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SOME CONTRIBUTIONS TO THE CLASS OF TWO-SEX BRANCHING PROCESSES DEPENDING ON THE NUMBER OF COUPLES IN THE POPULATION*

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We consider the class of two-sex branching processes with offspring and mating depending on the number of couples in the population introduced in Molina *et al.* (2008). In addition to its theoretical interest, this class also has clear practical implications, especially in population dynamics. We investigate its extinction probability and limiting behavior. By considering different probabilistic approaches, necessary and sufficient conditions for its almost sure extinction are determined. Assuming the nonextinction, some limiting results are derived.

1. Introduction

Branching processes are widely used as appropriate mathematical models to describe the probabilistic evolution of systems whose components (cells, particles, individuals in general) after certain life period reproduce and die. Nowadays, branching process theory is an active research area of interest and applicability to such fields as biology, demography, ecology, epidemiology, genetics, medicine,

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population dynamics, and others. We emphasize here that branching processes have interesting applications in biological populations, playing an increasingly important role in molecular biology and microbiology. One may cite, for example, the monographs by Jagers (1975), Kimmel and Axelrod (2002), Pakes (2003), or Haccou *et al.* (2005), which include practical applications to cell kinetics, drug resistance and chemotherapy, gene amplification, polymerase chain reaction, and so on.

In particular, with the purpose to model the probabilistic evolution of populations where females and males coexist and form couples several classes of discrete time two-sex branching processes have been investigated, including the bisexual Galton-Watson process (see Alsmeyer and Rösler (1996, 2002), Bruss (1984), Daley (1968a), Daley *et al.* (1986)); processes with immigration (see González *et al.* (2000, 2001), Ma and Xing (2006)); processes in varying or random environments (see Ma (2006), Ma and Molina (2009), Molina *et al.* (2003)); processes with population-size depending mating (see Molina *et al.* (2002, 2004, 2006), Xing and Wang (2005)), or processes with a control function (see Molina *et al.* (2007)). Also, a general class of continuous time two-sex branching processes was introduced in Molina and Yanev (2003). For a more detailed information about two-sex branching processes we refer the reader to the surveys by Hull (2003) or Molina (2010).

In this work, we continue the research about the class of two-sex branching processes with offspring and mating depending on the number of couples in the population introduced in Molina *et al.* (2008). We investigate necessary and sufficient conditions for its almost sure extinction and, assuming nonextinction, we derive some limiting results. The paper is organized as follow: In Section 2, we describe formally and we interpret intuitively the class of two-sex branching process we study. In Section 3, we state and discuss the main results. In order to achieve a more comprehensible reading of the paper, we include the proofs in Section 4.

2. The two-sex process

On a probability space $(\Omega, \mathcal{F}, P)$, let us consider the two-sex branching process $\{(F_n, M_n)\}_{n\geq 1}$ defined in the form:

$$(1)\qquad (F_n, M_n) = \sum_{i=1}^{Z_{n-1}} (f_{n,i;Z_{n-1}}, m_{n,i;Z_{n-1}}),\ Z_n = L_{Z_{n-1}}(F_n, M_n),\ n = 1, 2, \ldots$$

where the empty sum is considered to be $(0, 0)$. The random vector (F_n, M_n) represents the number of females and males in the nth generation. These females and males form Z_n couples. A couple is formed by one female and one male of the same generation who came together with the purpose of generating offspring. Initially, we assume that there is a positive number, N_0, of couples in the population, i.e., $Z_0 = N_0$. Let Z^+ and R^+ be, respectively, the nonnegative integer and real numbers. Given that, $Z_{n-1} = N$:

(a) $(f_{n,i;N}, m_{n,i;N})$, $i = 1, \ldots, N$ are independent and identically distributed nonnegative integer valued random vectors on $(\Omega, \mathcal{F}, P)$. Intuitively, $(f_{n,i;N}, m_{n,i;N})$ represents the number of females and males descending from the ith couple of the $(n-1)$th generation. Its probability law is referred as the offspring probability distribution when there are N progenitor couples in the population. When $N = 0$, it is clear that $P(f_{1,1;0} = 0, m_{1,1;0} = 0) = 1$.

