\ \mathbb{1}_{]-\infty,0[}(\gamma) & 1+\gamma x \leq 0,\ \gamma \neq 0 \\ \exp\left(-e^{-x}\nu(x)\right) & \gamma = 0,\ x \in \mathbb{R} \end{cases},$$

where $\gamma \in \mathbb{R}$ and ν is a positive, bounded and periodic function with period $p = \log r$. The parameters r and γ are related with parameters a and b in (1), the following way: $a = r^\gamma$ and $b = \log r$ if $\gamma = 0$.

A characterization of the max-semistable domains of attraction, depending on a d.f. F_0 in a max-stable domain of attraction, is presented in [6].

In current language, we can say that a max-semistable d.f. is a max-stable d.f. perturbed by a log-periodic function. This relation can be easily seen when we construct a QQ-plot of a max-semistable d.f. against a max-stable one, with the same parameter γ. In fact, as it is easily derived, the respective quantiles, y and x, are related through the equations $y = \frac{1}{\gamma}(1+\gamma x)\,[\nu(\log(1+\gamma x)^{-1/\gamma}]^{-\gamma} - \frac{1}{\gamma}$, for $\gamma \neq 0$ and $y = x - \log(\nu(x))$, for $\gamma = 0$, meaning that its graph is a log-periodic (periodic) curve along a straight line, when $\gamma \neq 0$ ($\gamma = 0$). Figure 1 shows the QQ-plot of the max-semistable d.f. $F(x) = \exp\{-x^{-2}(8 + \cos(4\pi \log x))\}$, for $x > 0$, against the max-stable d.f. $G(x) = \exp(-x^{-2})$, for $x > 0$.

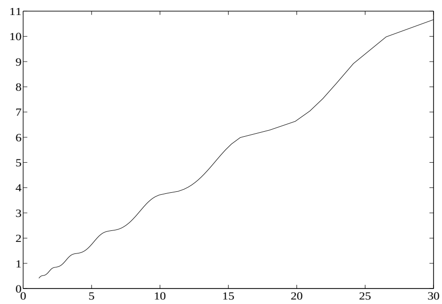

Figure 1: QQ-Plot of the d.f. $F(x) = \exp\{-x^{-2}(8 + \cos(4\pi \log x))\}$ against the d.f. $G(x) = \exp(-x^{-2})$

A new characterization of the limit functions G was established by Canto e Castro *et al.* [1]. Assuming, without lost of generality, that

$$(4) \qquad \begin{cases} G(0) = e^{-1} \\ G(1) = \exp(-r^{-1}) \\ G \text{ continuous in } x = 0 \end{cases},$$

this authors obtained the following representation

$$(5) \qquad -\log(-\log G(a^n x + s_n)) = n \log r + y(x),\ \forall\, x \in [0,1],\ n \in \mathbb{Z},$$

where the function $y : [0,1] \to [0, \log\ r]$ is non decreasing, right continuous and continuous at $x = 1$, and $s_n = (a^n - 1)/(a-1)$ if $a \neq 1$ and $a > 0$ or $s_n = n$ if $a = 1$.

With this representation those authors obtained a new characterization of the max-semistable domain of attraction not dependent from the existence of some d.f. F_0 in a max-stable domain of attraction.

Theorem 1. *Let G be a max-semistable d.f. satisfying* (4)*. In order that* (2) *holds for some d.f. F and some sequence $\{k_n\}$ satisfying* (3)*, it is necessary and sufficient that, for $U = -\log(-\log F)$, there exists a real sequence $\{b_n\}$ such that*

$$\lim_{n\to+\infty} \frac{b_{n+1}-b_n}{b_n-b_{n-1}} = a,$$

$$\lim_{n\to+\infty} U(b_{n+1}) - U(b_n) = \log r, \tag{6}$$

and

$$\lim_{n\to+\infty} U(b_n + x(b_{n+1}-b_n)) - U(b_n) = y(x),$$

for all continuity points of y in $[0,1]$. Furthermore, the convergence (2) is verified with $a_n := b_{n+1} - b_n$ and $k_n = [e^{U(b_n)}]$.

