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SUBCRITICAL RANDOMLY INDEXED BRANCHING PROCESSES*

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The paper continues the study of the randomly indexed branching processes in the subcritical case. The asymptotic behavior of the moments and the probability for non-extinction is investigated. Conditional limiting distributions are obtained.

1. Introduction

Randomly indexed branching processes (RIBP) were introduced by Epps [2] for modelling of daily stock prices as an alternative of the geometric Brownian motion. He considered a Bienaymé-Galton-Watson (BGW) branching process indexed by a Poisson process, assuming four particular discrete offspring distributions. Under these conditions, Epps obtained the asymptotic behavior of the moments, submitted certain estimates of the parameters of the process, and made the calibration of the model using real data from the New York Stock Exchange (NYSE). Assuming this stock price process, two formulas for pricing of European Call Option and Up-and-Out Barrier Option were derived in [5] and [8], respectively.

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Dion and Epps [1] noted that if the subordinator is a Poisson process then the a RIBP are particular case of branching processes in random environments. Therefore, one can derive their general properties from the results for branching processes in random environments. On the other hand, the particular assumptions provide some important characteristics which are not exhibited in the general framework.

In the present paper we continue the investigation of the randomly indexed branching processes initiated in [4], [6], and [7]. Let us briefly recall the definition.

Assume that on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ are given:

(i) The set $X = \{X_i(n), n = 1, 2, \dots; i = 1, 2, \dots\}$ of i.i.d. nonnegative integer valued random variables (r.v.) with the probability generating function (p.g.f.) $f(s) = \mathbf{E}(s^{X_i(n)}) = \sum_{k=0}^{\infty} p_k s^k, \quad s \in [0, 1].$

(ii) An independent of X set $J = \{J_1, J_2, \dots\}$ of positive i.i.d. r.v. with the cumulative distribution function (c.d.f.) $F(x) = \mathbf{P}\{J_n \le x\}$.

The classical BGW branching process is defined as follows

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_i(n+1), \quad n = 0, 1, 2, \dots. \tag{1}$$

It is well known that the p.g.f. $f_n(s) = \mathbf{E}(s^{Z_n}|Z_0 = 1), |s| \le 1$, is the n-fold iteration of $f(s)$; that is $f_n(s) = f(f_{n-1}(s))$, $f_1(s) = f(s)$, $f_0(s) = s$ (see e.g. [11]).

Define also the ordinary renewal process

$$S_0 = 0, \quad S_n = \sum_{j=1}^{n} J_j, \quad n = 0, 1, \dots, \tag{2}$$

and the corresponding counting process

$$N(t) = \max\{n \ge 0 : S_n \le t\}, \quad t \ge 0. \tag{3}$$

Denote the renewal function of $N(t)$ by

$$H(t) = \mathbf{E}(N(t)) = \sum_{n=0}^{\infty} F^{*n}(t), \quad t \ge 0,$$

$$P_k(t) = \mathbf{P}\{N(t) = k\}, \quad k = 0, 1, 2, \dots,$$

and the p.g.f. of $N(t)$ by

$$\Psi(t,s) = \mathbf{E}(s^{N(t)}) = \sum_{k=0}^{\infty} P_k(t)s^k, \quad s \in [0,1].$$

Here and later $F^{*n}(t)$ denotes the $n-$fold convolution of the distribution function $F(t)$; that is $F^{*1}(t) = F(t)$, $F^{*n}(t) = \int_0^t F^{*(n-1)}(t-u)dF(u)$, $F^{*0}(t) = 1$.

Definition 1. *The continuous time process* $\{Y(t), t \geq 0\}$ *defined by*

$$Y(t) = Z_{N(t)}, \quad t \geq 0$$

is called a randomly indexed BGW branching process.

