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NONLINEAR NORMALIZATION IN LIMIT THEOREMS FOR EXTREMES

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It is well known that under linear normalization the maxima of iid random variables converges in distribution to one of the three types of max-stable laws: Frechet, Gumbel and Weibull. During the last two decades the first author and her collaborators worked out a limit theory for extremes and extremal processes under non-linear but monotone normalizing mappings. In this model there is only one type of max-stable distributions and all continuous and strictly increasing df's belong to it. In a recent paper on General max-stable laws, Sreehari points out two "confusing" results in Pancheva (1984). They concern the explicit form of a max-stable df with respect to a continuous one-parameter group of max-automorphisms, and domain of attraction conditions. In the present paper the first claim is answered by a detailed explanation of the explicit form, while for the second we give a revised proof. The rate of convergence is also discussed.

1. Introduction

Any limit theorem for convergence of normalized maxima of iid random variables to a max-stable law G separates a subclass of distribution functions (d.f.'s) $MDA(G)$ called max-domain of attraction of G. Thus, if we use a wider class of normalizing mappings than the linear ones, we get a wider class of limit laws which can be used in solving approximation problems. Another reason for using

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nonlinear normalization concerns the problem of refining the accuracy of the approximation in the limit theorems: by using monotone mappings in certain cases we can achieve a better rate of convergence. In the last two decades E. Pancheva and her collaborators investigated various limit theorems for extremes and extremal processes using as normalizing mappings the so-called max-automorphisms (see e.g. Pancheva (2010)): continuous and strictly increasing in each coordinate. The max-automorphisms $L : \mathbb{R}^d \to \mathbb{R}^d,\ d \geq 1$ preserve the max-operation, i.e. $L(X \vee Y) = L(X) \vee L(Y)$, there exist inverse mappings L^{-1} and they form a group w.r.t. the composition. We denote this group by GMA.

Choosing mappings from GMA for normalization in the limit theorems, we are forced to change the notion of type (F) for a nondegenerate d.f. F: We say that a d.f. G belongs to *type*(F) if there exists $T \in GMA$ such that $G = F \circ T$. Now the three extreme value distributions (Fréchet's Φ_α, Gumbel's Λ, and Weibull's Ψ_α) belong to the same type, the max-stable type $(\mathcal{MS})$.

Let us recall the major notions of the univariate max-model: A nondegenerate d.f. G is called max-stable if there exists a continuous one-parameter group (c.o.g.) $\mathcal{L} = \{\mathbb{L}_t : t > 0\}$ in GMA such that for all $t > 0$

$$G^t(x) = G(\mathbb{L}_t(x)), \quad x \in \mathbb{R}. \tag{1}$$

A d.f. G satisfying (1) has to be continuous and strictly increasing. Conversely, given G, let us consider (1) as a functional equation for the unknown $\mathcal{L}$. By solving it one obtains that there exists a continuous and strictly increasing mapping

$$h : Support(G) \leftrightarrow (-\infty, \infty)$$

such that

$$\mathbb{L}_t(x) = h^{-1}(h(x) - \log t), \quad t > 0. \tag{2}$$

Substituting (2) in (1) and solving w.r.t. $G(.)$, we get

$$G(x) = \exp\{-e^{-h(x)}\}, \tag{3}$$

and this is the general explicit form of any max-stable d.f. We denote for any d.f. G,

$$l_G = \inf\{x : G(x) > 0\}, \quad r_G = \sup\{x : G(x) < 1\}.$$

Remark. Note that representation (3) can be expressed also in the form

$$G(x) = \exp\{-e^{-c_1 h_1(x)}\}, \quad c_1 > 0 \tag{3'}$$

or in the form

$$G(x) = \exp\{-c_2 e^{-h_2(x)}\}, \quad c_2 > 0 \tag{3”}$$

for parametrization of the class $\mathcal{MS}$. Under this parametrization the c.o.g. $\mathcal{L}$ remains the same since $h(x) = c_1 h_1(x) = h_2(x) - \log c_2$. Then

$$\mathbb{L}_t(x) = h^{-1}(h(x) - \log t) = h_1^{-1}(h_1(x) - \frac{1}{c_1}\log t) = h_2^{-1}(h_2(x) - \log t).$$

In this connection, the claim in Sreehari (2009), Remark 1, p.191, is unfounded.

The convergence to type theorem (CTT) is the main tool for proving limit theorems for cumulative extremes. A convergence to type takes place if both convergences $F_n \xrightarrow{w} F$ and $F_n \circ T_n \xrightarrow{w} G$, with $T_n \in GMA$, imply $G \in type(F)$, i.e. there exists a $T \in GMA$ such that $G = F \circ T$. Using here max-automorphisms we are confronted with similar difficulties as if we were working in a space with infinite dimension. Let $f : [0,1] \to [0,\infty)$ be continuous and vanish in zero, and let $\mathcal{R}_f$ be the set of all sequences $\{T_n\} \subset GMA$ satisfying the conditions

a) $T_n(x) \geq x$,

b) $h \leq T_n(x+h) - T_n(x) \leq f(h) \to 0, \quad h \to 0.$

Denote $\mathcal{R} = \bigcup_f \mathcal{R}_f$. The sequences $\{T_n\}$ from $\mathcal{R}$ are equicontinuous and bounded from below. If in addition there exists a limit mapping T, then the right-hand side of b) gives the continuity of T and the left-hand side of b) supplies its strong monotony, i.e. $T \in GMA$.