(b) L_N is the function which governs the mating between females and males. It is a nonnegative real function, defined on $\mathrm{R}^+ \times \mathrm{R}^+$, assumed to be nondecreasing in each argument, integer valued on the integers, and such that, for $x, y \in \mathrm{R}^+$, $L_N(x, 0) = L_N(0, y) = 0$.

Process (1) may be interpreted as a branching model developing in an environment which changes stochastically in time according to the number of couples in the population. In each generation, both the offspring probability distribution and the mating function are affected by the number of couples in the previous generation. For certain animal populations, it is reasonable to assume that, by environmental, social, or other factors, the offspring and the mating between females and males may be affected by the number of couples in the population. Indeed, the motivation behind the class of processes considered in this paper is the interest in developing two-sex models to describe such behaviors. It is a general class of models which includes, as particular cases, the two-sex models introduced in Daley (1968), Molina *et al.* (2002) and Xing and Wang (2005).

Subsequently, in order to establish results about the extinction probability and the asymptotic behavior for the class of processes (1), we will introduce the following requirements on the mating functions and the offspring probability distributions:

$(a1)$: $\{L_N\}_{N \geq 0}$ is such that L_N is a superadditive function, namely,

$$L_N(x_1 + x_2, y_1 + y_2) \geq L_N(x_1, y_1) + L_N(x_2, y_2), \;\; x_i, y_i \in \mathrm{R}^+, \;\; i = 1, 2.$$

($a2$) For $x, y \in \mathrm{R}^+$ fixed, $\{L_N(x,y)\}_{N\geq 0}$ is a nondecreasing sequence.

($a3$) $f_{1,1;N} \preceq f_{1,1;N+1}$ and $m_{1,1;N} \preceq m_{1,1;N+1}$, $N \in \mathrm{Z}^+$.

Remark 2.1. Assumption ($a1$) expresses the intuitive notion that $x_1 + x_2$ females and $y_1 + y_2$ males coexisting together will form a number of couples that is at least as great as the total number of couples formed by x_1 females and y_1 males, and x_2 females and y_2 males, living separately. Most of mating functions considered in two-sex branching process theory are superadditive. Assumption ($a2$) represents the usual behavior in many biological populations in which the mating is promoted as the number of couples grows. Some classical sequences of mating functions verifying conditions ($a1$) and ($a2$) are, for example: $L_N(x,y) = x\min\{N, y\}$, or $L_N(x,y) = \min\{x, Ny\}$. According to ($a3$), the variables $f_{1,1;N}$ and $m_{1,1;N}$ take large values with a lower probability than $f_{1,1;N+1}$ and $m_{1,1;N+1}$ do, respectively. This expresses the intuitive fact that when the number of couples in the population grows then the numbers of originated females and males take large values with higher probabilities.

Throughout this work, we will assume the classical duality extinction-explosion in branching process theory, namely, for $N \geq 1$,

$$P\left(\lim_{n\nearrow\infty} Z_n = 0 \mid Z_0 = N\right) + P\left(\lim_{n\nearrow\infty} Z_n = \infty \mid Z_0 = N\right) = 1. \tag{2}$$

Under this framework, some general setting which guarantee (2) holds were investigated in Molina *et al.* (2008). Also, it was proved that the asymptotic growth rate $R = \lim_{N\to\infty} R_N$ exists where

$$R_N = N^{-1}E[Z_n \mid Z_{n-1} = N],\ N = 1, 2, \ldots$$

R_N represents the expected growth rate per couple when there are N couples in the population. Next, we continue the research about the class of two-sex branching processes presented in (1), investigating results concerning its extinction probability and asymptotic behavior.