This characterization was used in the development of parameter estimators by Canto e Castro and Dias [3]. In fact, those estimators were based on the sequence of statistics, suggested by Temido [10],

$$Z_s(m) := \frac{X_{(m/s^2)} - X_{(m/s)}}{X_{(m/s)} - X_{(m)}},$$

where $X_{(m)} := X_{N-[m]+1,N}$ are the order statistics of an independent and identically distributed random sample of size N drawn from a population distributed as X and $m := m_N$ is an intermediate sequence (that is, m_N is an integer sequence verifying $m_N \to +\infty$ and $m_N/N \to 0$, as $N \to +\infty$). The analysis of the asymptotic behavior of this sequence when the d.f. of X is in a max-semistable domain of attraction was done in Dias and Canto e Castro [2]. It was proved that it converges in probability to a^c if and only if $s = r^c, c \in \mathbb{N}$. Furthermore, if $s \neq r^c, c \in \mathbb{N}$, then $Z_s(m)$ as an oscillatory behaviour. These results allows to prove that if $s = r^c, c \in \mathbb{N}$, then the following sequences of statistics converge in probability to some constant, namely:

- $R_s(m) := \dfrac{Z_{s^2}(m)}{(Z_s(m))^2} \xrightarrow{P} 1, \quad n \to +\infty;$
- $\widehat{\gamma}_s(m) := \dfrac{\log(Z_s(m))}{\log\ s} \xrightarrow{P} \gamma, \quad n \to +\infty.$

Based on these results, to estimate r, the authors proposed an heuristic method which uses the sequence of statistics $R_s(m)$, considering as an estimate of r the value

$$\widehat{r} = \text{mode}\left\{\underset{s=1.1,(0.1),3.0}{\arg\max} B_s(\epsilon), \epsilon = 0.01, (0.01), 0.1\right\} \tag{7}$$

where

$$B_s(\epsilon) := \frac{1}{k}\sum_{i=1}^{k} \mathbb{1}_{\{m\in A_n:(R_s(m)-1)^2<\epsilon\}}(m^{(i)})$$

(that is, $B_s(\epsilon)$ is the percentage of time that the sequence of statistics R_s spends in a ϵ neighborhood of 1), $A_n = \{m^{(1)}, m^{(2)}, \ldots, m^{(k)}\}$ is a set of suitable values of m.

Having obtained an estimate of r, γ can also be estimated by

$$\widehat{\gamma} = \frac{1}{\log \widehat{r}}\frac{1}{k}\sum_{m\in A_n} \log Z_{\widehat{r}}(m). \tag{8}$$

2. The estimation of the function y

In this section we describe a method to estimate the function y. This method is based in the characterization of the limit d.f. G obtained by Canto e Castro *et al.* [1]. Combining the estimator of y with the estimators of the parameters r and γ we can characterize completely the d.f. G.

Let $X_{i,n}$ be the i_th ascending order statistic from a sample of size n with common d.f. F and consider the empirical d.f. F_n.

Notice that it is possible to establish a characterization similar to (5) for the general case of a function G, continuous at 0 but not satisfying the other two conditions in (4), taking into account that, by (1), we have

$$-\log(-\log G(ax+b)) = -\log(-\log G(x)) + \log r.$$

With $z_0 = -\log(-\log G(0))$ we obtain $-\log(-\log G(b)) = z_0 + \log r$ and, more generally,

$$-\log(-\log G(a^n x + s_n^*)) = y^*(x) + n\log r, \qquad \forall x \in [0, b],$$

where $y^* : [0, b] \to [z_0, z_0 + \log r]$ is the restriction of $-\log(-\log G)$ to $[0, b]$ and $s_n^* = s_n b$. In fact, this type of characterization can be achieved starting the process in any continuity point x_0. To simplify the notation we denote by y any of the functions y^* that can appear in the previous characterization.