Now, the CTT in our model claims: *The compactness (w.r.t. the pointwise convergence) of the normalizing sequence $\{T_n\} \subset \mathcal{R}$ is necessary and sufficient for a convergence to type (cf. Pancheva (1993)).* Unfortunately, this new formulation of CTT makes it difficult for application. This is the reason for **restricting** our investigation to **regular** normalizing sequences only. In this way we lose in generality but win in clarity.

Definition 1. *We refer to a sequence $\{L_n\} \subset GMA$ as regular on a set $S \times T$ if for every $x \in S$ and $t \in T$ there exists a limiting max-automorphism*

$$\mathbb{L}_t(x) = \lim_{n\to\infty} L^{-1}_{[nt]} \circ L_n(x) \tag{4}$$

uniformly on compact subsets of T and the mapping $t \to \mathbb{L}_t$ is one-to-one.

The main advantage of the restriction to regular normalizing sequences is that instead of using CTT we use the continuity of the composition.

Let $X_1, X_2, \ldots$ be iid r.v.s with d.f. F. Let G be a nondegenerate d.f. Suppose that there exists a regular normalizing sequence $\{L_n\}$ on $(l_G, r_G) \times (0, \infty)$ such that

$$F^n(L_n(x)) \xrightarrow{w} G(x). \tag{5}$$

Using the regularity of the sequence we see immediately that the limit law G satisfies functional equation (1), hence G is max-stable w.r.t. the c.o.g. $\mathcal{L} = \{\mathbb{L}_t, t > 0\}$ determined by (4). If (5) is met we say that F belongs to the max-domain of attraction of G w.r.t. $\mathcal{L}$, briefly $F \in MDA(G)$.

We underline that the normalizing sequence has to be regular on (l_G, r_G) with the following example.

Example 1. Let $F(x) = \Lambda(x)$ and $L_n(x) = \begin{cases} e^n \left(x - \dfrac{1}{n}\right) & \text{if} \quad x \le \dfrac{1}{n} \\ \log nx & \text{if} \quad x > \dfrac{1}{n}. \end{cases}$

Then (5) is met with $G(x) = \Phi_1(x)$ and

$$\lim_{n \to \infty} L^{-1}_{[nt]} \circ L_n(x) = \mathbb{L}_t(x)$$

where

$$\mathbb{L}_t(x) = \begin{cases} \begin{cases} \dfrac{x}{t}, & x > 0, \\ -\infty, & x \le 0 \end{cases} & \text{for} \quad t \in (0,1) \\ \begin{cases} \dfrac{x}{t}, & x > 0, \\ 0, & x \le 0 \end{cases} & \text{for} \quad t \ge 1. \end{cases}$$

Hence $\{L_n\}$ is regular on $(0, \infty) \times (0, \infty)$, Φ_1 is max-stable with $h(x) = \log x$, $\mathbb{L}_t(x) = \exp(\log x - \log t)$.

By the use of regular normalizing sequences one preserves the well-known classical structures of limit theory: the class of the limit df's in (5) coincides with the class of all df's satisfying functional equation (1), coincides with the class of all df's having the explicit form (3) (i.e. strictly increasing and continuous df's).

Theorem 5 in Pancheva (1984) says: *A nondegenerate d.f. F belongs to $MDA(G)$ iff*

$$1 - F(x) = [1 + o(1)]R(h(x))e^{-h(x)}, \quad x \to r_F,$$

where $R(x)$ is a regularly varying function at infinity. The normalizing mappings can be chosen as

$$L_n(x) = h^{-1}\{h(x) + \log[nL(\log n)]\}.$$

Sreehari (2009) pointed out that the necessary part of the above statement is wrong and proposed the following theorem: *If a nondegenerate d.f. $F \in MDA(G)$ then there exists a sequence of positive functions $\{L^*(x;n)\}$ such that*

$$\frac{K\{h(x) + \log(nL^*(x;n))\}}{L^*(x;n)} \to 1, \text{ as } n \to \infty, \text{ for } x \in (l_F, r_F), \tag{6}$$

where $K(x) = [1 - F \circ h^{-1}(x)]e^x$. Conversely, if for some strictly increasing continuous function $h(x)$ and a sequence of positive functions $\{L^*(x;n)\}$ equation (6) holds then $F \in MDA(G)$, $G(x) = e^{-e^{-h(x)}}$. In this case $L_n(x)$ can be chosen as

$$L_n(x) = h^{-1}\{h(x) + \log[nL^*(x;n)]\}. \tag{7}$$

We are thankful to Sreehari M. for discovering the annoying mistake. Yet, in the framework of our max-model the suggested normalization (7) cannot be adopted: the variables x and n in L^* are not separated and in general one cannot check if (7) defines (or does not define) a regular normalizing sequence. The aim of the present paper is to give a revised answer to the domain of attraction problem if using regular norming sequences. We start with several illustrative examples, then in Section 3 we state and prove our main results.

