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TWO-TYPE AGE-DEPENDENT BRANCHING PROCESSES WITH INHOMOGENEOUS IMMIGRATION AS MODELS OF RENEWING CELL POPULATION*

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Two-type reducible age-dependent branching processes with inhomogeneous immigration are considered to describe the kinetics of renewing cell populations. This class of processes can be used to model the generation of oligodendrocytes in the central nervous system *in vivo* or the kinetics of leukemia cells. The asymptotic behavior of the first and second moments, including the correlation, of the process is investigated.

1. Introduction

Biological problems have motivated a vast body of work on the theory of branching processes, including the first asymptotic result which is attributed to Kolmogorov [22]. Kolmogorov coined also the terminology "branching process"in 1946 when he established the famous seminar devoted to Branching Processes at Moscow State University. This seminar lead to numerous developments of the theory of branching processes and their multitype extensions due to himself and his Ph.D. students (see e.g. Sevastyanov [25]). Overviews of the theory of branching processes can be found also in Harris [10], Mode [24], Athreya and Ney

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[1], and Jagers [19], whereas comprehensive expositions of branching processes as applied to biology can be found in Jagers [19], Yakovlev and Yanev [32], Kimmel and Axelrod [21], and Haccou *et al.* [9]. References dealing with statistical inference for branching processes include Guttorp [8] and Yanev [34].

The objective of this paper is to investigate asymptotic properties for a class of two-type reducible age-dependent branching processes. We consider branching processes both with and without immigration. The immigration component is formulated as a non-homogeneous Poisson process but the time-homogeneous case is also investigated in detail. This work is a generalization of previous work by Hyrien and Yanev [16] who considered similar processes with a single type of cells.

The process under consideration can be used to model the kinetics of renewing cell populations that consist of two (observable) cell types. Two examples of cellular systems that can be modeled using this process include the pool of oligodendrocytes and of their progenitor cells, which play an important role in the central nervous system, and the leukemia progenitor and blast cells. It is worth noting that Yanev *et al.* [33] investigated a two type Markov branching process with homogeneous Poisson immigration to model the proliferation of leukemia cells. The results presented herein offers an extension of this previous work to age-dependent processes.

The paper is organized as follows. The biological background is presented in *Section 2.* This motivating material will lead to the construction of the general process in *Section 3* where the basic integral equations for the p.g.f. and for the means are derived. Some asymptotic results for renewal-type equations and for the means of the process submitted in Hyrien and Yanev [17] are presented in *Section 4.* These results are applied in *Section 5* to investigate the asymptotic behaviour of the second order moments and the correlations for the processes with or without immigration. This study finds also motivation in parameter estimation using asymptotic approximations to the moments.

2. Biological background and motivation

Recent advances in experimental techniques have made it possible to collect unprecedented information about the state of individual cells isolated from dissociated tissues. For instance, the power of high-thoughput flow cytometry allows experimentalists to identify different cell types by combining together expression levels of multiple surface or intra-cellular protein or DNA markers measured simultaneously in thousands or millions of individual cells. When collected repeatedly over time, such data provide snapshots about the temporal organization of multitype cellular populations (e.g., Hyrien and Zand [18], Hyrien, Chen and Zand [13]).

One cellular system that can be studied using this experimental setup is that of oligodendrocytes, the myelin-forming cells of the central nervous system, and their immediate progenitor cells, called oligodendrocyte type-2 astrocyte progenitor cells (thereafter simply referred to as O2A-OPCs). This cellular system has been extensively studied using multitype age-dependent branching processes (Yakovlev *et al.* [27, 28, 29], von Collani *et al.* [6], Boucher *et al.* [2, 3], Zorin *et al.* [35], Hyrien *et al.* [11 - 15, 17], Chen *et al.* [4]). These publications dealt with analyses of *in vitro* experiments where the generation of oligodendrocytes was observed at the clonal level in purified cultures of O2A-OPCs, and the proposed models did not account for the influx of precursor/stem cells into the pool of O2A-OPCs. In order to investigate the processes of division and differentiation of these cells *in vivo*, we will propose a two-type age-dependent branching process with immigration. In this model, O2A-OPCs and oligodendrocytes are referred to as type-1 and type-2 cells, respectively.

A second example is provided by the progression of leukemia. The initiation and perpetuation of leukemia is believed to rest on a pool of leukemic stem cell (LSC), whereas the propagation of leukemic disease itself is ensured by the immediate downstream progeny of LSC, the leukemic progenitor (LP) population. Although stem cells are not always observable, it is generally possible to experimentally quantify the number of LP cells (type-1 cells) and of their further differentiated progenies, referred to as blast cells (BC) and corresponding to type-2 cells in our model. The influx of stem cells into the pool of LP cells can be modeled as an immigration process.

Motivated by the above examples, we investigated some asymptotic properties of a class of two-type reducible age-dependent branching processes with time-inhomogeneous immigration. One of the ultimate goal of this work is to develop associated statistical methods to estimate cell kinetics parameters using flow cytometry experiments. The process that we will consider is general enough so it includes models of the generation of oligodendrocytes and of the progression of leukemia as particular cases.

3. Branching process models and equations

The kinetics of two-type renewing cell populations (*in vivo)* is described by a branching process that starts off when the first zero age immigrant (*stem cells*) appears in the population of type-1 (*progenitors*) cells. The lifespan of every type-1 cell is modeled by a r.v. τ_1 with c.d.f. $G_1(t) = P(\tau_1 \le t)$, satisfying $G_1(0+) = 0$.

Upon completion of its lifespan every type-1 cell produces a random number of offsprings $\xi = (\xi_1, \xi_2)$, where ξ_k, $k = 1, 2$, denotes the number of type-k cells arising from any type-1 cell. Let $h_1(s_1, s_2) = E\{s_1^{\xi_1} s_2^{\xi_2}\}$, $| s_i |\leq 1$, $i = 1, 2$, denote the p.g.f. of ξ. Notice that $h_1(1, 1) = 1$. The lifespan of every type-2 cell is modeled by a r.v. τ_2 with c.d.f. $G_2(t) = P(\tau_2 \leq t)$, satisfying $G_2(0+) = 0$. Upon completion of its lifespan, every type-2 cells generates a random number η_2 of type-2 cells. Write $h_2(s_2) = E\{s_2^{\eta_2}\}$ for the p.g.f. of η_2. We assume that every cell is of zero age at birth, and that all cells complete their evolutions independently of every other cell. The above formulated process is therefore an age-dependent branching process (without immigration).

Let us introduce the following notation associated with the offspring distribution:

$$a_1 = E\xi_1 = \left.\frac{\partial h_1(s_1, s_2)}{\partial s_1}\right|_{s_1=s_2=1}, a_2 = E\xi_2 = \left.\frac{\partial h_1(s_1, s_2)}{\partial s_2}\right|_{s_1=s_2=1},$$

$$a_{11} = E\xi_1(\xi_1 - 1) = \left.\frac{\partial^2 h_1(s_1, s_2)}{\partial s_1^2}\right|_{s_1=s_2=1}, a_{12} = E\xi_1\xi_2 = \left.\frac{\partial^2 h_1(s_1, s_2)}{\partial s_1 \partial s_2}\right|_{s_1=s_2=1},$$

$$a_{22} = E\xi_2(\xi_2 - 1) = \left.\frac{\partial^2 h_1(s_1, s_2)}{\partial s_2^2}\right|_{s_1=s_2=1},$$

$$b_2 = E\eta_2 = \left.\frac{dh_2(s_2)}{ds_2}\right|_{s_2=1}, b_{22} = E\eta_2(\eta_2 - 1) = \left.\frac{d^2 h_2(s_2)}{ds_2^2}\right|_{s_2=1}.$$

Let

$$\mu_1 = \int_0^\infty x dG_1(x) < \infty \quad \text{and} \quad \mu_2 = \int_0^\infty x dG_2(x) < \infty$$

denote the lifespan means, assumed finite. The above defined parameters will play a critical role in the asymptotic behavior of the process.