Given the random variables X and Y, we say that X is stochastically smaller than Y, written $X \preceq Y$, if $P(X > t) \leq P(Y > t)$, $t \in \mathrm{R}$.

3. The main results

First, we provide some necessary and sufficient conditions for the almost sure extinction of the two-sex process. To this end, we will use two probabilistic approaches: (i) by considering the concept of asymptotic growth rate (Theorem 3.1) and (ii) through the stochastic comparison with a two-sex process with only mating depending on the number of couples (Theorem 3.2). Then, assuming the nonextinction, we derive some asymptotic results (Theorems 3.3 and 3.4).

Note that, if for some $n \geq 1$, $Z_n = 0$ then, from (1), one deduces that $(F_{n+m}, M_{n+m}) = (0,0)$ and $Z_{n+m} = 0$, $m \geq 1$. Hence, the two-sex process does not survive.

Definition 3.1. *Let* $Q_N = P\left(\lim_{n \nearrow \infty} Z_n = 0 \mid Z_0 = N\right)$ *be the extinction probability when initially there are* N *couples in the population,* $N \geq 1$.

Theorem 3.1. *Assume* $(a1)$, $(a2)$, *and* $(a3)$.

(i) *If* $R \leq 1$ *then* $Q_N = 1$ *for* $N \geq 1$.

(ii) *If* $R > 1$ *then there exists* $K_0 \geq 1$ *such that* $Q_N < 1$ *for* $N \geq K_0$.

In the next result, by using a methodology based in the stochastic comparison with a two-sex process with only mating depending on the number of couples in the population, necessary and sufficient conditions for the almost sure extinction of the two-sex process are also determined. To this end, we introduce the following modification in requirement $(a3)$:

$(a4)$: For $N \in \mathrm{Z}^+$, $f_{1,1;N} \preceq f_{1,1;N+1}$, $m_{1,1;N} \preceq m_{1,1;N+1}$ and there exist random variables $f_{1,1}$ and $m_{1,1}$ such that $\lim_{N \nearrow \infty} f_{1,1;N} = f_{1,1}$ and $\lim_{N \nearrow \infty} m_{1,1;N} = m_{1,1}$ almost surely.

Remark 3.1. From $(a4)$, by stochastic order properties, one deduces that $f_{1,1;N} \preceq f_{1,1}$ and $m_{1,1;N} \preceq m_{1,1}$, $N \in \mathrm{Z}^+$. Let us write by $(\mu_{f;N}, \mu_{m;N})$ and (μ_f, μ_m), respectively, the mean vectors of $(f_{1,1;N}, m_{1,1;N})$ and $(f_{1,1}, m_{1,1})$, both assumed to be finite. Again, by $(a4)$, $\{\mu_{f;N}\}_{N \geq 0}$ and $\{\mu_{m;N}\}_{N \geq 0}$ are nondecreasing sequences. By monotone convergence theorem, one derives that $\lim_{N \nearrow \infty} \mu_{f;N} = \mu_f$ and $\lim_{N \nearrow \infty} \mu_{m;N} = \mu_m$.

Let $\{(F_n^*, M_n^*)\}_{n \geq 1}$ be the two-sex process, initiated with $Z_0^* = N_0$ couples:

$$(3) \qquad (F_n^*, M_n^*) = \sum_{i=1}^{Z_{n-1}^*} (f_{n,i}, m_{n,i}), \;\; Z_n^* = L_{Z_{n-1}^*}(F_n^*, M_n^*), \;\; n = 1, 2, \ldots$$

where $(f_{n,i}, m_{n,i})$ are independent and identically distributed random vectors with the same probability distribution of $(f_{1,1}, m_{1,1})$.