2. Examples

Example 2. Let $X_1, X_2, \dots$ be i.i.d. r.v. with c.d.f. $F(x) = 1 - x^{-x}$, $x \geq 1$. Denote $M_n = \max\{X_1, X_2, \dots, X_n\}$. We want to find a normalizing sequence $L_n(x)$ such that

$$\mathbf{P}\{M_n \leq L_n(x)\} = \mathbf{P}\{L_n^{-1}(M_n) \leq x\} \to \text{ proper limit distribution.}$$

It is natural to assume that the function $U(x) := \dfrac{1}{1-F(x)} = x^x, \quad x \geq 1$ will play an important role. Let us check some properties of $U(x)$. We have that $U'(x) = x^x(1+\ln x) > 0$ for every $x \geq 1$. So, $U(x)$ is strictly increasing and continuous on the interval $[1,\infty)$, $U(1) = 1$, and $U(x) \uparrow \infty$ as $x \to \infty$, and $U : [1,\infty) \to [1,\infty)$.

Therefore, there exists the inverse function $U^{-1}(x)$, that is

$$U(U^{-1}(x)) = U^{-1}(x)^{U^{-1}(x)} = x \text{ and } U^{-1}(U(x)) = x$$

for every $x \geq 1$. The function $U^{-1}(x)$ is also strictly increasing on $[1,\infty)$, $U^{-1}(1) = 1$, $U^{-1}(x) \to \infty$ as $x \to \infty$, and $U^{-1} : [1,\infty) \to [1,\infty)$.

2.1. Fréchet limit distribution. Let us denote $L_n(x) = U^{-1}(nx)$ and then $L_n^{-1}(x) = \dfrac{U(x)}{n}$, for every $x > 0$ and $n = 1, 2, \ldots$

We prove that as $n \to \infty$,

$$\mathbf{P}\left\{L_n^{-1}(M_n) \leq x\right\} = \mathbf{P}\left\{\frac{U(M_n)}{n} \leq x\right\} \quad \to \quad \exp(-1/x), \qquad x > 0.$$

Indeed

$$\begin{aligned}\mathbf{P}\left\{\frac{U(M_n)}{n} \leq x\right\} &= \mathbf{P}\left\{M_n \leq U^{-1}(nx)\right\} = \left(\mathbf{P}\left\{X_1 \leq U^{-1}(nx)\right\}\right)^n \\ &= \left(1 - U^{-1}(nx)^{-U^{-1}(nx)}\right)^n = \left(1 - \frac{1}{U^{-1}(nx)^{U^{-1}(nx)}}\right)^n \\ &= \left(1 - \frac{1}{nx}\right)^n \to \exp(-1/x) = \Phi_1(x), \quad n \to \infty.\end{aligned}$$

The sequence $L_n(x) = U^{-1}(nx)$ is regular. For this one has only to check that for $t > 0$, $L_{[nt]}^{-1} \circ L_n(x) \to \mathbb{L}_t(x) = x/t$. Recall that $\Phi_1(x)$ is max-stable d.f. w.r.t. the c.o.g. $\{\mathbb{L}_t(x) = \frac{x}{t},\ t \geq 0\}$. Indeed

$$L_{[nt]}^{-1} \circ L_n(x) = \frac{U(L_n(x))}{[nt]} = \frac{U(U^{-1}(nx))}{[nt]} = \frac{nx}{[nt]} \to \mathbb{L}_t(x) = x/t,\ n \to \infty.$$

2.2. Gumbel limit distribution. It appears that there exists another nonlinear normalization for the sequence M_n which leads to the Gumbel limit distribution. In other words we find a normalizing sequence $L_n(x)$ such that

$$\mathbf{P}\{M_n \leq L_n(x)\} = \mathbf{P}\{L_n^{-1}(M_n) \leq x\} \to \exp(-e^{-x}) = \Lambda(x).$$

Let us denote $L_n(x) = U^{-1}(ne^x)$ and then $L_n^{-1}(x) = \log \dfrac{U(x)}{n}$, for every $x > 0$ and $n = 1, 2, \ldots$. Then as $n \to \infty$,

$$\mathbf{P}\left\{\log \frac{U(M_n)}{n} \le x\right\} \quad \to \quad \exp(-e^{-x}), x \in (-\infty, \infty).$$