From a biological standpoint, the most relevant models (for the offspring p.g.f. $h_1(s_1, s_2)$) include the following cases:

- *Model 1*: $h_1(s_1, s_2) = p_0 + p_1 s_1^2 + p_2 s_2$. Under this model, upon completion of its lifespan, every type-1 cell can either die with probability p_0, or it divides into two type-1 cells with probability p_1, or it differentiates into a single type-2 cell with probability p_2.

- *Model 2*: $h_1(s_1, s_2) = p_0 + p_1 s_1^2 + p_2 s_2^2$;
- *Model 3*: $h_1(s_1, s_2) = p_0 + p_1 s_1^2 + p_2 s_1 s_2$;
- *Model 4*: $h_1(s_1, s_2) = p_0 + p_1 s_1^2 + p_2 s_2^2 + p_3 s_1 s_2$.

The offspring p.g.f. $h_2(s_2)$ for type-2 cells is generally assumed to take the form $h_2(s_2) = 1 - q + q s_2^2$, $0 \le q \le 1$. This p.g.f. is interpreted as follows: upon completion of its lifespan, every type-2 cell either dies with probability $1 - q$ or it divides into two new type-2 cells with probability q.

Let us define a stochastic process $\mathbf{Z}(t) = (Z_1(t), Z_2(t))$, where $Z_1(t)$ and $Z_2(t)$ denote the numbers of type-1 and type-2 cells at any time $t \ge 0$. Introduce the associated p.g.f.s

$F_1(t; s_1, s_2) = E\{s_1^{Z_1(t)} s_2^{Z_2(t)} \mid Z_1(0) = 1\}, F_2(t; s_2) = E\{s_2^{Z_2(t)} \mid Z_2(0) = 1\}$.

Under the above assumptions, it is not difficult to show that $F_1(t; s_1, s_2)$ and $F_2(t; s_1, s_2)$ are determined by the following system of nonlinear integral equations (e.g., Harris [10] and Athreya and Ney [1])

$$(1)\quad F_1(t; s_1, s_2) = \int_0^t h_1(F_1(t-u; s_1, s_2), F_2(t-u; s_2)) dG_1(u) + s_1(1 - G_1(t)),$$

$$(2)\qquad F_2(t; s_2) = \int_0^t h_2(F_2(t-u; s_2)) dG_2(u) + s_2(1 - G_2(t)),$$

with initial conditions $F_1(0; s_1, s_2) = s_1$ and $F_2(0; s_2) = s_2$. Let us define the associated expectations

$$A_1(t) = E\{Z_1(t) \mid Z_1(0) = 1\} = \frac{\partial}{\partial s_1} F_1(t; s_1, s_2) \mid_{s_1 = s_2 = 1},$$

$$A_2(t) = E\{Z_2(t) \mid Z_1(0) = 1\} = \frac{\partial}{\partial s_2} F_1(t; s_1, s_2) \mid_{s_1 = s_2 = 1},$$

$$B_2(t) = E\{Z_2(t) \mid Z_2(0) = 1\} = \frac{\partial}{\partial s_2} F_2(t; s_2) \mid_{s_2 = 1} .$$

It is not difficult to deduce from (1) and (2) the following system of renewal-type equations:

$$(3)\qquad A_1(t) = a_1 \int_0^t A_1(t-u) dG_1(u) + 1 - G_1(t),$$

$$(4) \qquad A_2(t) = a_1 \int_0^t A_2(t-u)dG_1(u) + a_2 \int_0^t B_2(t-u)dG_1(u),$$

$$(5) \qquad B_2(t) = b_2 \int_0^t B_2(t-u)dG_2(u) + 1 - G_2(t),$$

with the initial conditions: $A_1(0) = 1$, $A_2(0) = 0$, and $B_2(0) = 1$.

To describe the process with immigration, we let $0 = S_0 < S_1 < S_2 < S_3 < ...$ denote the random time points at which immigrants arrive in the pool of type-1 cells. We assume that this sequence forms a **non-homogeneous Poisson process** $\Pi(t)$ with rate $r(t)$ such that the cumulative rate of the process is given by $R(t) = \int_0^t r(u)du$, $r(t) \geq 0$, and $\Pi(t) \in Po(R(t))$. Let $U_i = S_i - S_{i-1}$ denote the inter-arrival times. Notice that $S_k = \sum_{i=1}^{k} U_i$, $k = 1, 2 \cdots$. We will assume also that associated with every time point S_k is an immigration component I_k denoting the number of immigrants arriving in the population of type-1 cells at time S_k. These immigrants are assumed to be of zero age upon arriving in the pool of type-1 cells. The collection $\{I_k\}_{k=1,2\cdots}$ form a sequence of independent and identically distributed (i.i.d.) random variables (r.v.) with p.g.f. $g(s) = Es^{I_k} = \sum_{i=0}^{\infty} g_i s^i$, $\mid s \mid \leq 1$.

Notice that if $\{U_i\}$ are i.i.d. exponentially distributed r.v. with c.d.f. $G_0(x) = P(U_i \leq x) = 1 - e^{-rx}$, $x \geq 0$, the immigration process $\Pi(t)$ reduces to an ordinary Poisson process with cumulative rate $R(t) = rt$.

Let $Y_1(t)$ and $Y_2(t)$ denote the number of type-1 cells at time t in the process with immigration. Put $\mathbf{Y}(t) = (Y_1(t), Y_2(t))$ and assume that $\mathbf{Y}(0) = (0, 0)$. Therefore, $\mathbf{Y}(t)$ admits the following representation

$$(6) \qquad \mathbf{Y}(t) = \begin{cases} \sum_{k=1}^{\Pi(t)} \mathbf{Z}^{I_k}(t - S_k) & \text{if } \Pi(t) > 0 \\ 0 & \text{if } \Pi(t) = 0, \end{cases}$$

where $\mathbf{Z}^{I_k}(t)$ denotes i.i.d. copies of the branching processes $\mathbf{Z}(t) = (Z_1(t), Z_2(t))$ started with a random number of ancestors I_k. In fact, the process $\mathbf{Y}(t)$ begins from the first non-zero immigrants. The process $\mathbf{Y}(t)$, $t \geq 0$, is time non-homogeneous and non-Markov process.

Define the p.g.f. $\Psi(t; s_1, s_2) = E\{s_1^{Y_1(t)} s_2^{Y_2(t)} \mid \mathbf{Y}(0) = (0, 0)\}$. It follows from (6) that

$$(7) \qquad \Psi(t; s_1, s_2) = \exp\{-\int_0^t r(t-u)[1 - g(F_1(u; s_1, s_2))]du\},$$

where $\Psi(0; s_1, s_2) = 1$ and where the p.g.f. $F_1(u; s_1, s_2)$ satisfies equation (1). The proof of identity (7) is similar to that for the one-dimensional case by Yakovlev and Yanev ([32], Theorem 1).