Applying the total probability law we obtain by the independence of the processes $Z(n)$ and $N(t)$ that

$$\Phi(t;s) = \mathbf{E}(s^{Y(t)}) = \sum_{n=0}^{\infty} P_n(t)f_n(s). \tag{4}$$

In the investigation of the limiting behavior of the process $Y(.)$, we need also the following functional equations

$$\Phi(t;s) = s(1-F(t)) + \int_0^t f(\Phi(t-u;s))dF(u), \tag{5}$$

$$\mathbf{P}\left\{Y(t) \leq x | Y(t) > 0\right\} = \int_0^{\infty} \mathbf{P}\left\{Z_{[y]} \leq x | Z_{[y]} > 0\right\} d\mathbf{P}\{N(t) \leq y | Z_{N(t)} > 0\}. \tag{6}$$

Further, we will suppose that the following conditions hold:

(B) $\{Z_n\}$ is subcritical, that is $m = f'(1) = \mathbf{E}(X_i(n)) < 1$ and

$$0 < b = f''(1) = \mathbf{E}(X_i(n)(X_i(n)-1)) < \infty.$$

(R1) The d.f. $F(t)$ is subexponential, i.e.

$$\lim_{t\to\infty} \frac{1-F^{*2}(t)}{1-F(t)} = 2.$$

(R2) There exist two positive numbers ζ_1 and ζ_2, such that

$$m\int_0^{\infty} e^{\zeta_1 t}dF(t) = 1, \quad m^2\int_0^{\infty} e^{\zeta_2 t}dF(t) = 1.$$

2. Subexponential interarrival times

In this section we assume the conditions (B) and (R1).

2.1. Moments

Differentiating (4) and setting $s = 1$ yields

$$\Phi'_s(t;1) = \sum_{k=0}^{\infty} P_k(t) f'_k(1), \quad \Phi''_{ss}(t;1) = \sum_{k=0}^{\infty} P_k(t) f''_k(1).$$

Using the well-known formulas for the first two moments of Z_n (see e.g. [11], p. 45) after simple calculations one can see that

$$M(t) = \mathbf{E}(Z(t)|Z(0) = 1) = \Psi(t;m). \tag{7}$$

and

$$B(t) = \mathbf{E}(Z(t)(Z(t) - 1)) = \frac{b}{m(1-m)}(\Psi(t;m) - \Psi(t;m^2)) \tag{8}$$

Theorem 1. *If the conditions* (B) *and* (R1) *are satisfied then*

$$M(t) \sim \frac{m}{1-m}(1 - F(t)), \quad t \to \infty$$

and

$$B(t) \sim \frac{b}{(1-m)^2(1+m)}(1 - F(t)), \quad t \to \infty.$$

P r o o f. It is not difficult to see that the p.g.f. $\Psi(t;s)$ can be written in the following form

$$\Psi(t;s) = (1-s)\sum_{n=1}^{\infty} s^{n-1}(1 - F^{*n}(t)), \quad s \in [0,1). \tag{9}$$

Therefore, for a fixed $s \in (0,1)$, $\Psi(t,s)$ is the tail of the d.f. of a sum of random number of i.i.d. r.v. with d.f. $F(t)$ where the number of summands has the geometric distribution $(1-s)s^{k-1}, k = 1, 2, \ldots$. Using this, we can apply Theorem 52, [10], to conclude that

$$\Psi(t;s) \sim \frac{s}{1-s}(1 - F(t)), \quad t \to \infty.$$

The last relation, (7), and (8) prove the assertions of the theorem. □

2.2. Probability for non-extinction

Let us denote by T the time to extinction of the process $Z_n, n = 0, 1, 2, \ldots$, that is

$$Z_0 = 1 > 0, Z_2 > 0, \ldots, Z_{T-1} > 0, Z_T = 0.$$

It is known that (see e.g. [11], §II.2, Theorem 2)

$$\mathbf{P}\{T > n\} = \mathbf{P}\{Z_n > 0\} = 1 - f_n(0) \sim Km^n, n = 0, 1, 2, \ldots.$$