Indeed

$$\mathbf{P}\left\{\log \frac{U(M_n)}{n} \le x\right\} = \mathbf{P}\left\{M_n \le U^{-1}(ne^x)\right\} = \left(\mathbf{P}\left\{X_1 \le U^{-1}(ne^x)\right\}\right)^n$$

$$= \quad \left(1 - U^{-1}(ne^x)^{-U^{-1}(ne^x)}\right)^n = \left(1 - \frac{1}{U^{-1}(ne^x)^{U^{-1}(ne^x)}}\right)^n$$

$$= \quad \left(1 - \frac{1}{ne^x}\right)^n \to \exp(-e^{-x}), \quad n \to \infty.$$

In order to prove that the sequence L_n is regular one has to check that for $t > 0$, $L_{[nt]}^{-1} \circ L_n(x) \to \mathbb{L}_t(x) = x - \log t$. Recall that $\Lambda(x)$ is max-stable w.r.t. c.o.g. $\{\mathbb{L}_t = x - \log t,\ t \ge 0\}$, hence $h(x) = x$. Indeed,

$$L_{[nt]}^{-1} \circ L_n(x) = \log \frac{U(L_n(x))}{[nt]}$$

$$= \quad \log \frac{U(U^{-1}(ne^x))}{[nt]} = \log \frac{ne^x}{[nt]} = x + \log \frac{n}{[nt]} \to x - \log t, \quad n \to \infty.$$

Remark. The nonlinear normalization $L_n(x) = U^{-1}(nx)$ in Example 2.1 and $L_n(x) = U^{-1}(ne^x)$ in Example 2.2 cannot be represented in an explicit form, but $U^{-1}(.)$ can be determined asymptotically as the solution of the equation $\log x + \log\log x + t = 0$ (see e.g. de Bruijn (1958)).

2.3. Linear normalization. Since the tail of the d.f. $F(x) = 1 - x^{-x}$, $x \ge 1$ is very light there should exist sequences $a_n > 0$ and b_n such that

$$\mathbf{P}\{M_n \le a_n x + b_n\} \to e^{-e^{-x}}, \quad x \in (-\infty, \infty).$$

Note that the above relation is equivalent to

$$n(1 - F(a_n x + b_n)) \to e^{-x}. \tag{8}$$

The normalizing sequences can be chosen as follows: $b_n = U^{-1}(n)$ and $a_n = \dfrac{1}{\log b_n}$, $n \ge 2$. For every $n \ge 2$ let us mention that $b_n^{b_n} = n$ and $b_n \log b_n = \log n$.

Then

$$n(1-F(a_nx+b_n)) = n\left(\frac{x}{\log b_n}+b_n\right)^{-\left(\frac{x}{\log b_n}+b_n\right)}$$

$$\begin{aligned}
&= n\left[b_n\left(\frac{x}{b_n\log b_n}+1\right)\right]^{-\left[b_n\left(\frac{x}{b_n\log b_n}+1\right)\right]} \\
&= n\left(b_n^{-b_n}\right)^{\left(\frac{x}{\log n}+1\right)}\left(\frac{x}{\log n}+1\right)^{-(\log n)\left[\frac{b_n}{\log n}\left(\frac{x}{\log n}+1\right)\right]} \\
&= n^{-\frac{x}{\log n}}\left(\frac{x}{\log n}+1\right)^{-(\log n)\left[\frac{b_n}{\log n}\left(\frac{x}{\log n}+1\right)\right]} \\
&= e^{-x}\left(\frac{x}{\log n}+1\right)^{-(\log n)\left[\frac{b_n}{\log n}\left(\frac{x}{\log n}+1\right)\right]}.
\end{aligned}$$

We observe that

$$\frac{b_n}{\log n} = \frac{1}{\log b_n} \to 0, \quad n\to\infty$$

because $b_n \uparrow \infty$, as $n \to \infty$. Using this and the fact that

$$\left(\frac{x}{\log n}+1\right)^{-\log n} \to e^{-x},$$

we obtain

$$\left(\frac{x}{\log n}+1\right)^{-(\log n)\left[\frac{b_n}{\log n}\left(\frac{x}{\log n}+1\right)\right]} \to \left(e^{-x}\right)^0 = 1,$$

which completes the proof. After some standard calculations one can see that the sequence of linear transforms $L_n(x) = \dfrac{x}{\log b_n} + b_n$ is regular.

Recall that there is only one type of limit df's in (5). Thus, if $F \in MDA(\Phi_\alpha)$ w.r.t. a regular normalizing sequence $\{L_n\}$, then one can always find another regular normalizing sequence $\{T_n\}$ such that $F \in MDA(\Lambda)$.