Introduce the expectations of the process with immigration

$$(8) \qquad M_1(t) = E\{Y_1(t) \mid \mathbf{Y}(0) = (0,0)\} = \frac{\partial}{\partial s_1}\Psi(t; s_1, s_2) \mid_{s_1=s_2=1},$$

$$(9) \qquad M_2(t) = E\{Y_2(t) \mid \mathbf{Y}(0) = (0,0)\} = \frac{\partial}{\partial s_2}\Psi(t; s_1, s_2) \mid_{s_1=s_2=1}.$$

Let $\gamma = E\{I_k\} = \left.\frac{dg(s)}{ds}\right|_{s=1}$ denote the immigration mean, and let $\gamma_2 = \left.\frac{d^2g(s)}{ds^2}\right|_{s=1} = E\{I_k(I_k-1)\}$. It follows from (7) that

$$(10) \qquad M_1(t) = \gamma \int_0^t r(t-u)A_1(u)du,$$

$$(11) \qquad M_2(t) = \gamma \int_0^t r(t-u)A_2(u)du,$$

with $M_1(0) = 0$ and $M_2(0) = 0$ for the initial conditions, and where $A_1(t)$ and $A_2(t)$ are determined by equations (3) – (5).

4. Renewal type equations and asymptotic behaviour of the means

The moments of the process without immigration satisfy renewal equations assuming the general form

$$(12) \qquad U(t) = \varkappa \int_0^t U(t-x)dG(x) + f(t),$$

where $\varkappa$ denotes a strictly positive constant, where $G(x)$ denotes a c.d.f. with Laplace transform $\widehat{G}(\lambda) = \int_0^\infty e^{-\lambda x} dG(x)$.

Recall that when $\varkappa G(0+) < 1$, equation (12) admits a unique solution that is bounded on bounded intervals. The associated Malthus parameter α is defined as the root to the equation $\varkappa\widehat{G}(\alpha) = 1$. This parameters governs the asymptotic behavior of the function $U(t)$. When $\varkappa \geq 1$, the equation $\varkappa\widehat{G}(\alpha) = 1$ has a unique real solution $\alpha \geq 0$. When $\varkappa < 1$ a solution may not exist, but if it does it has to be negative. In what follows, we will assume that the Malthus parameter always exists. Introduce the c.d.f. $\widetilde{G}(t) = \varkappa \int_0^t e^{-\alpha x} dG(x)$, and define the means $\mu = \int_0^\infty x dG(x)$ and $\widetilde{\mu} = \int_0^\infty x d\widetilde{G}(x)$. Whenever needed, we will implicitly assume that they are finite.

The following theorem includes classical asymptotic results for renewal processes (see e.g. Feller [7]).

Theorem 4.1. *(i) Let $\varkappa = 1$ and $f(t)$ be a directly Riemann integrable (d.R.i) function. Then $\lim_{t\to\infty} U(t) = \int_0^\infty f(x)dx/\mu$.*

(ii) Let $\varkappa = 1$ and $\lim_{t\to\infty} f(t) = C$, $0 < C < \infty$. Then $U(t) \sim Ct/\mu$ as $t \to \infty$.

(iii) Let $\varkappa < 1$ and $\lim_{t\to\infty} f(t) = C$, $0 < C < \infty$. Then $\lim_{t\to\infty} U(t) = C/(1-\varkappa)$.

Note that case (i) of Theorem 4.1 is well-known as *Key Renewal Theorem*. Some further developments (including the case $\mu = \infty$) are given in Mitov and Yanev [23].

To investigate the asymptotic behavior of the first and second order moments of the process, we will also need the following results derived by Hyrien and Yanev [17].

Theorem 4.2. *Assume that $f(t) \sim Ct^\rho e^{\beta t}$, as $t \to \infty$, where $0 < C < \infty$ and $\rho \geq 0$. Then we have*

(a) If $\alpha < \beta$, then $U(t) \sim Ct^\rho e^{\beta t}/[1 - \varkappa\widehat{G}(\beta)]$;

(b) If $\alpha > \beta$, then $U(t) \sim e^{\alpha t} \int_0^\infty f(x)e^{-\alpha x}dx/\widetilde{\mu}$;

(c) If $\alpha = \beta$, then $U(t) \sim Ct^{\rho+1}e^{\alpha t}/\widetilde{\mu}(\rho+1)$.

Define the Malthus parameters α_1 and α_2 that solve the equations

$$a_1 \int_0^\infty e^{-\alpha_1 x} dG_1(x) = 1, b_2 \int_0^\infty e^{-\alpha_2 x} dG_2(x) = 1. \tag{13}$$

We will classify the one-dimensional processes $Z_i(t)$, $i = 1, 2$, as subcritical if $\alpha_i < 0$ ($a_1 < 1$ or $b_2 < 1$), critical if $\alpha_i = 0$ ($a_1 = 1$ or $b_2 = 1$) and supercritical if $\alpha_i > 0$ ($a_1 > 1$ or $b_2 > 1$) (see e.g. Harris (1963) or Athreya and Ney (1972)).

It is well known that in the critical case $A_1(t) \equiv 1$ and $B_2(t) \equiv 1$, whereas in the non-critical processes we have

$$(14) \qquad A_1(t) \sim C_1 e^{\alpha_1 t} \quad \text{and} \quad B_2(t) \sim C_2 e^{\alpha_2 t}, \qquad \text{as } t \to \infty,$$

where

$$(15) \qquad C_1 = (a_1 - 1)/\alpha_1 a_1 \widetilde{\mu}_1 < \infty \quad \text{and} \quad C_2 = (b_2 - 1)/\alpha_2 b_2 \widetilde{\mu}_2 < \infty,$$

assuming that the corresponding integrals

$$(16) \qquad \widetilde{\mu}_1 = a_1 \int_0^\infty x e^{-\alpha_1 x} dG_1(x) = \int_0^\infty x d\widetilde{G}_1(x),$$

$$(17) \qquad \widetilde{\mu}_2 = b_2 \int_0^\infty x e^{-\alpha_2 x} dG_2(x) = \int_0^\infty x d\widetilde{G}_2(x)$$

are finite. Notice that $\widetilde{G}_1(\infty) = \widetilde{G}_2(\infty) = 1$ which is direct consequence of equation (13).

The two-type process $(Z_1(t), Z_2(t))$ is of reducible type, which complicates the asymptotic behaviour of $A_2(t)$. In particular, as shown by Hyrien and Yanev (2010), it depends on both Malthus parameters:

Theorem 4.3. *Assume* (13) *and* (15) *are satisfied. Then, as* $t \to \infty$, *we have that*

	$\sim$	$K_1 t e^{\alpha_1 t}$	$\delta = 0$	$K_1 = a_2 C_2 / a_1 \widetilde{\mu}_1,$
$A_2(t)$	$\sim$	$K_2 e^{\alpha_1 t}$	$\delta < 0$	$K_2 = \dfrac{a_2}{a_1 \widetilde{\mu}_1} \displaystyle\int_0^\infty e^{\delta x} \overline{B}_2(x) dx,$
	$\sim$	$K_3 e^{\alpha_2 t}$	$\delta > 0$	$K_3 = a_2 C_2 \widehat{G}_1(\alpha_2) / [1 - a_1\ \widehat{G}_1(\alpha_2)]$

where $\delta = \alpha_2 - \alpha_1$, $\overline{B}_2(t) = e^{-\alpha_2 t} B_2(t)$ *and* $\widehat{G}_1(\alpha_2) = \int_0^\infty e^{-\alpha_2 x} dG_1(x)$.