Due to (5) if we decompose the image set of the function $-\log(-\log G)$ into disjoint intervals of length $\log r$, after a suitable change of location and scale in each correspondent segment of the function's domain, the function y is obtained.

Then, in a first step, we apply the same decomposition to the empirical function $U_n := -\log(-\log F_n)$, starting the construction in the right tail. Indeed we use the maximum, $X_{n,n}$, as the upper bound of the first segment. Therefore, the first segment has domain $[X_{m_1,n}, X_{n,n}]$ where

$$X_{m_1,n} := \max\{X_{l,n} : U_n(X_{n,n}) - U_n(X_{l,n}) \geq \log r\}.$$

Recursively, we obtain the domain of the i-th segment, $[X_{m_i,n}, X_{m_{i+1},n}]$, $i = 1, \ldots, k$ where k is the total number of segments, with

$$X_{m_i,n} := \max\{X_{l,n} : U_n(X_{m_{i-1},n}) - U_n(X_{l,n}) \geq \log r\},$$

(see Figure 2 for an illustration).

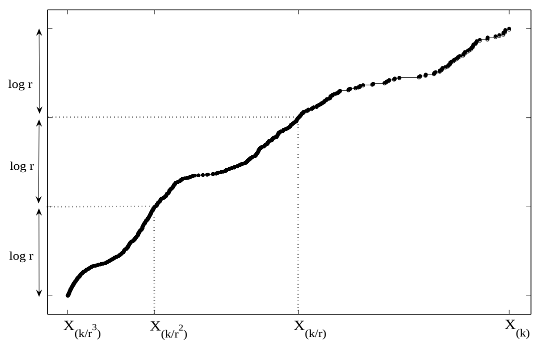

Figure 2: Decomposition of the right tail of the empirical function U_n

In a second step, we obtain an equivalent representation of each segment of U_n (changing scale and location) which will be called *empirical version of* y. This change of location and scale is such that each empirical version of y has domain $[0, 1]$ and image set $[0, \log r]$. Consequently, for each $X_{l,n} \in [X_{m_i,n}, X_{m_{i-1},n}]$, we consider the jump point of the i-th empirical version of y given by

$$t^{(i)}_{l-m_i+1} = \frac{X_{l,n} - X_{m_i,n}}{X_{m_{i-1},n} - X_{m_i,n}} \in [0, 1],$$

whose images are

$$y^{(i)}_{l-m_i+1} = U_n(X_{l,n}) - U_n(X_{m_i,n}) \in [0, \log r].$$

Thus, the i-th empirical version of y is defined by

$$y^{(i)}(x) = \sum_{j=1}^{n_i} y_j^{(i)} \ \mathbb{I}_{[t_j^{(i)}, t_{j+1}^{(i)}[}(x), \ x \in [0,1],$$

where n_i is the number of jump points $t_j^{(i)}$ (see Figure 3).

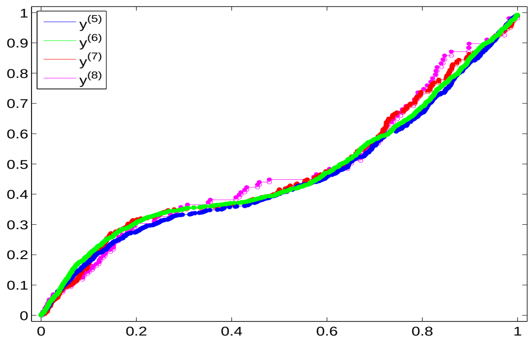

Figure 3: Empirical versions of the function y

The k empirical versions of the function y obtained in the second step allow us to estimate y in the grid $\{t_j^{(i)}, i = 1, 2, \ldots, k, \ j = 1, 2, \ldots, n_i\}$, using the following method:

- for each $t_j^{(i)} \in [0, 1]$, with $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n_i$, we calculate the values $y^{(l)}(t_j^{(i)})$, with $l = 1, \ldots, k$;

- the estimates of y in $t_j^{(i)}$ are equal to the sum, for each l, of the values obtained in the previous step weighted by the percentage of jump points that each empirical versions has (taking into account that the empirical version of the function y will be closer to the function y as the number of jump points increases, so that points from empirical versions close to y have more weight in the construction of the estimated function y). Namely, considering the total number of points $t_j^{(i)}$ given by $n_t = \sum_{l=1}^{k} n_l$, we define

$$\widehat{y}(t_j^{(i)}) = \sum_{l=1}^{k} \frac{n_l}{n_t} \ y^{(l)}(t_j^{(i)}).$$

  Using these points we apply linear interpolation taking into account that the function $\widehat{y}$ must be non decreasing.

3. High quantiles estimation

After the estimation of the parameters r and γ as well as the function y we can characterize completely the limit d.f. G and then proceed to the estimation of high quantiles.

Given q close to zero, let us consider a high quantile of probability $1-q$ of F, that is, a real number x_q that satisfies $F(x_q) = 1-q$.

Assuming that (2) holds, we obtain

$$F(x_q) \cong G^{1/k_n}\left(\frac{x_q - b_n}{a_n}\right),$$

or equivalently

$$(9) \qquad -\log\left(-\log G\left(\frac{x_q - b_n}{a_n}\right)\right) \cong -\log k_n - \log(-\log(1-q)).$$

Taking into account all the possible specifications of the sequences $\{a_n\}$, $\{b_n\}$ and $\{k_n\}$ presented in Theorem 1, we can estimate these sequences from the empirical function of F. In order to do that we will use the order statistics $X_{\eta_1,n}$ and $X_{\eta_2,n}$ that are, respectively, the lower and upper bounds of any of the intervals $[X_{m_i,n}, X_{m_{i-1},n}]$ constructed in the last section.

Thus, using these order statistics we consider

$$\widehat{k}_n = \exp\{U_n(X_{\eta_1,n})\}, \qquad \widehat{b}_n = X_{\eta_1,n}$$

and, in view of $U_n(X_{\eta_2,n}) - U_n(X_{\eta_1,n}) \cong \log \widehat{r}$, from (6) we obtain $\widehat{b}_{n+1} = X_{\eta_2,n}$ and then $\widehat{a}_n = X_{\eta_2,n} - X_{\eta_1,n}$.

Let $\widehat{G}$ be the limit d.f. estimated using the arguments presented before. Since the d.f. $\widehat{G}$ is max-semistable, due to (5) a function $y_1 : [\widehat{b}_n, \widehat{b}_{n+1}] \to [U_n(\widehat{b}_n), U_n(\widehat{b}_{n+1})]$ exists such that

$$-\log(-\log \widehat{G}(\widehat{a}^m x + \widehat{s}_m)) = y_1(x) + m \log \widehat{r}.$$

The function y_1 is related with the estimated function $\widehat{y}$ as it follows

$$y_1(x) = \widehat{y}\left(\frac{x - \widehat{b}_n}{\widehat{b}_{n+1} - \widehat{b}_n}\right) + U_n(\widehat{b}_n),$$

and so we also have

$$(10) \qquad -\log(-\log \widehat{G}(\widehat{a}^m x + \widehat{s}_m)) = \widehat{y}\left(\frac{x - \widehat{b}_n}{\widehat{b}_{n+1} - \widehat{b}_n}\right) + U_n(\widehat{b}_n) + m \log \widehat{r}.$$