Remark 3.2. Process (3) was studied in Molina *et al.*(2002). It was proved that $R^* = \lim_{k\nearrow\infty} R_k^*$ exists, with $R_k^* = k^{-1}E[Z_n^* \mid Z_{n-1}^* = k],\; k \geq 1$, and $R^* \leq 1$ if and only if $P\left(\lim_{n\nearrow\infty} Z_n^* = 0 \mid Z_0^* = N\right) = 1,\; N \geq 1$.

Theorem 3.2. *Assume* $(a1)$, $(a2)$, *and* $(a4)$.

(*i*) *If* $R^* \leq 1$ *then* $Q_N = 1$ *for* $N \geq 1$.

(*ii*) *If* $R^* > 1$ *then there exists* $K_0 \geq 1$ *such that* $Q_N < 1$ *for* $N \geq K_0$.

Remark 3.3. Note that assumption $(a4)$ is stronger than $(a3)$, so Theorem 3.2 is more restrictive than Theorem 3.1. However, if $(a4)$ holds then, in order to prove the almost sure extinction of the process $\{(F_n, M_n)\}_{n\geq 1}$, the sufficient condition given in Theorem 3.2 is easier to check than that provided in Theorem 3.1.

From now on, we will assume N_0 large enough such that:

$$P\left(\lim_{n\nearrow\infty} Z_n = \infty \mid Z_0 = N_0\right) > 0 \quad \text{and} \quad P\left(\lim_{n\nearrow\infty} Z_n^* = \infty \mid Z_0^* = N_0\right) > 0$$

It can be verified that the sequences $\{W_n\}_{n\geq 0}$, $W_n = R^{-n}Z_n$, and $\{W_n^*\}_{n\geq 0}$, $W_n^* = {R^*}^{-n} Z_n^*$, are nonnegative supermartingales relative to the families of σ-algebras $\{\sigma(Z_0, \ldots, Z_n)\}_{n\geq 0}$ and $\{\sigma(Z_0^*, \ldots, Z_n^*)\}_{n\geq 0}$, respectively. Hence, it is derived that there exist nonnegative and finite random variables W and W^* such that $\{W_n\}_{n\geq 0}$ and $\{W_n^*\}_{n\geq 0}$ converge almost surely to W and W^*, respectively.

Theorem 3.3. *Assume* $(a1)$, $(a2)$, *and* $(a4)$. *If* $\{W_n^*\}_{n\geq 0}$ *converges in* L^p *to* W^*, *for some* $p > 0$, *then* $\{W_n\}_{n\geq 0}$ *converges in* L^α *to* W, *for* $\alpha \in (0, p)$.

Remark 3.4. Sufficient conditions for the convergence of $\{W_n^*\}_{n\geq 0}$ to W^* in L^p, for $p = 1$ and $p = 2$, were investigated in Molina *et al.* (2004, 2006). According to Theorem 3.3, such conditions will also be sufficient in order to derive that $\{W_n\}_{n\geq 0}$ converges to W in L^α, for $\alpha \in (0,1)$ and $\alpha \in (0,2)$, respectively.

Next result establishes sufficient conditions which guarantee that W is a non-degenerate at 0 random variable. Let $\{\varepsilon_N\}_{N\geq 1}$ where $\varepsilon_N = R - R_N$.

Theorem 3.4. *Assume* $(a1)$, $(a2)$, *and* $(a4)$. *If* $\{\varepsilon_N\}_{N\geq 1}$ *is nonincreasing and* $\sum_{N=1}^{\infty} N^{-1}\varepsilon_N < \infty$ *then,* $\lim_{n\nearrow\infty} E[W_n \mid Z_0 = N_0] > 0$.

4. Proofs

Proof of Theorem 3.1. From $(a1)$, $(a2)$, and $(a3)$, one deduces, see Molina *et al.* (2008), that $R = \sup_{N\geq 1} R_N$.