Corollary 4.1 (Markov case). *When* $G_1(t) = 1 - e^{-\beta_1 t}$ *and* $G_2(t) = 1 - e^{-\beta_2 t}$, *the Malthus parameters are given by* $\alpha_1 = \beta_1(a_1 - 1)$ *and* $\alpha_2 = \beta_2(b_2 - 1)$. *It follows from* (3)–(5) *that* $A_1(t) = e^{\alpha_1 t}$ *and* $B_2(t) = e^{\alpha_2 t}$. *Similarly* $A_2(t) =$

$a_2\beta_1 te^{\alpha_1 t}$ *when* $\alpha_1 = \alpha_2$ *and* $A_2(t) = \dfrac{a_2\beta_1}{\alpha_2 - \alpha_1}(e^{\alpha_2 t} - e^{\alpha_1 t})$ *when* $\alpha_1 \neq \alpha_2$. *Therefore, in accordance with the value of* $\delta = \alpha_2 - \alpha_1$, *we deduce from Theorem* 4.3 *that*

	$\sim$	$K_1 te^{\alpha_1 t}$	$\delta = 0$	$K_1 = a_2\beta_1$
$A_2(t)$	$\sim$	$K_2 e^{\alpha_1 t}$	$\delta < 0$	$K_2 = a_2\beta_1/(-\delta)$
	$\sim$	$K_3 e^{\alpha_2 t}$	$\delta > 0$	$K_3 = a_2\beta_1/\delta.$

The asymptotic behaviour of the means for the process with immigration can be obtained using (8)-(11). Let $\alpha = \max\{\alpha_1, \alpha_2\}$, and define the following quantities:

$$\overline{R}(t) = \int_0^t R(u)du,\ \widehat{R}_z(t) = \int_0^t e^{-zx} dR(x),\ \widetilde{R}_{\alpha_1}(t) = \int_0^t \widehat{R}_{\alpha_1}(u)du,$$

$$K_\alpha = \{K_1, \text{ if } \alpha = \alpha_1 = \alpha_2\} \vee \{K_2, \text{ if } \alpha = \alpha_1 > \alpha_2\} \vee \{K_3, \text{ if } \alpha = \alpha_2 > \alpha_1\},$$

$$R_\alpha(t) = \{\widetilde{R}_{\alpha_1}(t),\ \alpha = \alpha_1 = \alpha_2\} \vee \{\widehat{R}_{\alpha_1}(t),\ \alpha = \alpha_1\} \vee \{\widehat{R}_{\alpha_2}(t),\ \alpha = \alpha_2\}.$$

Theorem 4.4. *Assume that the conditions of Theorem* 4.3 *are satisfied. Then, as* $t \to \infty$, *we have*

$$M_1(t) \sim \begin{cases} \gamma R(t) & \textit{if } \alpha_1 = 0 \\ \gamma C_1 e^{\alpha_1 t}\ \widehat{R}_{\alpha_1}(t) & \textit{if } \alpha_1 \neq 0, \end{cases} \tag{18}$$

and

$$M_2(t) \sim \gamma K_\alpha e^{\alpha t} R_\alpha(t). \tag{19}$$

Corollary 4.2 (time-homogeneous immigration). *When* $R(t) = rt$, *we have, as* $t \to \infty$, *that*

$$M_1(t) \sim \begin{cases} r\gamma C_1 e^{\alpha_1 t}/\alpha_1 & \textit{if } \alpha_1 > 0 \\ M_1(t) \to r\gamma C_1/(-\alpha_1) & \textit{if } \alpha_1 < 0 \\ r\gamma t & \textit{if } \alpha_1 = 0. \end{cases} \tag{20}$$

When $\alpha_1 \neq \alpha_2$, it follows from Theorem 4.4 that, as $t \to \infty$,

$$M_2(t) \sim \begin{cases} r\gamma K_\alpha e^{\alpha t}/\alpha & \text{if } \alpha > 0 \\ r\gamma K_\alpha t & \text{if } \alpha = 0 \\ r\gamma K_\alpha/(-\alpha) & \text{if } \alpha < 0. \end{cases} \tag{21}$$

Similarly to (20) and (21), in the case where $\alpha = \alpha_1 = \alpha_2$, we have, as $t \to \infty$,

$$M_2(t) \sim \frac{r\gamma}{\alpha} K_1 t e^{\alpha t},\ \alpha > 0;\ M_2(t) \sim \frac{r\gamma}{2\mu_1} t^2,\ \alpha = 0;\ M_2(t) \sim -\frac{r\gamma}{\alpha} K_1,\ \alpha < 0.$$

Comment 4.1. The two-type process can be classified as *subcritical* if $\alpha < 0$, *critical* if $\alpha = 0$ and *supercritical* if $\alpha > 0$. In particular, for time-homogeneous immigration ($R(t) = rt$), the means increase exponentially in the supercritical case; they converge to some constants in the subcritical case, and they may increase linearly or even quadratically in the critical case. For time-inhomogeneous immigration, the asymptotic behaviour of the means is more complicated as it depends on the form of the immigration rate $R(t)$.

The role of the critical parameter $\alpha = \max\{\alpha_1, \alpha_2\}$ is further confirmed by the following results. Note that the probability for extinction at t is given by

$$Q_{12}(t) = P\{Z_1(t) = 0, Z_2(t) = 0 \mid Z_1(0) = 1, Z_2(0) = 0\} = F_1(t; 0, 0)$$

and it is well determined through equations (1) and (2).

Proposition 4.1. *If $\alpha \le 0$ then $q_{12} = \lim_{t\to\infty} Q_{12}(t) = 1$, and if $\alpha > 0$ then $q_{12} < 1$.*

Proof. First it is not difficult to obtain that the probability for extinction $q_{12} = \lim_{t\to\infty} Q_{12}(t)$ satisfies the equation $q_{12} = h_1(q_{12}, q_2)$, where $q_2 = h_2(q_2)$. Note that $q_1 = P(Z_1(t) \to 0 \mid Z_1(0) = 1\}$ satisfies the equation $q_1 = h_1(q_1, 1)$, where $q_1 = 1$ if $\alpha_1 \le 0$ and $q_1 < 1$ if $\alpha_1 > 0$. On one hand, if $\alpha_2 \le 0$, then $q_2 = 1$ and by the equation $q_{12} = h_1(q_{12}, 1)$ it follows that $q_{12} = q_1$. On the other hand, if $\alpha_2 > 0$ then $q_2 < 1$ and $q_{12} = h_1(q_{12}, q_2) < h_1(q_1, 1) = q_1 \le 1$. Finally, the proof is completed by combining the above results. $\square$

Note that the asymptotic behavior of $Q_{12}(t)$ can also be deduced using the results of Vatutin [26].

