PLISKA
STUDIA MATHEMATICA
BULGARICA

ПЛИСКА
БЪЛГАРСКИ
МАТЕМАТИЧЕСКИ
СТУДИИ

Pliska Stud. Math. Bulgar. **20** (2011), 221–232

PLISKA
STUDIA MATHEMATICA
BULGARICA

PROPERTIES OF THE BELLMAN GAMMA DISTRIBUTION

Evelina Veleva

The Bellman gamma distribution is a matrix variate distribution, which is a generalization of the Wishart distribution. In practice it arises as a distribution of the empirical normal covariance matrix for samples with monotone missing data. The exact distributions of determinants and quotient of determinants of some submatrices of Bellman gamma distributed random matrices are obtained. The method, considered in this paper, gives the possibility to derive the distribution of products and quotient of products of principal minors of a Bellman gamma matrix, and in particular, of a Wishart matrix.

1. Introduction

The Bellman gamma distribution is a matrix variate distribution, which is a generalization of the Wishart and the matrix gamma distributions (see [3]). In practice it arises as a distribution of the empirical normal covariance matrix for samples with monotone missing data (see [6]). Some of the results in this paper are analogous to already known properties of the Wishart distribution.

Theorem 3.3 gives a representation of the elements of a Bellman gamma matrix as algebraic functions of independent random variables. It can be used for generation of Bellman gamma matrices and is applied for establishing properties of Bellman gamma matrices, and in particular, of Wishart matrices. A property of

2000 *Mathematics Subject Classification:* 62H10.

Key words: exact distribution, sample covariance matrix, monotone missing data, Wishart distribution.

a Wishart distribution, analogous to Theorem 3.5 is proved by a similar technique in [5], deriving the exact distribution of the likelihood ratio test for diagonality of a covariance matrix, when the last column of the sample correlation matrix has missing elements.

In this paper, the exact distributions of determinants and quotient of determinants of some submatrices of Bellman gamma distributed random matrices are obtained. The presented technique gives the possibility to derive the distribution of products and quotient of products of principal minors of a Bellman gamma matrix, and in particular, of a Wishart matrix.

Definitions of the Bellman gamma type I and II distributions are given in the next section. Section 2 also contains some notations and preliminary notes. The main results are given in Section 3.

2. Preliminary notes

We denote the four parameter Beta distribution (see [1]) by $Beta(a, b, c, d)$, where c and d represent the minimum and maximum values of the distribution. Let $\zeta \sim Gamma(a, b)$ denote that a random variable ζ has Gamma distribution (see [1]) with shape parameter a and scale parameter b. The next properties of the Beta and Gamma distribution can be easily checked by transforming variables.

Proposition 2.1. *If* $\zeta \sim Beta(a, a, -1, 1)$, *then* $1 - \zeta^2 \sim Beta(a, 1/2,\ 0, 1)$.

Proposition 2.2. *Let* ζ_1 *and* ζ_2 *be independent random variables,* $\zeta_1 \sim Beta(a, b, 0, 1)$, $\zeta_2 \sim Beta(a + b, c, 0, 1)$. *Then the product* $\zeta_1\zeta_2$ *has distribution* $Beta(a, b + c, 0, 1)$.

Proposition 2.3. *Let* ζ_1 *and* ζ_2 *be independent random variables,* ζ_1 *is* $Gamma(a, b)$ *and* $\zeta_2 \sim Beta(a - c, c, 0, 1)$. *Then* $\zeta_1\zeta_2 \sim Gamma(a - c, b)$.

Let A be a real $n \times n$ matrix. Let α and β be nonempty subsets of the set $N_n = \{1, \ldots, n\}$. By $\mathrm{A}[\alpha, \beta]$ we denote the submatrix of A, composed of the rows with numbers from α and the columns with numbers from β. When $\beta \equiv \alpha$, $\mathrm{A}[\alpha, \alpha]$ is denoted simply by $\mathrm{A}[\alpha]$. For the complement of α in N_n is used the notation α^c. Let $i, j \in N_n$ and $i, j \notin \alpha$. Suppose that in the submatrix $\mathrm{A}[\alpha \cup \{i\}, \alpha \cup \{j\}]$ of the matrix $\mathrm{A} = (a_{i,j})$ we replace the element $a_{i,j}$ by 0. We shall denote the obtained matrix by $\mathrm{A}[\alpha \cup \{i\}, \alpha \cup \{j\}]^0$.