Since $m < 1$, then

$$\mathbf{E}(T) = \sum_{n=1}^{\infty} \mathbf{P}\{T > n\} < \infty. \tag{10}$$

The probability for non extinction $\mathbf{P}\{Y(t) > 0\}$ can be written az follows

$$\mathbf{P}\{Y(t) > 0\} = \mathbf{P}\{Z_{N(t)} > 0\} = \mathbf{P}\{T > N(t)\} = \mathbf{P}\{S_T > t\}. \tag{11}$$

Theorem 2. *If the conditions* (B) *and* (R1) *hold then*

$$\mathbf{P}\{Y(t) > 0\} \sim \mathbf{E}(T)(1 - F(t)), \quad t \to \infty.$$

Proof. The most right hand side of (11), the independence of the T and J_n, and (10) allow us to apply (Theorem 47, [10]), to complete the proof of the theorem. □

2.3. Limit theorem

Since $\mathbf{E}(T) < \infty$ then for any $y \geq 0$, $\mathbf{E}(\min\{T, [y] + 1\}) < \infty$ and

$$\mathbf{E}(\min\{T, [y] + 1\}) \uparrow \mathbf{E}(T) < \infty, \qquad \text{as } y \to \infty.$$

Therefore

$$\pi(y) = \mathbf{E}(\min\{T, [y] + 1\})/\mathbf{E}(T), \quad y \geq 0 \tag{12}$$

is a proper distribution on $[0, \infty)$.

Lemma 1. *Under the conditions* (B) *and* (R1)

$$\lim_{t\to\infty} \mathbf{P}\{N(t) < y | Z_{N(t)} > 0\} = \pi(y),$$

where $\pi(.)$ *is the d.f. defined by* (12).

P r o o f. The events $\{Z_{N(t)} > 0\}$ and $\{N(t) < T\}$ are equivalent. Therefore, for any fixed $t \geq 0$ and $y \geq 0$,

$$\begin{aligned} \pi_t(y) &= \mathbf{P}\{N(t) \leq y | Z_{N(t)} > 0\} \\ &= \mathbf{P}\{N(t) \leq y | N(t) < T\} = \mathbf{P}\{N(t) \leq y, N(t) < T\} / \mathbf{P}\{N(t) < T\}. \end{aligned}$$

From $\mathbf{P}\{N(t) < T\} = \mathbf{P}\{S_T > t\}$ and the proof of Theorem 2 we have

$$\mathbf{P}\{N(t) < T\} \sim \mathbf{E}(T)(1 - F(t)), \quad t \to \infty.$$

Similarly, we have for any fixed $y \geq 0$,

$$\begin{aligned} & \mathbf{P}\{N(t) \leq y, N(t) < T\} = \mathbf{P}\{N(t) < [y] + 1, N(t) < T\} \\ = \ & \mathbf{P}\{N(t) < \min(T, [y] + 1)\} = \mathbf{P}\{S_{\min\{T, [y]+1\}} > t\} \\ \sim \ & \mathbf{E}(\min\{T, [y] + 1\})(1 - F(t)), \quad t \to \infty. \end{aligned}$$

Therefore, for $y \geq 0$,

$$\lim_{t \to \infty} \pi_t(y) = \mathbf{E}(\min\{T, [y] + 1\}) / \mathbf{E}(T).$$

□

Now we are ready to prove the conditional limit theorem for $Y(t)$.

Theorem 3. *Assume* (B) *and* (R1). *Then*

$$\lim_{t \to \infty} \mathbf{P}\{Y(t) \leq x | Y(t) > 0\} = \int_0^\infty \mathbf{P}\{Z_{[y]} \leq x | Z_{[y]} > 0\} d\pi(y).$$

P r o o f. Let $x > 0$ be fixed. By the independence of Z_n and $N(t)$ one gets

$$\begin{aligned} & \mathbf{P}\{Y(t) \leq x | Y(t) > 0\} = \mathbf{P}\left\{Z_{N(t)} \leq x | Z_{N(t)} > 0\right\} \\ = \ & \mathbf{P}\left\{Z_{N(t)} \leq x, Z_{N(t)} > 0\right\} / \mathbf{P}\{Z_{N(t)} > 0\} \\ = \ & \sum_{n=0}^{\infty} \mathbf{P}\{Z_n \leq x, Z_n > 0, N(t) = n\} / \mathbf{P}\{Z_{N(t)} > 0\} \\ = \ & \sum_{n=0}^{\infty} \mathbf{P}\{Z_n \leq x | Z_n > 0\} \mathbf{P}\{N(t) = n | Z_{N(t)} > 0\} \\ = \ & \int_0^\infty \mathbf{P}\left\{Z_{[y]} \leq x | Z_{[y]} > 0\right\} d\pi_t(y). \end{aligned}$$