5. Asymptotic Behaviour of the Second Moments

Introduce the second order moments for the process without immigration

$$\begin{aligned} A_{ij}(t) &= \frac{\partial^2}{\partial s_i \partial s_j} F_1(t; s_1, s_2) \mid_{s_1 = s_2 = 1} \\ &= E\{Z_i(t)(Z_j(t) - \delta_{ij}) \mid Z_1(0) = 1, Z_2(0) = 0\}, i \le j = 1, 2, \end{aligned}$$

$$B_{22}(t) = \frac{\partial^2}{\partial s_2^2} F_2(t; s_2) \mid_{s_2=1} = E\{Z_2(t)(Z_2(t) - 1) \mid Z_2(0) = 1\}.$$

We deduce from equations (1) and (2) that these moments satisfy the following equations:

$$
\begin{aligned}
A_{11}(t) &= a_1 \int_0^t A_{11}(t-u)dG_1(u) + a_{11} \int_0^t A_1^2(t-u)dG_1(u), \\
A_{12}(t) &= a_1 \int_0^t A_{12}(t-u)dG_1(u) \\
&\quad + \int_0^t A_1(t-u)[a_{11}A_2(t-u) + a_{12}B_2(t-u)]dG_1(u), \\
A_{22}(t) &= a_1 \int_0^t A_{22}(t-u)dG_1(u) + a_2 \int_0^t B_{22}(t-u)dG_1(u) \\
&\quad + \int_0^t [a_{11}A_2^2(t-u) + 2a_{12}A_2(t-u)B_2(t-u) + a_{22}B_2^2(t-u)]dG_1(u), \\
B_{22}(t) &= b_2 \int_0^t B_{22}(t-u)dG_2(u) + b_{22} \int_0^t B_2^2(t-u)dG_2(u).
\end{aligned}
$$

Introduce now the second moments for the process with immigration

$$
\begin{aligned}
M_{ij}(t) &= \frac{\partial^2}{\partial s_i \partial s_j} \Psi(t; s_1, s_2) \mid_{s_1=s_2=1} \\
&= E\{Y_i(t)(Y_j(t) - \delta_{ij}) \mid \mathbf{Y}(0) = (0,0)\}, i \le j = 1, 2.
\end{aligned}
$$

It follows easily from equation (7) that the above moments satisfy the integral equations

$$
\begin{aligned}
M_{11}(t) &= \gamma \int_0^t r(t-u)A_{11}(u)du + [\gamma \int_0^t r(t-u)A_1(u)du]^2 \\
&\quad + \gamma_2 \int_0^t r(t-u)A_1^2(u)du, \\
M_{12}(t) &= \gamma \int_0^t r(t-u)A_{12}(u)du + \gamma^2 \int_0^t r(t-u)A_1(u)du \int_0^t r(t-u)A_2(u)du \\
&\quad + \gamma_2 \int_0^t r(t-u)A_1(u)A_2(u)du,
\end{aligned}
$$

$$M_{22}(t) = \gamma \int_0^t r(t-u)A_{22}(u)du + [\gamma \int_0^t r(t-u)A_2(u)du]^2$$
$$+ \gamma_2 \int_0^t r(t-u)A_2^2(u)du.$$

In what follows, the asymptotic behaviour of the second order moments is investigated by applying the limiting results presented in *Section 4.*

Let us first consider the second order factorial moments $A_{11}(t)$, $A_{22}(t)$, $B_{22}(t)$, and the mixed moment $A_{12}(t)$ for the process without immigration. The functional equations satisfied by these moments are particular cases of the general renewal-type equation (12) where the asymptotic behaviour of the function $f(t)$ is determined by the asymptotic properties of the first order moments $A_1(t)$ and $B_2(t)$ as given in (14)–(17) in the non-critical cases, and $A_1(t) \equiv B_2(t) \equiv 1$ in the critical case. It is not difficult to check that the asymptotic behaviour of the functions $f(t)$ associated with these equations satisfies the conditions of Theorems 4.1 – 4.4. Therefore, by applying the corresponding Theorems, we obtain the following results as $t \to \infty$:

	$\sim$	$C_{01}e^{\alpha_1 t}$,	$\alpha_1 < 0$,
$A_{11}(t)$	$\sim$	$\dfrac{a_{11}}{\mu_1}t$,	$\alpha_1 = 0$,
	$\sim$	$C_{11}e^{2\alpha_1 t}$,	$\alpha_1 > 0$,

where $C_{01} = a_{11}C_1^2\ \widehat{G}_1(2\alpha_1)/(-\alpha_1)\widetilde{\mu}_1, C_{11} = a_{11}C_1^2\ \widehat{G}_1(2\alpha_1)/[1 - a_1\ \widehat{G}_1(2\alpha_1)]$ and

	$\sim$	$C_{02}e^{\alpha_2 t}$,	$\alpha_2 < 0$,
$B_{22}(t)$	$\sim$	$\dfrac{b_{22}}{\mu_2}t$,	$\alpha_2 = 0$,
	$\sim$	$C_{22}e^{2\alpha_2 t}$,	$\alpha_2 > 0$,

where $C_{02} = b_{22}C_2^2\ \widehat{G}_2(2\alpha_2)/(-\alpha_2)\widetilde{\mu}_2, C_{22} = b_{22}C_2^2\ \widehat{G}_2(2\alpha_2)/[1 - b_2\ \widehat{G}_2(2\alpha_2)]$.

The asymptotic behaviour of the moments $A_{12}(t)$ and $A_{22}(t)$ is more complicated because the asymptotic properties of the corresponding functions $f(t)$ depend of the Malthus parameters α_1 and α_2. Nevertheless, by applying the limiting statements from Section 4 to the functional equations satisfied by $A_{12}(t)$ and $A_{22}(t)$, we obtain the following asymptotically equivalence as $t \to \infty$:

Table 1

Malthus parameters	$A_{12}(t) \sim$	$A_{22}(t) \sim$
1) $\alpha_1 > \alpha_2, \alpha_1 > 0$	$D_{11}e^{2\alpha_1 t}$	$D_{21}e^{2\alpha_1 t}$
2) $\alpha_1 > \alpha_2, \alpha_1 = 0$	$D_{12}t$	$D_{22}t$
3) $\alpha_1 > \alpha_2, \alpha_1 < 0$	$D_{13}e^{\alpha_1 t}$	$D_{23}e^{\alpha_1 t}$
4) $\alpha_1 < \alpha_2, \alpha_2 > 0$	$D_{14}e^{(\alpha_1+\alpha_2)t}$	$D_{24}e^{2\alpha_2 t}$
5) $\alpha_1 < \alpha_2, \alpha_2 = 0$	$D_{15}te^{\alpha_1 t}$	$D_{25}t$
6) $\alpha_1 < \alpha_2, \alpha_2 < 0$	$D_{16}e^{\alpha_1 t}$	$D_{26}e^{\alpha_2 t}$
7) $\alpha_1 = \alpha_2 = 0$	$D_{17}t^2$	$D_{27}t^3$
8) $\alpha_1 = \alpha_2 < 0$	$D_{18}e^{\alpha_1 t}$	$D_{28}e^{\alpha_1 t}$
9) $\alpha_1 = \alpha_2 > 0$	$D_{19}te^{2\alpha_1 t}$	$D_{29}t^3e^{2\alpha_1 t}$