The next definitions of Bellman gamma type I and II distributions are given in [3]. By $\Gamma^*_n(a_1, \ldots, a_n)$ is denoted the generalized multivariate gamma function,

$\Gamma_n^*(a_1,\ldots,a_n) = \pi^{n(n-1)/4}\prod_{j=1}^{n}\Gamma\Big(a_j-(j-1)/2\Big),\quad a_j>(j-1)/2, j=1,\ldots,n.$
The trace of a matrix A is denoted by $tr(\mathrm{A})$.

Definition 2.1. *A random positive definite $n\times n$ matrix* $\mathbf{U}$ *follows Bellman gamma type I distribution, denoted by* $\mathbf{U}\sim BG_n^I(a_1,\ldots,a_n;\mathrm{C})$, *if its probability density function is given by*

$$f_{\mathbf{U}}(\mathrm{U}) = \frac{\left(\prod_{i=1}^{n}(\det\mathrm{C}[\{i,\ldots,n\}])^{a_i-a_{i-1}}\right)(\det\mathrm{U})^{a_n-(n+1)/2}}{\Gamma_n^*(a_1,\ldots,a_n)\prod_{i=2}^{n}(\det\mathrm{U}[\{1,\ldots,i-1\}])^{a_i-a_{i-1}}}e^{-tr(\mathrm{CU})}, \tag{2.1}$$

where C *$(n\times n)$ is a positive definite constant matrix, $a_0=0$ and $a_j>(j-1)/2$, $j=1,\ldots,n$, are constants.*

Definition 2.2. *A random positive definite $n\times n$ matrix* $\mathbf{U}$ *follows Bellman gamma type II distribution, denoted by* $\mathbf{U}\sim BG_n^{II}(b_1,\ldots,b_n;\mathrm{B})$, *if its probability density function is given by*

$$f_{\mathbf{U}}(\mathrm{U}) = \frac{\left(\prod_{i=1}^{n}(\det\mathrm{B}[\{1,\ldots,i\}])^{b_{n-i+1}-b_{n-i}}\right)(\det\mathrm{U})^{b_n-(n+1)/2}}{\Gamma_n^*(b_1,\ldots,b_n)\prod_{i=1}^{n-1}(\det\mathrm{U}[\{i+1,\ldots,n\}])^{b_{n-i+1}-b_{n-i}}}e^{-tr(\mathrm{BU})},$$

where B *$(n\times n)$ is a positive definite constant matrix, $b_0=0$ and $b_j>(j-1)/2$, $j=1,\ldots,n$, are constants.*

The next five Propositions are proved in [7]. We denote by I_n the identity matrix of size n. We shall denote by $\tilde{\mathrm{I}}_n$ the square matrix of size n with units on the anti-diagonal and zeros elsewhere.

Proposition 2.4. *Let* $\mathbf{U}\sim BG_n^I(a_1,\ldots,a_n;\mathrm{C})$. *Then the matrix* $\mathbf{V}=\tilde{\mathrm{I}}_n\mathbf{U}\tilde{\mathrm{I}}_n$ *is Bellman gamma type II distributed* $BG_n^{II}(a_1,\ldots,a_n;\mathrm{B})$, $\mathrm{B}=\tilde{\mathrm{I}}_n\mathrm{C}\tilde{\mathrm{I}}_n$.

Proposition 2.5. *Let* $\mathbf{U}\sim BG_n^I(a_1,\ldots,a_n;\mathrm{C})$ *and* L *be an arbitrary lower triangular constant matrix of size n. Then the matrix* $\mathbf{W}=\mathrm{L}\mathbf{U}\mathrm{L}^t$ *has distribution* $BG_n^I(a_1,\ldots,a_n;(\mathrm{L}^t)^{-1}\mathrm{C}\mathrm{L}^{-1})$.

For an arbitrary positive definite matrix U there exist a unique lower triangular matrix V with positive diagonal elements, such that $\mathrm{U} = \mathrm{V}\,\mathrm{V}^{\mathrm{t}}$. The matrix V is called the Cholesky triangle (see [2]).

Proposition 2.6. *Let* $\mathrm{U} = (u_{i,j})$ *be an arbitrary positive definite matrix of size* n. *Then* $\mathrm{U} = \mathrm{V}\,\mathrm{V}^{\mathrm{t}}$, *where* $\mathrm{V} = (v_{i,j})$ *is a lower triangular matrix,*

$$v_{j,i} = \frac{\det \mathrm{U}[\{1,\ldots,i\},\{1,\ldots,i-1,j\}]}{v_{i,i}\det \mathrm{U}[\{1,\ldots,i-1\}]},\ 2 \le i < j \le n,$$

$$v_{1,1} = \sqrt{u_{1,1}},\ \ v_{j,j} = \sqrt{\frac{\det \mathrm{U}[\{1,\ldots,j\}]}{\det \mathrm{U}[\{1,\ldots,j-1\}]}},\ v_{j,1} = \frac{u_{1,j}}{v_{1,1}},\ \ j = 2,\ldots,n.$$