(i) Assume $R \leq 1$. Then, for $n \in \mathrm{Z}^+$,

$$E[Z_{n+1}] = E\left[E[Z_{n+1} \mid Z_n]\ \right] = E\left[Z_n R_{Z_n}\right] \leq E\left[Z_n R\right] \leq E[Z_n].$$

Hence,

$$P\left(\lim_{n\nearrow\infty} Z_n = \infty \mid Z_0 = N\right) = 0, \quad N \geq 1.$$

By (2), $Q_N = 1,\ N \geq 1$.

(ii) Assume $R > 1$. Since $R = \lim_{N\nearrow\infty} R_N$, there exists a positive integer K such that, for $N \geq K$, $R_N > 1$.

Let $\{Z'_n\}_{n\geq 0}$ be the process defined in the form:

$$Z'_0 = N_0, \quad Z'_n = Z_n I_{\{Z'_{n-1}\leq\ K\}} + L_K(F'_n, M'_n) I_{\{Z'_{n-1} >\ K\}}, \quad n = 1, 2, \ldots$$

where

$$(F'_n, M'_n) = (F_n, M_n) I_{\{Z'_{n-1}\leq\ K\}} + \sum_{i=1}^{Z'_{n-1}} (f_{n,i;K}, m_{n,i;K}) I_{\{Z'_{n-1} >\ K\}},$$

I_A denoting the indicator function of the set A. It can be verified that $Z'_n \preceq Z_n,\ n \in \mathrm{Z}^+$. Thus, see Müller and Stoyan (2002), p. 3, one deduces, for $N \geq 1$,

$$P\left(\lim_{n\nearrow\infty} Z_n=\infty \mid Z_0=N\right) \geq P\left(\lim_{n\nearrow\infty} Z'_n=\infty \mid Z'_0=N\right). \tag{4}$$

Let $\left\{(F_n^{(K)}, M_n^{(K)})\right\}_{n\geq 1}$ be the two-sex process:

$$\left(F_n^{(K)}, M_n^{(K)}\right)=\sum_{i=1}^{Z_{n-1}^{(K)}}\left(f_{n,i;K}, m_{n,i;K}\right),\ Z_n^{(K)}=L_K\left(F_n^{(K)}, M_n^{(K)}\right),\ n=1,2,\ldots$$

with $Z_0^{(K)}=N_0$.
By Daley *et al.* (1986), one has that $R^{(K)}=\lim_{N\nearrow\infty} R_N^{(K)}=\sup_{N\geq 1} R_N^{(K)}$, where

$$R_N^{(K)}=N^{-1}E\left[Z_n^{(K)} \mid Z_{n-1}^{(K)}=N\right],\quad N=1,2,\ldots$$

Clearly $R^{(K)}\geq R_K^{(K)}$. Now,

$$R_K^{(K)}=K^{-1}E\left[Z_n^{(K)} \mid Z_{n-1}^{(K)}=K\right]=K^{-1}E\left[Z_n \mid Z_{n-1}=K\right]=R_K>1.$$

Consequently, $R^{(K)}>1$. By bisexual Galton-Watson process theory, there exists $K^*\in \mathrm{Z}^+$ such that, for $N\geq K^*$,

$$P\left(\lim_{n\nearrow\infty} Z_n^{(K)}=\infty \mid Z_0^{(K)}=N\right)>0.$$

Taking $K_0=\max\{K,\ K^*\}$,

$$P\left(\lim_{n\nearrow\infty} Z_n^{(K)}=\infty,\ Z_n^{(K)}\geq K,\ n\geq 1 \mid Z_0^{(K)}=K_0\right)>0.$$

Hence,

$$P\left(\lim_{n\nearrow\infty} Z'_n=\infty \mid Z'_0=K_0\right)>0. \tag{5}$$

From (4) and (5),

$$P\left(\lim_{n\nearrow\infty} Z_n=\infty \mid Z_0=N\right)>0,\ N\geq K_0.$$

Finally, by (2), ones derives that $Q_N < 1$ for $N \geq K_0$. $\square$

P r o o f o f T h e o r e m 3.2. It is sufficient to prove that, under conditions in Theorem 3.2, $R = R^*$.