The constants D_{ij} appearing in the above Table are calculated as follows:

$$
\begin{aligned}
D_{11} &= a_{11}C_1K_2\ \widehat{G}_1(2\alpha_1)/[1 - a_1\ \widehat{G}_1(2\alpha_1)],\\
D_{21} &= a_{11}K_2^2\ \widehat{G}_1(2\alpha_1)/[1 - a_1\ \widehat{G}_1(2\alpha_1)],\\
D_{12} &= D_{22} = a_{11}a_2C_2/\mu_1^2(-\alpha_2),\\
D_{13} &= C_1[a_{11}(-\alpha_2)K_2 + a_{12}(-\alpha_1)C_2]/a_1\alpha_1\alpha_2,\\
D_{23} &= \frac{K_2[2C_2(-\alpha_1) + K_2(-\alpha_2)]}{a_1\alpha_1\alpha_2} + \frac{C_2}{a_1}\left[\frac{a_{22}C_2}{\alpha_1 - 2\alpha_2} + \frac{a_2b_2b_{22}}{\widetilde{\mu}_2\alpha_2(\alpha_1 - \alpha_2)}\right],\\
D_{14} &= [C_1(a_{11}K_3 + a_{12}C_2)\widehat{G}_1(\alpha_1 + \alpha_2)]/[1 - a_1\ \widehat{G}_1(\alpha_1 + \alpha_2)],\\
D_{24} &= \widehat{G}_1(2\alpha_1)(a_{11}K_3^2 + 2a_{11}K_3C_2 + a_{22}C_2^2 + a_2C_{22})/[1 - a_1\ \widehat{G}_1(2\alpha_1)],\\
D_{15} &= C_1(a_{11}K_3 + a_{12})/a_1\widetilde{\mu}_1, D_{25} = a_1b_2/\mu_2(1 - a_1),\\
D_{16} &= C_1(a_{11}K_1 + a_{12}C_2)/a_1\widetilde{\mu}_1(-\alpha_2), D_{26} = C_2b_2b_{22}\ \widehat{G}_1(\alpha_2)/[1 - a_1\widehat{G}_1(\alpha_2)],\\
D_{17} &= a_{11}K_1/2\mu_1, \quad D_{27} = a_{11}K_1^2/3\mu_1,\\
D_{18} &= [a_{11} - a_1a_{12}]/a_1\alpha_1^2\widetilde{\mu}_1, D_{28} = 2a_{11}K_1^2/\widetilde{\mu}_1(-\alpha_1)^3,\\
D_{19} &= a_{11}C_1K_1\ \widehat{G}_1(2\alpha_1)/[1 - a_1\widehat{G}_1(2\alpha_1)], D_{29} = a_{11}K_1^2/[1 - a_1\ \widehat{G}_1(2\alpha_1)].
\end{aligned}
$$

Comment 5.1. It is interesting to compare the asymptotic behaviour of both moments. In general it is similar with an exception of cases 4) and 5). In case 4) $A_{22}(t)$ increases exponentially while $A_{12}(t)$ can either decrease to zero, or convergence to a constant positive, or grow exponentially, when $\alpha_1+\alpha_2$ is negative, or zero, or positive, respectively. In case 5) $A_{12}(t)$ goes to zero whereas $A_{22}(t)$ increases linearly.

Introduce the notations:

$$\begin{aligned} V_1(t) &= Var\{Z_1(t) \mid (1,0)\} = A_{11}(t) + A_1(t)[1-A_1(t)], \\ V_2(t) &= Var\{Z_2(t) \mid (1,0)\} = A_{22}(t) + A_2(t)[1-A_2(t)], \\ C_{12}(t) &= Cov\{Z_1(t), Z_2(t) \mid (1,0)\} = A_{12}(t) - A_1(t)A_2(t), \\ \rho_{12}(t) &= Corr\{Z_1(t), Z_2(t) \mid (1,0)\} = C_{12}(t)/\sqrt{V_1(t)V_2(t)}. \end{aligned}$$

First of all, it is not difficult to obtain the asymptotic behavior for $V_1(t)$ as $t \to \infty$:

	$\sim$	$(C_{01}+C_1)e^{\alpha_1 t}$	$\alpha_1 < 0,$
$V_1(t)$	$\sim$	$\frac{a_{11}}{\mu_1}t$	$\alpha_1 = 0,$
	$\sim$	$(C_{11}-C_1^2)e^{2\alpha_1 t}$	$\alpha_1 > 0.$

Now using Table 1 one can obtain the following asymptotic relations as $t \to \infty$:

Table 2

Malthus parameters	$V_2(t) \sim$	$C_{12}(t) \sim$
1) $\alpha_1 > \alpha_2, \alpha_1 > 0$	$(D_{21}-K_2^2)e^{2\alpha_1 t}$	$(D_{11}-C_1K_2)e^{2\alpha_1 t}$
2) $\alpha_1 > \alpha_2, \alpha_1 = 0$	$D_{22}t$	$D_{12}t$
3) $\alpha_1 > \alpha_2, \alpha_1 < 0$	$(D_{23}+K_2)e^{\alpha_1 t}$	$D_{13}e^{\alpha_1 t}$
4) $\alpha_1 < \alpha_2, \alpha_2 > 0$	$(D_{24}-K_3^2)e^{2\alpha_2 t}$	$(D_{14}-C_1K_3)e^{(\alpha_1+\alpha_2)t}$
5) $\alpha_1 < \alpha_2, \alpha_2 = 0$	$D_{25}t$	$D_{15}te^{\alpha_1 t}$
6) $\alpha_1 < \alpha_2, \alpha_2 < 0$	$(D_{26}+K_3)e^{\alpha_2 t}$	$D_{16}e^{\alpha_1 t}$
7) $\alpha_1 = \alpha_2 = 0$	$D_{27}t^3$	$D_{17}t^2$
8) $\alpha_1 = \alpha_2 < 0$	$K_1te^{\alpha_1 t}$	$D_{18}e^{\alpha_1 t}$
9) $\alpha_1 = \alpha_2 > 0$	$D_{29}t^3e^{2\alpha_1 t}$	$(D_{19}-C_1K_1)te^{2\alpha_1 t}$

Therefore by Table 2 one obtains the asymptotic behaviour of the correlation as $t \to \infty$:

Table 3

Malthus roots	$\rho_{12}(t) \sim$
1) $\alpha_1 > \alpha_2, \alpha_1 > 0$	$(D_{11} - C_1K_2)/\sqrt{(C_{11} - C_1^2)(D_{21} - K_2^2)}$
2) $\alpha_1 > \alpha_2, \alpha_1 = 0$	$D_{12}/\sqrt{D_{22}a_{11}/\mu_1}$
3) $\alpha_1 > \alpha_2, \alpha_1 < 0$	$D_{13}/\sqrt{(C_{01} + C_1)(D_{23} + K_2)}$
4) $\alpha_1 < \alpha_2, \alpha_2 > 0$	$\alpha_1 > 0 : \dfrac{(D_{14} - C_1K_3)}{\sqrt{(C_{11} - C_1^2)(D_{24} - K_3^2)}}$ $\alpha_1 = 0 : t^{-1/2}\dfrac{(D_{14} - C_1K_3)}{\sqrt{(D_{24} - K_3^2)a_{11}/\mu_1}} \to 0$ $\alpha_1 < 0 : e^{\alpha_1 t/2}\dfrac{(D_{14} - C_1K_3)}{\sqrt{(C_{01} + C_1)(D_{24} - K_3^2)}} \to 0$
5) $\alpha_1 < \alpha_2, \alpha_2 = 0$	$\sqrt{t}e^{\alpha_1 t/2}D_{15}/\sqrt{(C_{01} + C_1)D_{25}} \to 0$
6) $\alpha_1 < \alpha_2, \alpha_2 < 0$	$e^{(\alpha_1-\alpha_2)t/2}D_{16}/\sqrt{(C_{01} + C_1)(D_{26} + K_3)} \to 0$
7) $\alpha_1 = \alpha_2 = 0$	$\sqrt{3/4}$
8) $\alpha_1 = \alpha_2 < 0$	$t^{-1/2}D_{18}/\sqrt{K_1(C_{01} + C_1)} \to 0$
9) $\alpha_1 = \alpha_2 > 0$	$t^{-1/2}(D_{19} - C_1K_1)/\sqrt{(C_{11} - C_1^2)D_{29}} \to 0$