Proposition 2.7. *Let* $\mathrm{C} = (c_{i,j})$ *be an arbitrary positive definite matrix of size* n. *Then* $\mathrm{C} = \mathrm{D}\mathrm{D}^{\mathrm{t}}$, *where* $\mathrm{D} = (d_{i,j})$ *is an upper triangular matrix,*

$$d_{i,j} = \frac{\det \mathrm{C}[\{i,j+1,\ldots,n\},\{j,\ldots,n\}]}{d_{j,j}\det \mathrm{C}[\{j+1,\ldots,n\}]},\ 1 \le i < j \le n-1,$$

$$d_{n,n} = \sqrt{c_{n,n}},\ \ d_{i,i} = \sqrt{\frac{\det \mathrm{C}[\{i,\ldots,n\}]}{\det \mathrm{C}[\{i+1,\ldots,n\}]}},\ d_{i,n} = \frac{c_{i,n}}{d_{n,n}},\ \ i = 1,\ldots,n-1.$$

Proposition 2.8 below is analogous to the Bartlett's decomposition of the Wishart distribution (see [3], [4]).

Proposition 2.8. *Let* $\mathbf{U} \sim BG_n^I(a_1,\ldots,a_n;\mathrm{I}_n)$ *and* $\mathbf{U} = \mathbf{V}\,\mathbf{V}^t$, *where* $\mathbf{V} = (V_{i,j})$ *is a lower triangular random matrix with* $V_{i,i} > 0$. *Then* $V_{i,j}$, $1 \le j \le i \le n$, *are independently distributed,* $V_{i,i}^2 \sim Gamma\Big(a_i - (i-1)/2, 1\Big)$, $i = 1,\ldots,n$, *and* $\sqrt{2}V_{i,j} \sim N(0,1)$, $1 \le j < i \le n$.

3. Main results

From Proposition 2.4 it follows that the properties of a Bellman gamma type I distributed random matrix can be reformulated for Bellman gamma type II matrices.

Using Proposition 2.5, the properties of $BG_n^I(a_1,\ldots,a_n;\mathrm{I}_n)$ distribution can be generalized for $BG_n^I(a_1,\ldots,a_n;\mathrm{C})$, where C is an arbitrary positive definite matrix.

From Definition 2.1 it can be seen that if $\mathbf{U} \sim BG_n^I(a_1, \dots, a_n; \mathrm{C})$ with $a_1 = \dots = a_n = m/2$, then $\mathbf{U}$ has Wishart distribution with m degrees of freedom and covariance matrix $\frac{1}{2}\mathrm{C}^{-1}$, denoted by $W_n\left(m, \frac{1}{2}\mathrm{C}^{-1}\right)$. Using Proposition 2.5 with $\mathrm{L} = \sqrt{2}\mathrm{I}_n$ we obtain that if $\mathbf{U} \sim BG_n^I(a_1, \dots, a_n; \mathrm{C})$, then $2\mathbf{U} \sim BG_n^I\left(a_1, \dots, a_n; \frac{1}{2}\mathrm{C}\right)$. Hence if $\mathbf{U} \sim BG_n^I\left(\frac{m}{2}, \dots, \frac{m}{2}; \mathrm{C}\right)$, then $2\mathbf{U} \sim W_n(m, \mathrm{C}^{-1})$. In particular, if $\mathbf{U} \sim BG_n^I\left(\frac{m}{2}, \dots, \frac{m}{2}; \mathrm{I}_n\right)$, then $2\mathbf{U} \sim W_n(m, \mathrm{I}_n)$.

Theorem 3.1. *Let* $\mathbf{U} \sim BG_n^I(a_1, \dots, a_n; \mathrm{C})$ *and* η_i, $i = 1, \dots, n$ *be the random variables*

$$\eta_i = \frac{\det \mathbf{U}[\{1, \dots, i\}]}{\det \mathbf{U}[\{1, \dots, i-1\}]} \frac{\det \mathrm{C}[\{i, \dots, n\}]}{\det \mathrm{C}[\{i+1, \dots, n\}]}, \ i = 2, \dots, n-1,$$

$$\eta_1 = \det \mathbf{U}[\{1\}] \frac{\det \mathrm{C}}{\det \mathrm{C}[\{2, \dots, n\}]}, \ \eta_n = \frac{\det \mathbf{U}}{\det \mathbf{U}[\{1, \dots, n-1\}]} \det \mathrm{C}[\{n\}].$$

Then η_i, $i = 1, \dots, n$, *are mutually independent and* η_i *is gamma distributed* $Gamma\Big(a_i - (i-1)/2, 1\Big)$, $i = 1, \dots, n$.