Example 3. Let $X_1, X_2, \ldots$ be i.i.d. r.v.s with standard exponential d.f.

$$F(x) = \begin{cases} 1 - e^{-x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

3.1. Linear normalization. It is well known that the sequences $a_n = 1$, $b_n = \log n$, $n = 1, 2, \ldots$ provide that for every fixed $x \in \mathbb{R}$,

$$(F(a_n x + b_n))^n = \left(1 - e^{-(a_n x + b_n)}\right)^n = \left(1 - e^{-(x + \log n)}\right)^n$$

$$= \left(1 - e^{-x} e^{-\log n}\right)^n = \left(1 - \frac{e^{-x}}{n}\right)^n \to e^{-e^{-x}}, \ n \to \infty.$$

The sequence $L_n(x) = x + \log n$ is regular.

3.2. Nonlinear normalization.

3.2.1. Gumbel limit distribution. Let us define

$$U(x) = \frac{1}{1 - F(x)} = \begin{cases} e^x, & x > 0, \\ 1, & x \le 0 \end{cases}$$

and its inverse

$$U^{-1}(x) = \begin{cases} \log x & x > 1, \\ -\infty, & x \le 1. \end{cases}$$

Take the normalizing monotone transforms $L_n(x) = U^{-1}(ne^x)$. Assume that $x \in \mathbb{R}$ is fixed, then $ne^x > 1$ for every $n > e^{-x}$ and then $L_n(x) = \log(ne^x) = x + \log n$, which coincides with the linear normalization given above and $F^n(L_n(x)) \to \Lambda(x)$. This is not surprising because the exponential distribution belongs to the normal max-domain of attraction of Λ ($NMDA(\Lambda)$). Hence the normalizing sequence

$$L_n(x) = \mathbb{L}_{1/n}(x) = x + \log n$$

cannot be other than linear (or asymptotically equivalent to a linear one).

Recall that F belongs to the normal max-domain of attraction of Λ if F belongs to the max-domain of attraction of Λ with respect to the same normalizing sequence $\{L_n\}$ as Λ itself.

3.2.2. Fréchet limit distribution. Recall that Φ_α is max-stable w.r.t. $\mathbb{L}_t(x) = xt^{-1/\alpha} = \exp\{\frac{1}{\alpha}(\alpha\log x - \log t)\}$. Hence for $\alpha > 0$,

$$h(x) = \begin{cases} \alpha\log x, & x > 0, \\ -\infty, & x \leq 0. \end{cases} \tag{9}$$

Now we take the following monotone normalizing sequence

$$L_n(x) = U^{-1}(ne^{h(x)}) = \log(nx^\alpha) = \alpha\log x + \log n \tag{10}$$

and obtain the convergence

$$(F(L_n(x)))^n = (1-e^{-L_n(x)})^n = (1-e^{-\log(nx^\alpha)})^n = (1-\frac{1}{nx^\alpha})^n \to e^{-x^{-\alpha}}, \ \ n \to \infty.$$

Therefore under the regular normalizing sequence (10) the exponential distribution belongs to the $MDA(\Phi_\alpha)$.

Remark. Note that if using nonlinear normalizing sequences, the classical relation

$$F \in MDA(\Phi_\alpha) \Leftrightarrow 1 - F \in RV_{-\alpha}$$

is not true, as Examples 2 and 3 show.

Example 4. Let $X_1, X_2, \ldots$ be i.i.d. r.v.s with Pareto distribution, i.e.

$$F(x) = \begin{cases} 1 - (1+x)^{-\alpha}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

where $\alpha > 0$.

4.1. Linear normalization. It is well known that the sequences $a_n = n^{1/\alpha}$, $b_n = -1$, $n = 1, 2, \ldots$ provide that for every fixed $x > 0$,

$$(F(a_n x + b_n))^n = \left(1 - (1 + a_n x + b_n)^{-\alpha}\right)^n = \left(1 - (1 + n^{1/\alpha}x - 1)^{-\alpha}\right)^n$$

$$= \left(1 - n^{-1}x^{-\alpha}\right)^n = \left(1 - \frac{x^{-\alpha}}{n}\right)^n \to e^{-x^{-\alpha}}, \ n \to \infty,$$

i.e. Pareto distribution belongs to $NMDA(\Phi_\alpha)$ with the regular normalizing sequence $L_n(x) = n^{1/\alpha}x - 1$.

4.2. Nonlinear normalization.
4.2.1. Fréchet limit distribution. Let us define

$$U(x) = \frac{1}{1-F(x)} = \begin{cases} (1+x)^\alpha, & x > 0, \\ 1, & x \le 0 \end{cases}$$

and its inverse

$$U^{-1}(x) = \begin{cases} x^{1/\alpha} - 1 & x > 1, \\ -\infty, & x \le 1. \end{cases}$$

Take $h(x)$ as in (9) and define the monotone normalizing sequence

$$L_n(x) = U^{-1}(ne^{h(x)}) = n^{1/\alpha}x - 1.$$

It is in fact the linear transform given above.