Consider now the process with immigration where $R^*(t) = \int_0^t u dR(u)$. One can obtain the following asymptotic behaviour using the corresponding expression for $M_{11}(t)$:

	$\sim e^{2\alpha_1 t}\{C_{01}e^{-\alpha_1 t}\widehat{R}_{\alpha_1}(t) + \gamma^2 C_1^2 \widehat{R}^2_{\alpha_1}(t) + \gamma_2 C_1^2 \widehat{R}_{2\alpha_1}(t)\},$	$\alpha_1 < 0,$
$M_{11}(t)$	$\sim \gamma a_{11}\{tR(t) - R^*(t)\}/\mu_1 + \gamma^2 R^2(t),$	$\alpha_1 = 0,$
	$\sim e^{2\alpha_1 t}\{C_{11}\widehat{R}_{2\alpha_1}(t) + \gamma^2 C_1^2 \widehat{R}^2_{\alpha_1}(t) + \gamma_2 C_1^2 \widehat{R}_{2\alpha_1}(t)\},$	$\alpha_1 > 0,$

It is not difficult to show that, in the homogeneous case $R(t) = rt$, the asymptotic behavior of $M_{11}(t)$ as $t \to \infty$ simplifies to:

	$\to r\{C_{01}(-\alpha_1)+2r\gamma^2C_1^2+\gamma_2C_1^2(-\alpha_1)\}/2\alpha_1^2$	$\alpha_1<0,$
$M_{11}(t)$	$\sim r\gamma(a_{11}+2\gamma\mu_1)t^2/2\ \mu_1$	$\alpha_1=0,$
	$\sim re^{2\alpha_1 t}\{C_{11}\alpha_1+2r\gamma^2C_1^2+\gamma_2C_1^2\alpha_1\}/2\alpha_1^2$	$\alpha_1>0.$

Denote $R^{**}(t)=\int_0^t udR^*(u)$ and $\widehat{R}^*_{\alpha_1}(t)=\int_0^t ud\widehat{R}_{\alpha_1}(u)$. Applying now asymptotic results from Table 1 to the corresponding relation for $M_{12}(t)$ we obtain, as $t\to\infty$, that

Table 4

Malthus roots	$M_{12}(t)\sim$
$0<\alpha_1>\alpha_2$	$e^{2\alpha_1 t}\{(\gamma D_{11}+\gamma_2C_1K_2)\widehat{R}_{2\alpha_1}(t)+\gamma^2C_1K_2\widehat{R}^2_{\alpha_1}(t)\}$
$0=\alpha_1>\alpha_2$	$\gamma D_{12}[tR(t)-R^*(t)]+\gamma^2K_2R^2(t)$
$0>\alpha_1>\alpha_2$	$e^{2\alpha_1 t}\{\gamma D_{13}e^{-\alpha_1 t}\widehat{R}_{\alpha_1}(t)+C_1K_2[\gamma^2\widehat{R}^2_{\alpha_1}(t)+\gamma_2\widehat{R}_{2\alpha_1}(t)]\}$
$\alpha_1<\alpha_2>0$	$e^{(\alpha_1+\alpha_2)t}\{\widehat{R}_{\alpha_1+\alpha_2}(t)[\gamma D_{14}+\gamma_2C_1K_3]$ $+\gamma^2C_1K_3\widehat{R}_{\alpha_1}(t)\widehat{R}_{\alpha_2}(t)\}$
$\alpha_1<\alpha_2=0$	$e^{\alpha_1 t}\{\gamma D_{15}[t\widehat{R}_{\alpha_1}(t)-\widehat{R}^*_{\alpha_1}(t)]+\gamma^2C_1K_3\widehat{R}_{\alpha_1}(t)R(t)\}$
$\alpha_1<\alpha_2<0$	$e^{(\alpha_1+\alpha_2)t}\{\gamma D_{16}e^{-\alpha_2 t}\widehat{R}_{\alpha_1}(t)$ $+C_1K_3[\gamma^2\widehat{R}_{\alpha_1}(t)\widehat{R}_{\alpha_2}(t)+\gamma_2\widehat{R}_{\alpha_1+\alpha_2}(t)]\},$
$\alpha_1=\alpha_2=0$	$\gamma D_{17}\{t^2R(t)+R^{**}(t)-2tR^*(t)\}$
$\alpha_1=\alpha_2<0$	$e^{2\alpha_1 t}\{\gamma D_{18}e^{-\alpha_1 t}\widehat{R}_{\alpha_1}(t)+\gamma^2C_1K_1\widehat{R}_{\alpha_1}(t)[t\widehat{R}_{\alpha_1}(t)-\widehat{R}^*_{\alpha_1}(t)]$ $+\gamma_2C_1K_1[t\widehat{R}_{2\alpha_1}(t)-\widehat{R}^*_{2\alpha_1}(t)]\},$
$\alpha_1=\alpha_2>0$	$e^{2\alpha_1 t}\{(\gamma D_{19}+\gamma_2C_1K_1)[t\widehat{R}_{2\alpha_1}(t)-\widehat{R}^*_{2\alpha_1}(t)]$ $+\gamma^2C_1K_1\widehat{R}_{\alpha_1}(t)[t\widehat{R}_{\alpha_1}(t)+\widehat{R}^*_{\alpha_1}(t)]\},$

It follows from Table 4 that, for time-homogeneous processes ($R(t)=rt$), the asymptotic behavior of $M_{12}(t)$ can be presented in the following Table 5.

Table 5

M. Roots	$M_{12}(t) \sim$
$0 < \alpha_1 > \alpha_2$	$e^{2\alpha_1 t} r\{\alpha_1(\gamma D_{11} + \gamma_2 C_1 K_2) + 2r\gamma^2 C_1 K_2\}/2\alpha_1^2,$
$0 = \alpha_1 > \alpha_2$	$t^2 r[\gamma D_{12}/2 + r\gamma^2 K_2],$
$0 > \alpha_1 > \alpha_2$	$r\{\gamma D_{13} + C_1 K_2[2r\gamma^2 + \gamma_2(-\alpha_1)]\}/2\alpha_1^2,$
$\alpha_1 < \alpha_2 > 0$	$\alpha_1 > 0 : e^{(\alpha_1+\alpha_2)t} \left\{ \frac{r}{\alpha_1+\alpha_2}[\gamma D_{14} + \gamma_2 C_1 K_3] + \frac{r^2}{\alpha_1\alpha_2}\gamma^2 C_1 K_3 \right\}$ $\alpha_1 = 0 : te^{\alpha_2 t} r\gamma^2 C_1 K_3/\alpha_2$ $\alpha_1 < 0 : e^{\alpha_2 t} r^2 \gamma^2 C_1 K_3/(-\alpha_1)\alpha_2$
$\alpha_1 < \alpha_2 = 0$	$tr^2\gamma^2 C_1 K_3/(-\alpha_1) + r\gamma D_{15}/\alpha_1^2,$
$\alpha_1 < \alpha_2 < 0$	$r\{\gamma D_{16}/(-\alpha_1) + C_1 K_3[r\gamma^2/\alpha_1\alpha_2 - \gamma_2/(\alpha_1+\alpha_2)]\},$
$\alpha_1 = \alpha_2 = 0$	$r\gamma D_{17} t^3/3,$
$\alpha_1 = \alpha_2 < 0$	$r\{\gamma D_{18}/(-\alpha_1) + r\gamma^2 C_1 K_1/(-\alpha_1^3) + \gamma_2 C_1 K_1/4\alpha_1^2\},$
$\alpha_1 = \alpha_2 > 0$	$te^{2\alpha_1 t} r\{\alpha_1(\gamma D_{19} + \gamma_2 C_1 K_1) + 2r\gamma^2 C_1 K_1\}/2\alpha_1^2.$