For each $N \in \mathrm{Z}^+$, let $\left\{(F_n^{(N)}, M_n^{(N)})\right\}_{n\geq 1}$ be the process defined, for $n \geq 1$, in the form:

$$\left(F_n^{(N)}, M_n^{(N)}\right) = \sum_{i=1}^{Z_{n-1}^{(N)}} (f_{n,i;N}, m_{n,i;N}),\ Z_n^{(N)} = L_{Z_{n-1}^{(N)}}\left(F_n^{(N)}, M_n^{(N)}\right).$$

where $Z_0^{(N)} = N_0$. It is a two-sex process with only mating depending on the number of couples, being the offspring probability distribution the law of $(f_{1,1;N}, m_{1,1;N})$. Hence, for $N \in \mathrm{Z}^+$, there exists $R^{(N)} = \lim_{k\nearrow\infty} R_k^{(N)}$ and,

$$R^{(N)} = \sup_{k\geq 1} R_k^{(N)},\ R_k^{(N)} = k^{-1} E[Z_n^{(N)} \mid Z_{n-1}^{(N)} = k],\ k = 1, 2, \ldots$$

Taking into account $(a4)$, by stochastic order properties (see Müller and Stoyan (2002)),

$$E\left[L_N\left(\sum_{i=1}^N f_{n,i;N}, \sum_{i=1}^N m_{n,i;N}\right)\right] \leq E\left[L_N\left(\sum_{i=1}^N f_{n,i}, \sum_{i=1}^N m_{n,i}\right)\right].$$

Therefore,

$$R = \limsup_{N\nearrow\infty} R_N \leq \limsup_{N\nearrow\infty} R_N^* = R^*.$$

On the other hand, given $j \geq 1$ fixed, for $N \geq j$,

$$E\left[L_N\left(\sum_{i=1}^N f_{n,i;N}, \sum_{i=1}^N m_{n,i;N}\right)\right] \geq E\left[L_N\left(\sum_{i=1}^N f_{n,i;j}, \sum_{i=1}^N m_{n,i;j}\right)\right]$$

Thus,

$$R = \liminf_{N\nearrow\infty} R_N \geq \liminf_{N\nearrow\infty} R_N^{(j)} = R^{(j)}$$

Taking limit, as $j \nearrow \infty$, $R \geq \lim_{j\nearrow\infty} R^{(j)}$.

Finally, it is matter of straightforward calculation to deduce that

$$\lim_{j\nearrow\infty} R^{(j)} = R^*.$$

$\square$

P r o o f o f T h e o r e m 3.3. First, we will proved that, under conditions in Theorem 3.3, if ϕ is an increasing function then $E[\phi(W_n)] \leq E[\phi(W_n^*)]$, $n \in$ Z$^+$, whenever such expectations exist. In fact, by ($a4$) and using that L_{N_0} is monotonic nondecreasing in each argument,

$$L_{N_0}\left(\sum_{i=1}^{N_0} f_{1,i;N_0}, \sum_{i=1}^{N_0} m_{1,i;N_0}\right) \preceq L_{N_0}\left(\sum_{i=1}^{N_0} f_{1,i}, \sum_{i=1}^{N_0} m_{1,i}\right).$$

Hence,

$$P(Z_1 > t \mid Z_0 = N_0) \leq P(Z_1^* > t \mid Z_0^* = N_0),\ t \in \mathbb{R}.$$

Now, by ($a1$), ($a2$), and ($a4$), ones derives, see Molina *et al.* (2008), that $\{Z_n\}_{n\geq 0}$ and $\{Z_n^*\}_{n\geq 0}$ are stochastically monotone sequences, namely, given $N_1, N_2 \in \mathbb{Z}^+$ with $N_1 < N_2$, it is verified, for $t \in \mathbb{R}$ and $n \geq 1$,

$$P(Z_n \leq t \mid Z_{n-1} = N_2) \leq P(Z_n \leq t \mid Z_{n-1} = N_1)$$
$$P(Z_n^* \leq t \mid Z_{n-1}^* = N_2) \leq P(Z_n^* \leq t \mid Z_{n-1}^* = N_1).$$