4.2.2. Gumbel limit distribution. Let us now set $h(x) = x$ for $x \in \mathbb{R}$ and define the regular normalizing transforms $L_n(x) = (ne^x)^{1/\alpha} - 1$. Then we have

$$F^n(L_n(x)) = (1 - \frac{e^{-x}}{n})^n \to e^{-e^{-x}}, \quad n \to \infty,$$

thus the Pareto d.f. belongs to $MDA(\Lambda)$ w.r.t. the above normalizing sequence.

3. Main results

Let F be an arbitrary nondegenerate d.f. Denote again $U(x) = \dfrac{1}{1-F(x)}$. The mapping $U : (l_F, r_F) \to (1, \infty)$ is monotone increasing.

Lemma 1. *There exists a continuous and strictly increasing function $g(x)$ such that*

$$\frac{g(x)}{U(x)} \to 1, \quad \textit{as } x \to r_F, \tag{11}$$

if and only if U is asymptotically continuous at r_F, i.e.

$$\frac{U(x+0)}{U(x-0)} \to 1, \quad \textit{as } x \to r_F. \tag{12}$$

This statement is a light modification of Lemma 2, Faktorovich (1989).

The following result answers the max-domain of attraction problem when using regular normalizing sequences (cf. Theorem 6.4. in Balkema and Embrechts (2007)).

Theorem 1 (On max-domain of attraction). *Let $F \in MDA(H)$, where $H(x) = e^{-e^{-h(x)}}$. Then F is asymptotically continuous at r_F and the normalizing sequence L_n can be taken as*

$$L_n(x) = g^{-1}(ne^{h(x)}), \tag{13}$$

where g is continuous and strictly increasing on (l_F, r_F) and satisfies (11). Conversely, let F be asymptotically continuous at r_F and let $h : (l_F, r_F) \leftrightarrow (-\infty, \infty)$ be continuous and strictly increasing. Then there exists a continuous and strictly increasing function g, such that the sequence $g^{-1}(ne^{h(x)})$ is regular and normalizes the convergence

$$F^n(L_n(x)) \to \exp\{-e^{-h(x)}\}, \quad n \to \infty, \tag{14}$$

i.e. $F \in MDA(H)$.

Remark. Roughly speaking, Theorem 1 says that, given F is asymptotically continuous at its right endpoint, then $F^n(L_n(x)) \to H(x)$ iff the tail of F, the tail of H and the regular normalizing sequence L_n are connected by the asymptotic relation

$$L_n(x) \sim \left(\frac{1}{1-F}\right)^{\leftarrow} (n.e^{h(x)}), \quad n \to \infty. \tag{15}$$

Here $U^{\leftarrow}$ is a left continuous inverse of U. The latter is equivalent to

$$n(1 - F(L_n(x))) \to e^{-h(x)}, \quad n \to \infty.$$

P r o o f o f T h e o r e m 1. Let $F \in MDA(H)$. Assume that F is not asymptotically continuous at r_F. Then $p = F(r_F - 0) < 1$ or there is a constant $c > 1$ and a strictly increasing sequence $x_n \to r_F$ such that $(1 - F(x_n - 0))/(1 - F(x_n)) > c$. (cf. Theorem 3.1.3, Embrechts et al.(1997)). For x fixed and $n \to \infty$ the normalizing sequence $L_n(x) \uparrow r_F$, hence $F(L_n(x)) \to p$ and $F^n(L_n(x)) \sim p^n \to 0$ in contradiction to the assumption $F \in MDA(H)$. Thus F has to be asymptotically continuous at r_F, and, by Lemma 1, there exists a strictly increasing and continuous function g, with

$$g(x) \sim \left(\frac{1}{1-F}\right)(x) = U(x), \quad \text{as} \quad x \to r_F. \tag{16}$$

The inverse function $g^{-1}(x)$ exists. It is strictly increasing and $g^{-1}(x) \uparrow \infty$, as $x \to \infty$. Therefore

$$U(g^{-1}(x)) \sim g(g^{-1}(x)) \sim x, \quad \text{as } x \to \infty.$$

The sequence $L_n(x) = g^{-1}(ne^{h(x)})$ satisfies $F^n(L_n(x)) \to H(x),\ n \to \infty$. Indeed,

$$\begin{aligned} & \mathbf{P}\left\{M_n \le L_n(x)\right\} = \mathbf{P}\left\{M_n \le g^{-1}(ne^{h(x)})\right\} \\ = \quad & \mathbf{P}^n\left\{X_1 \le g^{-1}(ne^{h(x)})\right\} = \left[1-(1-F(g^{-1}(ne^{h(x)})))\right]^n \\ = \quad & \left[1-\frac{1}{U(g^{-1}(ne^{h(x)}))}\right]^n \sim \left[1-\frac{1}{ne^{h(x)}}\right]^n = \left[1-\frac{e^{-h(x)}}{n}\right]^n \to \exp\{-e^{-h(x)}\}, \end{aligned}$$

as $n \to \infty$.