Now we can apply Lemma 1 to obtain that

$$\lim_{t\to\infty}\int_0^\infty \mathbf{P}\left\{Z_{[y]}\le x|Z_{[y]}>0\right\}d\pi_t(y)=\int_0^\infty \mathbf{P}\left\{Z_{[y]}\le x|Z_{[y]}>0\right\}d\pi(y),$$

which completes the proof. □

3. $F(\cdot)$ has light tail

In this section we suppose that the conditions (B) and (R2) hold. Note that $0<\zeta_1<\zeta_2$.

3.1. Moments

Theorem 4. *Under the conditions* (B) *and* (R2) *and*

(13) $$\mu_1=m\int_0^\infty te^{\zeta_1 t}dF(t)<\infty,$$

we have

(14) $$M(t)\sim\frac{1-m}{m\zeta_1\mu_1}e^{-\zeta_1 t},\ t\to\infty.$$

If additionally

(15) $$\mu_2=m^2\int_0^\infty te^{\zeta_2 t}dF(t)<\infty,$$

then

(16) $$B(t)\sim\frac{b}{m^2\zeta_1\mu_1}e^{-\zeta_1 t},\ t\to\infty.$$

Proof. Now we will use the following representation of the p.g.f. $\Psi(t;s)$,

(17) $$\Psi(t;s)=\frac{1}{s}\left[1-(1-s)H_s(t)\right],\quad s\in(0,1),$$

where $H_s(t)=\sum_{n=0}^{\infty}s^nF^{*n}(t)$ is the renewal function corresponding to the unproper d.f. $sF(t)$, $(sF(t)\to s<1,\quad t\to\infty.)$

One gets from (7) and (17) that

$$M(t) = \Psi(t;m) = \frac{1}{m}\left[1-(1-m)H_m(t)\right], \tag{18}$$

Since $mF(t) \to m < 1$, as $t \to \infty$, then

$$H_m(t) \to \frac{1}{1-m}, \quad t \to \infty.$$

Using the conditions of the theorem we apply (Theorem 2, Sect. XI.6, (6.7), (6.16), [3]) to obtain

$$\lim_{t\to\infty} e^{\zeta_1 t}\left(\frac{1}{1-m} - H_m(t)\right) = \frac{1}{\zeta_1\mu_1}$$

or equivalently

$$1-(1-m)H_m(t) \sim \frac{1-m}{\zeta_1\mu_1}e^{-\zeta_1 t}, \quad t \to \infty,$$

which together with (18), yield

$$\Psi(t;m) \sim \frac{1-m}{m\zeta_1\mu_1}e^{-\zeta_1 t}, \quad t \to \infty. \tag{19}$$

This prove (14). The proof of (16) is very similar, we have only to use that

$$1-(1-m^2)H_{m^2}(t) \sim \frac{1-m^2}{\zeta_2\mu_2}e^{-\zeta_2 t}, \quad t \to \infty,$$

which yields (see (18)),

$$\Psi(t,m^2) \sim \frac{1-m^2}{m^2\zeta_2\mu_2}e^{-\zeta_2 t}, \quad t \to \infty. \tag{20}$$

Combine (20), (19), (8), and the fact that $0 < \zeta_1 < \zeta_2$ we complete the proof of the theorem. □

3.2. Probability for non extinction

In order to prove the asymptotic behaviour of the probability for non-extinction we need the following relation proved by Nagaev and Muhamedhanova [9] under the condition (B),

$$\mathbf{P}\{T > n\} = 1 - f_n(0) = Km^n + \frac{b}{2m}\frac{Km^{2n}}{1-m} + o(m^{2n}), \quad n \to \infty. \tag{21}$$