Proof. Suppose first that $\mathrm{C} = \mathrm{I}_n$. Let $\mathbf{V} = (V_{i,j})$ be the Cholesky triangle of $\mathbf{U}$. From Proposition 2.6 we have that $\eta_1 = V_{1,1}^2$, $\eta_i = V_{i,i}^2$, $i = 2, \dots, n$. The assertion of the theorem now follows from Proposition 2.8.

Let now C be an arbitrary $n \times n$ positive definite matrix. Let D be the upper triangular matrix, defined by Proposition 2.7. Then $\mathrm{DD}^t = \mathrm{C}$ and according to Proposition 2.5, the matrix $\mathbf{W} = \mathrm{D}^t\mathbf{U}\mathrm{D}$ has distribution $BG_n^I(a_1, \dots, a_n; \mathrm{I}_n)$. Since for $i = 1, \dots, n$

$$\mathbf{W}[\{1, \dots, i\}] = \mathrm{D}^t[\{1, \dots, i\}]\mathbf{U}[\{1, \dots, i\}]\mathrm{D}[\{1, \dots, i\}],$$

it can be seen that

$$\eta_1 = \det \mathbf{W}[\{1\}], \eta_i = \frac{\det \mathbf{W}[\{1, \dots, i\}]}{\det \mathbf{W}[\{1, \dots, i-1\}]}, i = 2, \dots, n.$$

Hence, by the first part of the proof, the Theorem follows. □

Corollary 3.1. *Let* $\mathbf{U} \sim BG_n^I(a_1, \dots, a_n; \mathrm{C})$. *Then the random variable* $\det \mathbf{U} \det \mathrm{C}$ *is distributed as the product* $\eta_1 \dots \eta_n$, *where* $\eta_1, \dots, \eta_n$ *are mutually independent random variables,* $\eta_i \sim Gamma\Big(a_i - (i-1)/2, 1\Big)$, $i = 1, \dots, n$.

P r o o f. Let η_i, $i = 1, \dots, n$, be defined as in Theorem 3.1. Since $\eta_1 \dots \eta_n = \det \mathbf{U} \det \mathrm{C}$, the corollary follows from Theorem 3.1. $\square$

Theorem 3.2. *Let $\mathbf{U} \sim BG_n^I(a_1, \dots, a_n; \mathrm{I}_n)$ and i be an integer, $1 < i < n$. Then for all integers j, $i < j \leq n$ the random variable*

$$\det \mathbf{U}[\{1, \dots, i\}, \{1, \dots, i-1, j\}] \tag{3.1}$$

is distributed as the product $\nu\, \eta_1 \dots \eta_{i-1} \sqrt{\eta_i}$, where ν, $\eta_1, \dots, \eta_i$ are mutually independent, $\sqrt{2}\nu \sim N(0,1)$, $\eta_k \sim Gamma\Big(a_k - (k-1)/2, 1\Big)$, $k = 1, \dots, i$.

P r o o f. Let $\mathbf{V} = (V_{i,j})$ be the Cholesky triangle of $\mathbf{U}$. Let us consider the random variables $\nu = V_{j,i}$, $\eta_1 = V_{1,1}^2$, $\eta_k = V_{k,k}^2$, $k = 2, \dots, i$. According to Proposition 2.6, for $i < j \leq n$ the random variable (3.1) is equal to $\nu\, \eta_1 \dots \eta_{i-1} \sqrt{\eta_i}$. Now, using Proposition 2.8 we complete the proof. $\square$

Corollary 3.2. *Let $\mathbf{U} \sim BG_n^I(a_1, \dots, a_n; \mathrm{I}_n)$ and i be an integer, $1 < i < n$. Then for all integers j, $i < j \leq n$*

$$\frac{\det \mathbf{U}[\{1, \dots, i\}, \{1, \dots, i-1, j\}]}{\det \mathbf{U}[\{1, \dots, i-1\}]} \sim \nu \sqrt{\eta}, \tag{3.2}$$

$$\frac{\det \mathbf{U}[\{1, \dots, i\}, \{1, \dots, i-1, j\}]}{\det \mathbf{U}[\{1, \dots, i\}]} \sim \frac{\nu}{\sqrt{\eta}}, \tag{3.3}$$

where ν and η are independent, $\sqrt{2}\nu \sim N(0,1)$ and $\eta \sim Gamma(a_i - (i-1)/2, 1)$.