On the other side, from (9), an estimate of x_q, denoted by $\widehat{x}_q$, should satisfy

$$-\log\left(-\log\widehat{G}\left(\frac{\widehat{x}_q-\widehat{b}_n}{\widehat{b}_{n+1}-\widehat{b}_n}\right)\right)=-U_n(\widehat{b}_n)-\log(-\log(1-q)). \tag{11}$$

For any $m\in\mathbb{Z}$ take $t_q^{(m)}\in[\widehat{b}_n,\widehat{b}_{n+1}]$ such that the left hand side of (10) and (11) are equal. Thus, we deduce

$$\widehat{y}\left(\frac{t_q^{(m)}-\widehat{b}_n}{\widehat{b}_{n+1}-\widehat{b}_n}\right)=-\log(-\log(1-q))-m\log\widehat{r}-2U_n(\hat{b}_n).$$

Now, choosing a particular value of m, say m_0, such that the right hand side of the previous expression belongs to the image set of $\widehat{y}$ (that is $[0,\log\widehat{r}]$) we obtain a value $\widehat{t}_q=t_q^{(m_0)}$, since the function $\widehat{y}$ is injective. Finally, we get

$$\widehat{x}_q=(\widehat{a}^{m_0}\widehat{t}_q+\widehat{s}_{m_0})(\widehat{b}_{n+1}-\widehat{b}_n)+\widehat{b}_n.$$

Observe that, if we do not reject the hypothesis $\gamma=0$, we can replace $\widehat{a}$ by 1 and $\widehat{s}_n$ by m_0, obtaining

$$\widehat{x}_q=(\widehat{t}_q+m_0)(\widehat{b}_{n+1}-\widehat{b}_n)+\widehat{b}_n.$$

A simulation study was designed to evaluate the procedure here described. In its implementation the same models were selected as in the simulation study concerning the estimation of the parameters r and γ (see [3]), that is, max-semistable models and models that are convenient "perturbations" of Pareto, Burr and Weibull models. More precisely, we have considered d.f.'s F in a max-semistable domain of attraction obtained in the following way: $1-F(x)=(1-F_0(x))\theta(x)$ where F_0 is in a max-stable domain of attraction and θ is a positive, bounded, and periodic function such that F is a d.f. (see [6]). In the specific choice of the d.f.'s were combined several orders of magnitude for the underlying parameters.

For each d.f., 1000 sample replicas were simulated with sample sizes 1000, 2000, 5000 and 10000.

The first aspect to point out respects the suitable choice of the interval $[X_{\eta_1,n},X_{\eta_2,n}]$ to use in the quantile estimation. Clearly we must use high order statistics but, in fact, not too high because we need empirical versions of y that are good representations of the true function. After some preliminary simulation studies we chose $[X_{\eta_1,n},X_{\eta_2,n}]$ as the first interval that contains more than

30 order statistics in the case $\gamma = 0$, and more than 50 order statistics otherwise (recall that we have different estimators according to $\gamma = 0$ or $\gamma \neq 0$). More details on the design and results of the simulation study can be seen in Dias [4]. As an overall conclusion, we can say that the procedure performs as expected (giving better results for lower values of r and moderately high quantiles) and the precision of the estimates are comparable with the ones obtained in analogous studies in max-stable contexts.

4. Application to seismic data

Given the social and economic impact of earthquakes, the study of their extreme observations is very important, especially in areas such as seismic risk, seismic hazard, insurance, etc. In literature there exist several studies about earthquakes where a d.f. is fitted to the earthquakes magnitudes or seismic moments. From the proposed laws the more well known is the Gutenberg-Richter law, that establishes a relation between the magnitude, M_w, and the frequency of earthquakes with magnitude larger than a previously fixed value, in a given geographic area. According to this law, conditionally to the magnitudes being larger than a given value a, the d.f. of the magnitudes is given by

$$F(x) = 1 - \exp((a - x)/b), \quad \text{for } x > a.$$

Although usually the size of an earthquake is defined using the magnitude, we can also use the seismic moment, M_0, which is related with the magnitude through $M_w = 2/3 \log_{10}(M_0) - 10.7$, if the seismic moments are defined in *dyne-cm*. Therefore, the Gutenberg-Richter law can be rewritten using the seismic moments, obtaining the d.f. of the seismic moments defined by

$$F(x) = 1 - (cx)^{-\beta}, \quad \text{para } x > 1/c,$$

where $\beta = 2/3b$ and b is approximately 1.