Besides, the sequence L_n is regular because for $t > 0$,

$$\begin{aligned} & L_{[nt]}^{-1} \circ L_n(x) = h^{-1}\left(\log\frac{g(L_n(x))}{[nt]}\right) = h^{-1}\left(\log\frac{g\left(g^{-1}(ne^{h(x)})\right)}{[nt]}\right) \\ = \quad & h^{-1}\left(\log\frac{ne^{h(x)}}{[nt]}\right) = h^{-1}\left(\log e^{h(x)} + \log\frac{n}{[nt]}\right) \\ = \quad & h^{-1}\left(h(x) + \log\frac{n}{[nt]}\right) \to h^{-1}\left(h(x) - \log t\right) = \mathbb{L}_t(x), \quad n \to \infty. \end{aligned}$$

Conversely, by Lemma 1 there exists a continuous and strictly increasing function g such that (16) is satisfied. Then the mapping $L_n(x) := g^{-1}(ne^{h(x)})$ belongs to the GMA and the sequence $\{L_n\}$ is regular. Thus, we have only to show (14). Since (16) we have

$$n(1-F(L_n(x))) = n(1-F(g^{-1}(ne^{h(x)}))) \sim n/g(g^{-1}(ne^{h(x)})) = e^{-h(x)}. \quad \square$$

Corollary 1.

1. *Let $F \in MDA(\Phi_\alpha)$. Then $h(x) = \alpha \log x$ and $L_n(x) \sim \left(\frac{1}{1-F}\right)^{\leftarrow}(nx^\alpha)$. The function $R(x) = nx^\alpha$ is regularly varying at infinity, hence $U(L_n(x)) \in RV_\alpha$. Since $U(y) = \dfrac{1}{1-F(y)}$ for $y \to r_F$ we conclude that $\bar{F} \circ L_n \in RV_{-\alpha}$.*

2. *In the max-model with monotone normalization, the necessary and sufficient condition for* $F \in MDA(\Phi_\alpha)$ *is* $\bar{F} \circ L_n \in RV_{-\alpha}$*. It differs from the necessary and sufficient condition* $\bar{F} \in RV_{-\alpha}$ *in the max-model with linear normalization (cf. Examples* 2, 3 *and* 4*).*

3. *Let* $F \in MDA(\Lambda)$*. Then* $h(x) = x$ *and* $L_n(x) \sim \left(\frac{1}{1-F}\right)^{\leftarrow}(ne^x)$*. Hence*

$$\frac{1}{n}\left(\frac{1}{1-F}\right)(L_n(x)) \to e^x, \qquad n \to \infty.$$

Choose $y_n \uparrow r_F$ *such that* $1 - F(y_n) = \frac{1}{n}$*. Then for* $U = \left(\frac{1}{1-F}\right)^{\leftarrow}$

$$\frac{U^{\leftarrow}(L_n(x))}{U^{\leftarrow}(y_n)} \to e^x, \quad n \to \infty.$$

The converse is also true (cf. de Haan (1970)).

Remark. Examples 2, 3, and 4 from Section 2 show that a distribution may belong to MDA of two different max-stable laws. Moreover, every continuous and strictly increasing df belongs to the max-domain of attraction of every max-stable df. Yet it will be wrong to conclude that "domains of attractions of different types are not disjoint if using monotone normalization" as read in some authors. In fact, if using monotone normalization, there is only one type of max-stable laws!

Theorem 2. *Let* $F \in MDA(H), H(x) = e^{-e^{-h(x)}}$*, w.r.t. the regular normalizing sequence* L_n*, defined in Theorem 1. If* $g(x) = U(x)$ *then*

$$|F^n(L_n(x)) - H(x)| = O\left(1/n\right), \quad n \to \infty.$$

Remark. If the function $g(x)$ is asymptotically equivalent to $U(x)$ then the rate of convergence depends also on the rate of convergence in the asymptotic relation $\frac{g(x)}{U(x)} \to 1$ as $x \to \infty$.

Proof. Since $F^n(L_n(x)) = \left[1 - \frac{e^{-h(x)}}{n}\right]^n$, we have to estimate

$$\left|\left[1 - \frac{e^{-h(x)}}{n}\right]^n - e^{-e^{-h(x)}}\right|$$

For $u > 0$ it follows from the power series that

$$n \log\left(1 - \frac{u}{n}\right) = -u + O\left(\frac{1}{n}\right), \quad n \to \infty. \tag{17}$$

Using the inequality $|e^{-a} - e^{-b}| \leq |a - b|, (a, b \geq 0)$, one gets

$$\begin{aligned}
&\left| \left[1 - \frac{e^{-h(x)}}{n}\right]^n - e^{-e^{-h(x)}} \right| \\
&= \left| \exp\left(n \log\left[1 - \frac{e^{-h(x)}}{n}\right]\right) - \exp\left(-e^{-h(x)}\right) \right| \\
&\leq \left| -n \log\left[1 - \frac{e^{-h(x)}}{n}\right] - e^{-h(x)} \right| = O\left(\frac{1}{n}\right), \quad n \to \infty.
\end{aligned}$$