Theorem 5. *If* (B) *and* (R2) *are satisfied then*

$$\mathbf{P}\{Y(t)>0|Y(0)=1\}\sim\frac{2(1-m)^2}{(2m(1-m)+b)\zeta_1\mu_1}e^{-\zeta_1 t}, \tag{22}$$

where $\zeta_1>0$ *and* μ_1 *are defined in* (R2) *and* (13) *respectively.*

P r o o f. Using the relation (21) we have

$$\begin{aligned} &\mathbf{P}\{Y(t)>0|Y(0)=1\}\\ =\ &K\sum_{n=0}^{\infty}P_n(t)m^n\\ +\ &\sum_{n=0}^{\infty}P_n(t)\frac{bK}{m(1-m)}m^{2n}(1+\alpha_n), \end{aligned} \tag{23}$$

where $\alpha_n\to 0,\ \ n\to\infty$. Under the condition of the theorem by (19) one gets

$$K\sum_{n=0}^{\infty}P_n(t)m^n=K\Psi(t;m)\sim K\frac{1-m}{m\zeta_1\mu_1}e^{-\zeta_1 t},\quad t\to\infty. \tag{24}$$

For the second sum in (23) there exists a positive constant C such that

$$\begin{aligned} &\sum_{n=0}^{\infty}P_n(t)\frac{bK}{m(1-m)}m^{2n}(1+\alpha_n)\\ &\qquad\le\frac{CbK}{m(1-m)}\Psi(t;m^2). \end{aligned}$$

This inequality and (20) yield

$$\sum_{n=0}^{\infty}P_n(t)\frac{bK}{m(1-m)}m^{2n}(1+\alpha_n)=O(e^{-\zeta_2 t}),t\to\infty.$$

The last relation, (24), (23), and the fact that $0<\zeta_1<\zeta_2$, complete the proof of the theorem. □

3.3. Limit theorem

Let us recall that under condition (B) there exists a proper conditional limiting distribution for the process Z_n, defined by

$$\lim_{n\to\infty}\mathbf{P}\{Z_n=k|Z_n>0\}=d_k,\ \text{for } k=1,2,\ldots, \tag{25}$$

and the p.g.f $D(s)=\sum_{k=1}^{\infty} d_k s^k$ satisfies the functional equation

$$D(f(s)) = mD(s) + (1-m). \tag{26}$$

(See e.g. [11]).

Theorem 6. *Assume* (B) *and* (R2). *Then*

$$\lim_{t\to\infty} \mathbf{P}\{Y(t)=k|Y(t)>0\} = d_k, k=1,2,\dots$$

where $d_k, k=1,2,\dots$ *are defined in* (25) *and* (26).

P r o o f. We have for a fixed integer k that

$$\begin{aligned}
&\mathbf{P}\{Y(t)=k|Y(t)>0\}\\
=\ & \frac{\mathbf{P}\{Y(t)=k, Y(t)>0\}}{\mathbf{P}\{Y(t)>0\}} = \frac{\mathbf{P}\{Y(t)=k\}}{\mathbf{P}\{Y(t)>0\}}\\
=\ & \frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=0}^{\infty}\mathbf{P}\{N(t)=l, Y(t)=k\}\\
=\ & \frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=0}^{\infty}\mathbf{P}\{N(t)=l, Z_{N(t)}=k\}\\
=\ & \frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=0}^{\infty}\mathbf{P}\{N(t)=l, Z_l=k\}\\
=\ & \frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=0}^{\infty}\mathbf{P}\{N(t)=l\}\mathbf{P}\{Z_l=k\}\\
=\ & \frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=0}^{\infty}\mathbf{P}\{N(t)=l\}\mathbf{P}\{Z_l=k|Z_l>0\}\mathbf{P}\{Z_l>0\}.
\end{aligned}$$