P r o o f. Let $\eta_1, \dots, \eta_i$ and ν be defined as in the proof of Theorem 3.2. Using Proposition 2.6, the left hand side of (3.2) is equal to $\nu \sqrt{\eta_i}$; the left hand side of (3.3) equals to $(\nu \sqrt{\eta_i})/\eta_i = \nu/\sqrt{\eta_i}$. The corollary now follows from Proposition 2.8. $\square$

Let $P(n, \Re)$ be the set of all real, symmetric, positive definite matrices of order n. Let us denote by $D(n, \Re)$ the set of all real, symmetric matrices of order n, with positive diagonal elements, whose off-diagonal elements are in the interval (-1,1). There exist a bijection (one-to-one correspondence) $\tilde{h} : \; D(n, \Re) \to P(n, \Re)$, constructed in [5]. The image of an arbitrary matrix $\mathrm{X} = (x_{i,j})$ from $D(n, \Re)$ by the bijection $\tilde{h}$ is a matrix $\mathrm{Y} = (y_{i,j})$ from $P(n, \Re)$, defined first on the main diagonal and then consecutively on the diagonals parallel to the main diagonal, by the recurrence formulas

$$y_{i,i} = x_{i,i}, \quad i = 1, \dots, n, \tag{3.4}$$

$$y_{i,i+1} = x_{i,i+1}\sqrt{y_{i,i}y_{i+1,i+1}}, \quad i = 1, \dots, n-1, \tag{3.5}$$

$$y_{i,j} = \frac{E}{\det \mathrm{Y}[\{i+1, \dots, j-1\}]}, \tag{3.6}$$

$$E = (-)^{j-i} \det \mathrm{Y}[\{i, \dots, j-1\}, \{i+1, \dots, j\}]^0 \\ + x_{i,j}\sqrt{\det \mathrm{Y}[\{i, \dots, j-1\}] \det \mathrm{Y}[\{i+1, \dots, j\}]},$$

$$j - i = 2, \dots, n-1.$$

The preimage $\mathrm{X} = \tilde{h}^{-1}(\mathrm{Y})$ of a matrix Y from $P(n, \Re)$ is defined by the equalities (see [5])

$$x_{i,i} = y_{i,i}, \quad i = 1, \dots, n,$$

$$x_{i,i+1} = \frac{y_{i,i+1}}{\sqrt{y_{i,i}y_{i+1,i+1}}}, \quad i = 1, \dots, n-1,$$

$$x_{i,j} = \frac{(-1)^{j-i-1} \det \mathrm{Y}[\{i, \dots, j-1\}, \{i+1, \dots, j\}]}{\sqrt{\det \mathrm{Y}[\{i, \dots, j-1\}] \det \mathrm{Y}[\{i+1, \dots, j\}]}}, \quad 2 \leq j - i \leq n-1.$$

For an arbitrary real square matrix A of order n and integers i, j, $1 \leq i < j \leq n$, the following identity holds

$$\det \mathrm{A} \;\det \mathrm{A}[\{i,j\}^c] = \det \mathrm{A}[\{i\}^c] \det \mathrm{A}[\{j\}^c] \\ - \det \mathrm{A}[\{i\}^c, \{j\}^c] \det \mathrm{A}[\{j\}^c, \{i\}^c]. \tag{3.7}$$

It is a special case of the identity (1) in [8]. Using (3.7), it is shown in [5] that

$$1 - x_{i,j}^2 = \frac{\det \mathrm{Y}[\{i, \dots, j\}] \det \mathrm{Y}[\{i+1, \dots, j-1\}]}{\det \mathrm{Y}[\{i, \dots, j-1\}] \det \mathrm{Y}[\{i+1, \dots, j\}]}, \; 2 \leq j - i \leq n-1, \tag{3.8}$$

$$1 - x_{i,i+1}^2 = \frac{\det \mathrm{Y}[\{i, i+1\}]}{y_{i,i} y_{i+1,i+1}}, \ i = 1, \dots, n-1, \tag{3.9}$$

$$\det \mathrm{Y}[\{i, \dots, j\}] = x_{i,i} \dots x_{j,j} \left(\prod_{i \le s < t \le j} (1 - x_{s,t}^2) \right), \ 1 \le i < j \le n. \tag{3.10}$$