Thus, see Daley (1968b) for details, for $n \geq 2$,

$$P(Z_n > t \mid Z_0 = N_0) \leq P(Z_n^* > t \mid Z_0^* = N_0),\ t \in \mathbb{R}.$$

Therefore, given that $Z_0 = N_0$, $Z_n \preceq Z_n^*$, $n \in \mathbb{Z}^*$ and using the fact that, under assumptions in Theorem 3.3, $R = R^*$, one obtains that $W_n \preceq W_n^*$. Taking into account that ϕ is an increasing function, $\phi(W_n) \preceq \phi(W_n^*)$ which implies that $E[\phi(W_n)] \leq E[\phi(W_n^*)]$ whenever such expected values exist.

We now prove the Theorem.

If $\{W_n^*\}_{n\geq 0}$ converges in L^p to W^*, for some $p > 0$,

$$\lim_{n\nearrow\infty} E[(W_n^*)^p] = E[(W^*)^p] < \infty.$$

Thus, $\sup_{n\geq 0} E[(W_n^*)^p] < \infty$. By previous result, $\sup_{n\geq 0} E[(W_n)^p] < \infty$.

Now, by Proposition A1 (see Appendix), for $\alpha \in (0, p)$, $\{W_n\}_{n\geq 0}$ is αth-order uniformly integrable, that is, $\{(W_n)^\alpha\}_{n\geq 0}$ is uniformly integrable.

Finally, using that $\{W_n\}_{n\geq 0}$ converges almost surely to W, by Proposition A2 in Appendix, one derives that $\{W_n\}_{n\nearrow\infty}$ converges in L^α to W, for $\alpha \in (0,p)$. $\square$

P r o o f o f T h e o r e m 3.4. It is proved by applying a similar methodology, suitable adapted to the class of processes (1), to that considered for the two-sex process with only mating depending on the number of couples in the population (see Theorem 7 in Molina *et al.* (2004)).

Appendix

Proposition A1. *Let* $\{X_\alpha : \alpha \in A\}$ *be a system of extended real valued random variables on a probability space* $(\Omega, \mathcal{F}, P)$*. If* $\sup_{\alpha\in A} \|X_\alpha\|_{p_0} < \infty$ *for some* $p_0 \in (0,\infty)$*, then* $\{X_\alpha : \alpha \in A\}$ *is pth-order uniformly integrable, that is,* $\{|X|^p : \alpha \in A\}$ *is uniformly integrable for every* $p \in (0, p_0)$*.*

Proposition A2. *Let* $X_n \in L_p(\Omega, \mathcal{F}, P)$, $n \in \mathbb{Z}^+$ *where* $p \in (0,\infty)$*. let* X *be an extended real valued random variable on* $(\Omega, \mathcal{F}, P)$ *and assume that* $\lim_{n\nearrow\infty} X_n = X$ *in probability. Then the following three conditions are equivalent:*

(*a*) $\{X_n : n \in \mathbb{Z}^+\}$ *is pth-order uniformly integrable.*

(*b*) $X \in L_p(\Omega, \mathcal{F}, P)$ *and* $\lim_{n\nearrow\infty} \|X_n - X\|_p = 0$.

(*c*) $X \in L_p(\Omega, \mathcal{F}, P)$ *and* $\lim_{n\nearrow\infty} \|X_n\|_p = \|X\|_p$.

where, given a random variable Y *on* $(\Omega, \mathcal{F}, P)$, $\|Y\|_p = (\int_\Omega |Y|^p dP)^{1/p}$.

P r o o f s. We refer the reader to the Theorems 4.12 and 4.16 in Yeh (1995).

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