Although the Gutenberg-Richter law is a good approximation for a large interval of magnitudes that ranges values from 4 ($M_0 = 10^{24}$) to near 7 ($M_0 = 10^{26.5}$), this law does not seem to be a good approximation for the remaining magnitudes, in particular for the case that is more interesting, the one of the larger magnitudes.

In [8] a Generalized Pareto model (GPD) and a threshold model (GPD below a chosen threshold and Pareto above it) were fitted to the seismic moments (larger than 10^{24} *dyce-cm*) of shallow earthquakes (deep ≤ 70 *km*) occurring in subduction zones in the Pacific region (using the seismic moments in the Harvard catalog from 1977 to 2000). The fitting of both models to the data was considered satisfactory and none could be preferred. As happens with the Gutenberg-Richter law, these models presented large deviations in the tail of the distribution, although they adjust well for intermediate values.

The study of this data was continued in [9], where a test statistic was constructed to test deviations from the Gutenberg-Richter law. The authors applied the test statistic to the seismic moments of earthquakes with seismic moments larger than 10^{24} *dyce-cm* in subduction zones. The data showed large deviations from the Pareto distribution not only in the tail, but elsewhere in the distribution. Furthermore, the behaviour of the test statistic was similar to the one presented by a mixture of two Pareto models with log-periodic oscillations. In [9] is also referred a physical explanation to justify the existence of log-periodic oscillations. Due to these considerations, it seems reasonable to assume that a better fitting can be provided by a max-semistable distribution.

We used data from Harvard catalog, registered by Peter Bird and Yan Y. Kagan in ftp://element.ess.ucla.edu/2003107-esupp/index.htm. This data includes the earthquakes in subduction zones for a larger period of time than the one in [8] and [9]. From this data we selected earthquakes that occurred from 1977/01/01 to 2000/05/31 in the zones refereed by [8], with seismic moment larger than 10^{24} *dyce-cm* and deep inferior to 70 *km*, obtaining a sample of 3865 seismic moments.

The adjustment of a max-semistable model to the data was performed, starting with the estimation of the parameter r. Using (7) we obtained the estimate $\widehat{r} = 1.3$, the value of s for which the trajectory of R_s seems to stabilize around 1 (see Figure 4). The estimate of the tail index parameter γ was calculated through (8) and using the estimated value for r, giving $\widehat{\gamma} = 1.5248$.

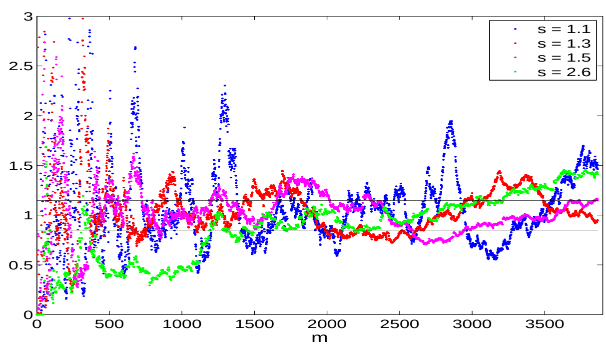

Figure 4: Sample trajectories of $R_s(m)$ for $s = 1.1, 1.3, 1.5$ and 2.6

Finally, we used the method described in Section 2 to estimate the function y. Its graph is depicted in Figure 5. The fact that it does not deviate significantly from a straight line is a sign that not a big improvement will be obtained when comparing with Generalized Pareto modelling.