Let $\varepsilon>0$ be fixed but arbitrary. From (25) we have that there exists $L>0$ such that for every $l>L$

$$d_k-\varepsilon \le \mathbf{P}\{Z_l=k|Z_l>0\} \le d_k+\varepsilon.$$

Using this we continue as follows

$$
\begin{aligned}
& \frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=0}^{\infty}\mathbf{P}\{N(t)=l\}\mathbf{P}\{Z_l=k|Z_l>0\}\mathbf{P}\{Z_l>0\} \\
= \quad & \frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=0}^{L}\mathbf{P}\{N(t)=l\}\mathbf{P}\{Z_l=k|Z_l>0\}\mathbf{P}\{Z_l>0\} \\
+ \quad & \frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=L+1}^{\infty}\mathbf{P}\{N(t)=l\}\mathbf{P}\{Z_l=k|Z_l>0\}\mathbf{P}\{Z_l>0\} \\
\leq \quad & \frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=0}^{L}\mathbf{P}\{N(t)=l\}\mathbf{P}\{Z_l=k|Z_l>0\}\mathbf{P}\{Z_l>0\} \\
+ \quad & \frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=L+1}^{\infty}\mathbf{P}\{N(t)=l\}(d_k+\varepsilon)\mathbf{P}\{Z_l>0\} \\
\leq \quad & \frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=0}^{L}\mathbf{P}\{N(t)=l\}\mathbf{P}\{Z_l=k|Z_l>0\}\mathbf{P}\{Z_l>0\} \\
+ \quad & (d_k+\varepsilon)\frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=0}^{\infty}\mathbf{P}\{N(t)=l\}\mathbf{P}\{Z_l>0\} \\
\leq \quad & \frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=0}^{L}\mathbf{P}\{N(t)=l\}\mathbf{P}\{Z_l=k|Z_l>0\}\mathbf{P}\{Z_l>0\} \\
+ \quad & (d_k+\varepsilon)\frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=0}^{\infty}\mathbf{P}\{N(t)=l, Z_{N(t)}>0\} \\
\leq \quad & \frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=0}^{L}\mathbf{P}\{N(t)=l\}\mathbf{P}\{Z_l=k|Z_l>0\}\mathbf{P}\{Z_l>0\} \\
+ \quad & (d_k+\varepsilon)\frac{1}{\mathbf{P}\{Y(t)>0\}}\mathbf{P}\{Y(t)>0\} \\
\leq \quad & \frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=0}^{L}\mathbf{P}\{N(t)=l\}\mathbf{P}\{Z_l=k|Z_l>0\}\mathbf{P}\{Z_l>0\}+(d_k+\varepsilon).
\end{aligned}
$$

Let us consider the first sum

$$\begin{aligned} & \frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=0}^{L}\mathbf{P}\{N(t)=l\}\mathbf{P}\{Z_l=k|Z_l>0\}\mathbf{P}\{Z_l>0\} \\ \leq\ & \frac{1}{\mathbf{P}\{Y(t)>0\}}\sum_{l=0}^{L}\mathbf{P}\{N(t)=l\} = \frac{1}{\mathbf{P}\{Y(t)>0\}}\mathbf{P}\{N(t)\leq L\} \\ =\ & \frac{1}{\mathbf{P}\{Y(t)>0\}}\mathbf{P}\{S_L\geq t\}. \end{aligned}$$

Now we will prove that under the condition (R2)

$$\mathbf{P}\{S_L \geq t\} = o(e^{-\zeta_1 t}), \quad t\to\infty$$

for any fixed L.