The Jacobian of the transformation from $(x_{i,j})$ to $(y_{i,j})$ is

$$\mathrm{J} = \frac{\partial(x_{1,1}, \dots, x_{n,n}, x_{1,2}, \dots, x_{n-1,n}, x_{1,3}, \dots, x_{n-2,n}, \dots, x_{1,n})}{\partial(y_{1,1}, \dots, y_{n,n}, y_{1,2}, \dots, y_{n-1,n}, y_{1,3}, \dots, y_{n-2,n}, \dots, y_{1,n})}.$$

¿From (3.6) it can be seen that $x_{i,j}$ depends only on $y_{k,s}$, $i \le k \le s \le j$. Consequently, all the elements above the main diagonal in J are zero. Therefore $\det \mathrm{J}$ is equal to the product of the diagonal elements, the first n of which are ones. ¿From (3.5) and (3.6) we find the rest of them

$$\frac{\partial x_{i,i+1}}{\partial y_{i,i+1}} = \frac{1}{\sqrt{y_{i,i} y_{i+1,i+1}}}, \quad i = 1, \dots, n-1,$$

$$\frac{\partial x_{i,j}}{\partial y_{i,j}} = \frac{\det \mathrm{Y}[\{i+1, \dots, j-1\}]}{\sqrt{\det \mathrm{Y}[\{i, \dots, j-1\}] \det \mathrm{Y}[\{i+1, \dots, j\}]}}, \ 2 \le j - i \le n-1.$$

After simplifications we obtain

$$\det \mathrm{J} = \left[\sqrt{y_{1,1} y_{n,n}} \left(\prod_{k=2}^{n-1} \sqrt{\det \mathrm{Y}[\{1, \dots, k\}] \det \mathrm{Y}[\{k, \dots, n\}]} \right) \right]^{-1}. \tag{3.11}$$

Theorem 3.3. *Let $a_1, \dots, a_n$ be real numbers, such that $a_i > (i-1)/2$, $i = 1, \dots, n$. Let $\xi = (\xi_{i,j})$ be a symmetric $n \times n$ random matrix. Suppose that $\xi_{i,j}$, $1 \le i \le j \le n$, are mutually independent, $\xi_{i,j} \sim Beta\Big(a_j - (j-i)/2, a_j - (j-i)/2, -1, 1\Big)$, $1 \le i < j \le n$, and $\xi_{i,i} \sim Gamma(a_i, 1)$, $i = 1, \dots, n$. Then the matrix $\mathbf{U} = \tilde{h}(\xi)$, where $\tilde{h}$ is the bijection defined by (3.4) - (3.6), has Bellman gamma type I distribution $BG_n^I(a_1, \dots, a_n; \mathrm{I}_n)$.*

Proof. The joint density function of $\xi_{i,j}$, $1 \le i \le j \le n$ has the form

$$f(x_{i,j}, 1 \le i \le j \le n) = K \left(\prod_{i=1}^{n} x_{i,i}^{a_i - 1} e^{-x_{i,i}} \right) \left(\prod_{1 \le i < j \le n} (1 - x_{i,j}^2)^{a_j - \frac{(j-i)}{2} - 1} \right),$$

$$K = \left(\prod_{i=1}^{n} \frac{1}{\Gamma(a_i)} \right) \left(\prod_{1 \le i < j \le n} \frac{\Gamma(2a_j - j + i)}{\left[\Gamma\left(a_j - (j-i)/2\right)\right]^2 2^{2a_j - j + i - 1}} \right),$$

$x_{i,i} > 0$, $i = 1, \ldots, n$, $x_{i,j} \in (-1, 1)$, $1 \le i < j \le n$. Using the duplication formula for the gamma function (see [4], p.154) $\Gamma(2x) = \pi^{-1/2} 2^{2x-1} \Gamma(x) \Gamma(x + 1/2)$, after simplification we get $K = 1/\Gamma_n^*(a_1, \ldots, a_n)$, where Γ_n^* denotes the generalized multivariate gamma function introduced on p. 3. The new variables are the elements $U_{i,j}$, $1 \le i \le j \le n$ of the matrix $\mathbf{U}$. Using (3.4), (3.8), (3.9) and (3.11) we obtain that the joint density of $U_{i,j}$, $1 \le i \le j \le n$ is equal to the right hand side of (2.1) with $\mathrm{C} = \mathrm{I}_n$. $\square$

Corollary 3.3. *Let* $\mathbf{U} \sim BG_n^I(a_1, \ldots, a_n; \mathrm{I}_n)$ *and let* p *and* q *be integers,* $1 \le p \le q \le n$. *Then the matrix* $\mathbf{U}[\{p, \ldots, q\}]$ *has* $BG_{q-p+1}^I(a_p, \ldots, a_q;\, \mathrm{I}_{q-p+1})$ *distribution.*