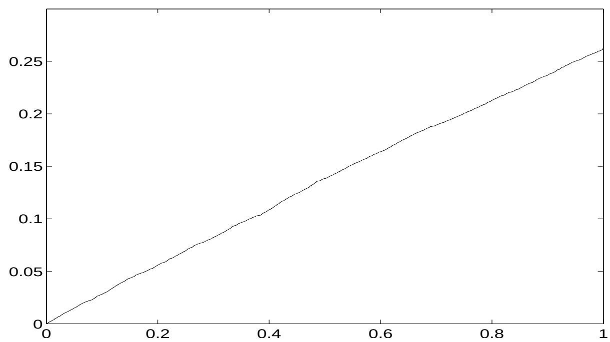

Figure 5: Estimated function y

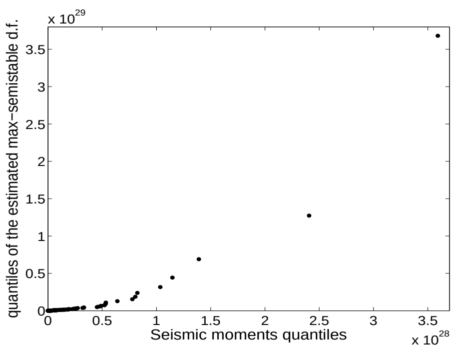

Figure 6: QQ-plot of the seismic moments against the estimated max-semistable $G_{1.52,\nu}$

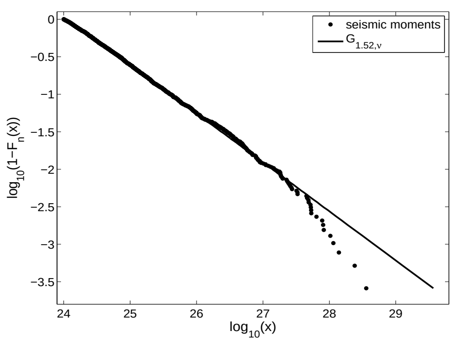

Figure 7: Survival function, in a logarithmic scale, of the seismic moments and of the estimated max-semistable function $G_{1.52,\nu}$

In Figure 6 the QQ-plot relating the observed data with the estimated max-semistable model is presented while Figure 7 confronts the empirical survival function of the seismic moments with the estimated one, in a logarithmic scale. As pointed out before, this fitting has the same kind of misadjustments as the ones observed in other studies. Nevertheless, observing some of the empirical

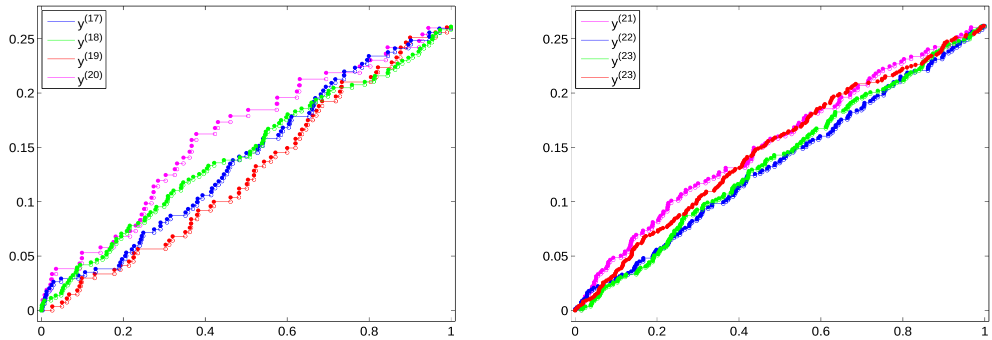

Figure 8: Empirical versions of y

versions of y in Figure 8, we notice that they differ quite a lot from each other, so that a truncated or a mixed model, involving max-semistable distributions, configure as appropriate alternatives.

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