The last relation follows from (17) with $u = e^{-h(x)} > 0$. $\square$

Corollary 2. *Let L_n and T_n be two normalizing sequences of max- automorphisms, such that*

$$n(1 - F(L_n(x))) \to e^{-h(x)}, \tag{18}$$

$$n(1 - F(T_n(x))) \to e^{-h(x)} \tag{19}$$

for h continuous and strictly increasing. Then both sequences are regular and asymptotically equivalent in the sense that

$$L_n^{-1}(T_n(x)) \to x, \quad n \to \infty.$$

Conversely, if (18) *holds and $\{T_n\}$ is asymptotically equivalent to $\{L_n\}$ in the above sense, then* (19) *also holds.*

P r o o f. Covergences (18) and (19) imply that

$$L_n(x) \sim \left(\frac{1}{1-F}\right)^{\leftarrow} \left(ne^{h(x)}\right) \sim T_n(x).$$

Take $L_n(x) = g^{-1}(ne^{h(x)})$ and $T_n(x) = f^{-1}(ne^{h(x)})$ where g and f are continuous and strictly increasing functions satisfying (11). Both g^{-1} and f^{-1} are asymptotically inverse to $U(x) = \dfrac{1}{1 - F(x)}$. Since $L_n^{-1}(y) = h^{-1}(\log g(y) - \log n)$ we have

$$L_n^{-1} \circ T_n(x) = h^{-1}\left\{\log \frac{g(T_n(x))}{n}\right\}$$

$$= h^{-1}\left\{\log \frac{g \circ f^{-1}(ne^{h(x)})}{n}\right\} \sim h^{-1}\left\{\log \frac{ne^{h(x)}}{n}\right\} = x.$$

Conversely, (19) can be rewritten as

$$n(1 - F(T_n(x))) = n\left\{1 - F(L_n[L_n^{-1} \circ T_n(x)])\right\} \sim n\left\{1 - F(L_n(x))\right\} \to e^{-h(x)}.$$

□

As a conclusion let us consider the normalization of maxima of normally distributed iid random variables.

Example 5 (Normal distribution). Let $X_1, X_2, \ldots$ be iid r.v.s with standard normal d.f.

$$\mathfrak{N}(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-u^2/2} du \ \text{ and density } \ \mathfrak{n}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in (-\infty, \infty).$$

By Theorem 1 the regular normalizing sequence $L_n(x) = U^{-1}(ne^x)$, where $U = \dfrac{1}{1-\mathfrak{N}}$, causes the weak convergence

$$\mathfrak{N}^n(L_n(x)) \to e^{-e^{-x}}. \tag{20}$$

Theorem 2 says that the rate of this convergence is $O(1/n)$. Unfortunately, the sequence $U^{-1}(ne^x)$ is not very useful in practice, because of the fact that the inverse function $\left(\dfrac{1}{1-\mathfrak{N}}\right)^{-1}$ is not explicitly known. Thus we go through the well known asymptotic relation

$$U(x) = \frac{1}{1-\mathfrak{N}(x)} \sim g(x) = \frac{x}{\mathfrak{n}(x)} = \sqrt{2\pi} x e^{\frac{x^2}{2}}, \quad x \to \infty \tag{21}$$

in order to find an asymptotic inverse of $g(x)$ and of $U(x)$, respectively. Following the same way as in the proof of Theorem 1.5.3 in Leadbetter et al. (1983), we obtain the following asymptotic inverse of $g(x)$ as $x \to \infty$

$$g^{-1}(x) = \sqrt{2\log x - \log 4\pi - \log\log x}$$

for which

$$U(g^{-1}(x)) \sim g(g^{-1}(x)) \sim x, \quad \text{as } x \to \infty.$$

Now we define the sequence

$$T_n(x) = g^{-1}(ne^x) = \sqrt{2\log n + 2x - \log 4\pi - \log(\log n + x)}.$$

Next we show that both sequences T_n and L_n are asymptotically equivalent. Indeed, since $L_n^{-1}(y) = \log \frac{U(y)}{n}$ we have

$$L_n^{-1} \circ T_n(x) = \log \frac{U(T_n(x))}{n} = \log \frac{U(g^{-1}(ne^x))}{n} = \log \frac{ne^x}{n} + o(1) \to x, n \to \infty.$$

Then one can use the sequence $T_n(x)$ for normalization in (20), thus

$$\mathfrak{N}^n(T_n(x)) \to \Lambda(x), \quad n \to \infty. \tag{22}$$

According to Theorem 2 the rate of convergence in the equation (20) is $O(1/n)$. On the other hand the rate of convergence in the equation (22) depends also on the rate of convergence in the asymptotic relation (21). It is not difficult to show that in this case the rate of convergence is equivalent to that in the linear case, namely $O\left(\frac{1}{\log n}\right)$ (see e.g. de Haan (1970)).

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