Proof. By Theorem 3.3, $\mathbf{U}$ can be considered as an image $\mathbf{U} = \tilde{h}(\xi)$. From formulas (3.4) – (3.6) it can be seen that if $\mathrm{Y} = \tilde{h}(\mathrm{X})$ and p, q are integers, $1 \le p \le q \le n$, then

$$\mathrm{Y}[\{p, \ldots, q\}] = \tilde{h}(\mathrm{X}[\{p, \ldots, q\}]). \tag{3.12}$$

Applying again Theorem 3.3 we complete the proof. $\square$

Corollary 3.4. *Let* $\mathbf{U} \sim BG_n^I(a_1, \ldots, a_n; \mathrm{I}_n)$ *and let* p *be an integer,* $1 \le p \le n$. *Then the random matrices* $\mathbf{U}[\{1, \ldots, p\}]$ *and* $\mathbf{U}[\{p+1, \ldots, n\}]$ *are independent.*

Proof. Using (3.12) we have $\mathbf{U}[\{1, \ldots, p\}] = \tilde{h}(\xi[\{1, \ldots, p\}])$, $\mathbf{U}[\{p + 1, \ldots, n\}] = \tilde{h}(\xi[\{p+1, \ldots, n\}])$. The corollary now follows from the independence of $\xi_{i,j}$, $1 \le i \le j \le n$. $\square$

Theorem 3.4. *Let* $\mathbf{U} \sim BG_n^I(a_1, \ldots, a_n; \mathrm{I}_n)$ *and* $\mathbf{U}$ *be partitioned with sub-matrices* $\mathbf{U}_{i,j}$, $i, j = 1, \ldots, k$, *where* $\mathbf{U}_{i,i}$ *are square matrices of size* n_i, $i = 1, \ldots, k$. *Then*

$$\frac{\det \mathbf{U}}{\det \mathbf{U}_{1,1} \ldots \det \mathbf{U}_{k,k}} \sim \beta_{n_1+1} \ldots \beta_n, \tag{3.13}$$

where β_j, $j = n_1 + 1, \ldots, n$, *are mutually independent,* $\beta_j \sim Beta\Big(a_j - (j-1)/2, (n_1 + \cdots n_{r_j})/2, 0, 1\Big)$; r_j *is the greatest integer such that* $n_1 + \cdots + n_{r_j} < j$, $j = n_1 + 1, \ldots, n$.

P r o o f. The matrix $\mathbf{U}$ can be considered as an image $\mathbf{U} = \tilde{h}(\xi)$, where ξ is the random matrix given in Theorem 3.3. Applying (3.10) to $\det \mathbf{U}$ and $\det \mathbf{U}_{i,i}$, $i = 1, \ldots, k$, we obtain that the left hand side of (3.13) equals

$$\prod_{j=n_1+1}^{n} \prod_{s=1}^{n_1+\cdots+n_{r_j}} (1 - \xi_{s,j}^2).$$

Let us substitute $\beta_j = \prod\limits_{s=1}^{n_1+\cdots+n_{r_j}} (1 - \xi_{s,j}^2)$, $j = n_1 + 1, \ldots, n$. Since $\xi_{s,j}$, $1 \leq s \leq j \leq n$ are mutually independent, $\beta_{n_1+1}, \ldots, \beta_n$ are also independent. Using Propositions 2.1 and 2.2, we obtain that for $1 \leq u \leq v < j \leq n$

$$(1 - \xi_{u,j}^2) \ldots (1 - \xi_{v,j}^2) \sim Beta\Big(a_j - (j - u)/2, (v - u + 1)/2, 0, 1\Big). \tag{3.14}$$

Using (3.14) we find the distribution of β_j and complete the proof. □

Theorem 3.5. *Let $\mathbf{U} \sim BG_n^I(a_1, \ldots, a_n; \mathrm{I}_n)$ and p, q be integers, $1 < p < q < n$. Then*

$$\frac{\det \mathbf{U}[\{1, \ldots, q\}] \det \mathbf{U}[\{p, \ldots, n\}]}{\det \mathbf{U}[\{p, \ldots, q\}]} \sim \eta_1 \ldots \eta_n, \tag{3.15}$$

where η_i, $i = 1, \ldots, n$, are mutually independent and $\eta_i \sim Gamma(a_i - (i - 1)/2, 1)$, $i = 1, \ldots, q$, $\eta_i \sim Gamma\Big(a_i - (i - p)/2, 1\Big)$, $i = q + 1, \ldots, n$.