We have that $\int_0^\infty e^{\zeta_1 t}dF(t) = \dfrac{1}{m} < \infty$, that is $\mathbf{E}(e^{\zeta_1 T_i}) = \dfrac{1}{m} < \infty$. Therefore, for any fixed L, (using the independence of T's),

$$\mathbf{E}(e^{\zeta_1(T_1+T_2+\ldots+T_L)}) = \Big[\mathbf{E}(e^{\zeta_1 T_i})\Big]^L = \int_0^\infty e^{\zeta_1 t}dF^{*L}(t) < \infty = m^{-L} < \infty.$$

Then

$$\int_x^\infty e^{\zeta_1 t}dF^{*L}(t) \downarrow 0, \quad x\to\infty. \tag{27}$$

Finally, using that $\zeta_1 > 0$ we get

$$\int_x^\infty e^{\zeta_1 t}dF^{*L}(t) \geq e^{\zeta_1 x}\int_x^\infty dF^{*L}(t) = e^{\zeta_1 x}(1-F^{*L}(x)). \tag{28}$$

From (27) and (28) we get

$$\mathbf{P}\{S_L\geq t\} = 1 - F^{*L}(t) = o(e^{-\zeta_1 t}), \quad t\to\infty.$$

Therefore if t is chosen large enough (using the asymptotic of $\mathbf{P}\{Y(t)>0\}$),

$$\frac{1}{\mathbf{P}\{Y(t)>0\}}\mathbf{P}\{S_L\geq t\} \leq \varepsilon.$$

Follow the same way we can prove that for any t large enough

$$\mathbf{P}\{Y(t)=k|Y(t)>0\} \geq -\varepsilon + (d_k - \varepsilon).$$

The theorem is proved, since ε was arbitrary. $\square$

Corollary 1. *Assume* (B), (R2), *and*

$$1-f(s)=\frac{m(1-s)}{1+\frac{b}{2m}(1-s)}.$$

Then

$$\lim_{t\to\infty}\mathbf{P}\{Y(t)=k|Y(t)>0\}=\frac{1}{1+R}\left(\frac{R}{1+R}\right)^{k-1}, k=1,2,\dots,$$

where $R=\dfrac{b}{2m(1-m)}$.

4. Conclusion remarks

From the obtained results it is seen that the asymptotic behavior of the randomly indexed BGW branching processes essentially depend on the life length of the particles. In case when the d.f. of the life length has light tail the behavior is close to that of the ordinary BGW branching processes and it is rather different in case when the d.f. is subexponential one.

REFERENCES

[1] J.-P. Dion, T. Epps. Stock prices as branching processes in random environment: estimation. *Commun. Stat.-Simulations* **28** (1999), 957–975.

[2] T. Epps. Stock prices as branching processes. *Commun. Stat.-Stochastic Models* **12** (1996), 529–558.

[3] W. Feller. An introduction to probability theory and its applications, Vol. 2, Wiley, New York, 1971.

[4] G. K. Mitov, K. V. Mitov. Randomly indexed branching processes. *Math. and Education in Math.* **35** (2006), 275–281.

[5] G. K. Mitov, K. V. Mitov. Option pricing by branching processes. *Pliska Stud. Math. Bulgar.* **18** (2007), 213–224.

[6] G. K. Mitov, K. V. Mitov, N. M. Yanev. Critical randomly indexed branching processes. *Stat. Probab. Letters* **79(13)** (2009), 1512-1521.

[7] G. K. Mitov, K. V. Mitov, N. M. Yanev. Limit theorems for critical randomly indexed branching processes In: Workshop on Branching Processes and Their Applications (Eds M. Gonzalez, R. Martinez, I. Del Puerto) Springer: Lecture Notes in Statistics, Vol. **197**, Part 2, 2010, 95–108.

[8] G. K. Mitov, S. T. Rachev, Y. S. Kim, F. Fabozzi. Barrier option pricing by branching processes. *International Journal of Theoretical and Applied Finance* **12(7)** (2009), 1055–1073.

[9] S. V. Nagaev, R. Muhamedhanova. A refinement of certain theorems in the theory of branching random processes. In: Limit theorems and Statistical Inferences, Fan, Tashkent, 1966, 90–112 (in Russian).

[10] E. A. M. Omey, F. Mallor. Univariate and Multivariate Weighted Renewal Theory. Public University of Navarre, Navarre, 2006.

[11] B. A. Sevastyanov. Branching Processes. Nauka, Moscow, 1971 (in Russian).

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