P r o o f. Applying (3.10) to $\det \mathbf{U}[\{1, \ldots, q\}]$, $\det \mathbf{U}[\{p, \ldots, n\}]$ and $\det \mathbf{U}[\{p, \ldots, q\}]$ we obtain that the left hand side of (3.15) equals to

$$\xi_{1,1} \ldots \xi_{n,n} \left(\prod_{t=2}^{q} \prod_{s=1}^{t-1} (1 - \xi_{s,t}^2)\right) \left(\prod_{t=q+1}^{n} \prod_{s=p}^{t-1} (1 - \xi_{s,t}^2)\right).$$

Let us substitute $\beta_t = \prod\limits_{s=1}^{t-1} (1 - \xi_{s,t}^2)$, $t = 2, \ldots, q$, $\beta_t = \prod\limits_{s=p}^{t-1} (1 - \xi_{s,t}^2)$, $t = q + 1, \ldots, n$. From (3.14) we have that $\beta_t \sim Beta\Big(a_t - (t - 1)/2, (t - 1)/2, 0, 1\Big)$, $t = 2, \ldots, q$, $\beta_t \sim Beta\Big(a_t - (t - p)/2, (t - p)/2, 0, 1\Big)$, $t = q + 1, \ldots, n$. Let $\eta_1 = \xi_{1,1}$, $\eta_i = \xi_{i,i}\beta_i$, $i = 2, \ldots, n$. Then the required assertion follows from Proposition 2.8. □

Theorem 3.6. *Let* $\mathbf{U} \sim BG_n^I(a_1, \ldots, a_n; \mathrm{I}_n)$ *and* p, q *be integers,* $1 < p < q < n$. *Then*

$$\frac{\det \mathbf{U} \det \mathbf{U}[\{p, \ldots, q\}]}{\det \mathbf{U}[\{1, \ldots, q\}] \det \mathbf{U}[\{p, \ldots, n\}]} \sim \beta_{q+1} \ldots \beta_n, \tag{3.16}$$

where β_{q+1}, …, β_n *are mutually independent and* $\beta_i \sim Beta\Big(a_i - (i-1)/2, (p-1)/2, 0, 1\Big)$, $i = q+1, \ldots, n$.

Proof. Using (3.10) we obtain that the left hand side of (3.16) equals $\prod_{t=q+1}^{n} \prod_{s=1}^{p-1} (1 - \xi_{s,t}^2)$. Let us substitute $\beta_t = \prod_{s=1}^{p-1} (1 - \xi_{s,t}^2)$, $t = q+1, \ldots, n$. Since $\xi_{s,t}$, $1 \leq s \leq t \leq n$ are mutually independent, β_{q+1}, …, β_n are also independent. Finally, applying (3.14) we complete the proof. □

The approach, used in the proofs of Theorems 3.4 - 3.6, can be also applied to derive the distribution of products and quotient of products of principal minors of the form $\mathbf{U}[\{i, \ldots, j\}]$ of a Bellman gamma matrix $\mathbf{U}$.

REFERENCES

[1] M. Evans, N. Hastings, B. Peacock. Statistical Distributions. John Wiley & Sons, Inc., N.Y., 3ed., 2000.

[2] G. H. Golub, Ch. F. Van Loan. Matrix Computations, Johns Hopkins University Press, 3ed., 1996.

[3] A. K. Gupta, D. K. Nagar. Matrix Variate Distributions. Chapman & Hall/CRC, 2000.

[4] R. J. Muirhead. Aspects of Multivariate Statistical Theory. John Wiley & Sons, N. Y., 2ed., 2005.

[5] E. Veleva. The exact distribution of the UMP test for diagonality of covariance matrices with missing elements, *J. Tech. Univ. Plovdiv Fundam. Sci. Appl. Ser. A Pure Appl. Math.*, **14**, 2009, 449–454.

[6] E. Veleva. Testing a normal covariance matrix for small samples with monotone missing data, *Applied Mathematical Sciences*, **3** (2009), No 54, 2671–2679.

[7] E. Veleva. Stochastic representations of the Bellman gamma distribution, Proc. of ICTAMI 2009, Alba Iulia, Romania, 2009, 463–474.

[8] E. W. Weisstein. Sylvester's Determinant Identity. From MathWorld – A Wolfram Web Resource (http://mathworld.wolfram.com/SylvestersDeterminantIdentity.html).

Evelina Veleva
Department of Numerical Methods and Statistics
University of Rousse "Angel Kanchev"
8, Studentska str.
7017 Rouse, Bulgaria
e-mail: eveleva@uni-ruse.bg