Note that in the case $\alpha_1 < 0$ and $\alpha_2 > 0$, we need the second terms in the asymptotic approximation:

	$\frac{e^{\alpha_2 t} r^2 \gamma^2 C_1 K_3}{(-\alpha_1)\alpha_2} + \frac{e^{(\alpha_1+\alpha_2)t} r[\gamma D_{14} + \gamma_2 C_1 K_3]}{\alpha_1+\alpha_2}$	$\alpha_1 + \alpha_2 > 0$
$M_{12}(t) \sim$	$\frac{e^{\alpha_2 t} r^2 \gamma^2 C_1 K_3}{(-\alpha_1)\alpha_2} + rt[\gamma D_{14} + \gamma_2 C_1 K_3]$	$\alpha_1 + \alpha_2 = 0$
	$\frac{e^{\alpha_2 t} r^2 \gamma^2 C_1 K_3}{(-\alpha_1)\alpha_2} + \frac{r[\gamma D_{14} + \gamma_2 C_1 K_3]}{-(\alpha_1+\alpha_2)}$	$\alpha_1 + \alpha_2 < 0$

Now, by applying results from Table 1 along with Theorem 4.3 to the corresponding relation for the moment $M_{22}(t)$ one can obtain as $t \to \infty$ the following relations where $\widehat{R}^{**}_{2\alpha_1}(t) = \int_0^t u^2 d\widehat{R}_{2\alpha_1}(u)$ and $\widehat{R}^{***}_{2\alpha_1}(t) = \int_0^t u^3 d\widehat{R}_{2\alpha_1}(u)$:

Table 6

Malthus roots	$M_{22}(t) \sim$
$0 < \alpha_1 > \alpha_2,$	$e^{2\alpha_1 t}\{(\gamma D_{21}\widehat{R}_{2\alpha_1}(t) + \gamma^2 K_2^2 \widehat{R}^2_{\alpha_1}(t) + \gamma_2 K_2^2 \widehat{R}_{2\alpha_1}(t)\},$
$0 = \alpha_1 > \alpha_2,$	$\gamma D_{22}[tR(t) - R^*(t)] + \gamma^2 K_2^2 R^2(t),$
$0 > \alpha_1 > \alpha_2,$	$e^{2\alpha_1 t}\{\gamma D_{23} e^{-\alpha_1 t}\widehat{R}_{\alpha_1}(t) + \gamma^2 K_2^2 \widehat{R}^2_{\alpha_1}(t) + \gamma_2 K_2^2 \widehat{R}_{2\alpha_1}(t)\},$
$\alpha_1 < \alpha_2 > 0,$	$e^{2\alpha_2 t}\{\widehat{R}_{2\alpha_2}(t)[\gamma D_{24} + \gamma_2 K_3^2] + \gamma^2 K_3^2 \widehat{R}^2_{\alpha_2}(t)\},$
$\alpha_1 < \alpha_2 = 0,$	$\gamma D_{25}[tR(t) - R^*(t)] + \gamma^2 K_3^2 R^2(t),$
$\alpha_1 < \alpha_2 < 0,$	$e^{2\alpha_2 t}\{\gamma D_{26} e^{-\alpha_2 t}\widehat{R}_{\alpha_2}(t) + K_3^2[\gamma^2 \widehat{R}^2_{\alpha_2}(t) + \gamma_2 \widehat{R}_{2\alpha_2}(t)]\},$
$\alpha_1 = \alpha_2 = 0,$	$\gamma D_{27}\{t^3 R(t) - 3t^2 R^*(t) + 3tR^{**}(t) - R^{***}(t)\}$ $+\gamma^2 K_1^2[tR(t) - R^*(t)]^2,$
$\alpha_1 = \alpha_2 < 0,$	$\gamma D_{28} e^{\alpha_1 t}\widehat{R}_{\alpha_1}(t) + \gamma^2 K_1 e^{2\alpha_1 t}[t\widehat{R}_{\alpha_1}(t) - \widehat{R}^*_{\alpha_1}(t)]^2$ $+\gamma_2 K_1^2 e^{2\alpha_1 t}[t^2 \widehat{R}_{2\alpha_1}(t) - 2t\widehat{R}^*_{2\alpha_1}(t) + \widehat{R}^{**}_{2\alpha_1}(t)]\},$
$\alpha_1 = \alpha_2 > 0,$	$\gamma D_{29} e^{2\alpha_1 t}[t^3 \widehat{R}_{2\alpha_1}(t) - 3t^2 \widehat{R}^*_{2\alpha_1}(t) + 3t\widehat{R}^{**}_{2\alpha_1}(t) - \widehat{R}^{***}_{2\alpha_1}(t)]$ $+\gamma^2 K_1^2 e^{\alpha_1 t}[t\widehat{R}_{\alpha_1}(t) - \widehat{R}^*_{\alpha_1}(t)]$

Table 7

Malthus roots	$M_{22}(t) \sim$
$0 < \alpha_1 > \alpha_2,$	$e^{2\alpha_1 t} r\{\alpha_1(\gamma D_{21} + \gamma_2 K_2^2) + 2r\gamma^2 K_2^2\}/2\alpha_1^2,$
$0 = \alpha_1 > \alpha_2,$	$t^2 r\gamma[D_{22}/2 + r\gamma K_2^2],$
$0 > \alpha_1 > \alpha_2,$	$r\{2\gamma D_{23} + 2r\gamma^2 K_2^2 + \gamma_2 K_2(-\alpha_1)\}/2\alpha_1^2,$
$\alpha_1 < \alpha_2 > 0,$	$e^{2\alpha_2 t} r\{2\alpha_2 \gamma D_{24} + 2r\gamma^2 K_3^2 + \alpha_2 \gamma_2 K_3^2\}/2\alpha_2^2,$
$\alpha_1 < \alpha_2 = 0,$	$t^2 r\gamma[D_{25}/2 + r\gamma K_3^2],$
$\alpha_1 < \alpha_2 < 0,$	$r\{2(-\alpha_2)\gamma D_{26} + 2r\gamma^2 K_3^2 + (-\alpha_2)\gamma_2\ K_3\}/2\alpha_2^2,$
$\alpha_1 = \alpha_2 = 0,$	$t^4 r\gamma[D_{27} + r\gamma K_1^2]/4,$
$\alpha_1 = \alpha_2 < 0,$	$r\{4(-\alpha_1^3)\gamma D_{28} + 4r\gamma^2 K_1^2 + \gamma_2 K_1^2(-\alpha_1)\}/4\alpha_1^4,$
$\alpha_1 = \alpha_2 > 0,$	$t^3 e^{2\alpha_1 t} r\gamma D_{29}/2\alpha_1.$

From Table 6 it is not difficult to deduce the asymptotic behaviour for the homogeneous immigration ($R(t) = rt$) (see Table 7)

Introduce the notation:

$W_1(t) = Var\{Y_1(t) \mid (0,0)\} = M_{11}(t) + M_1(t)[1 - M_1(t)]$,
$W_2(t) = Var\{Y_2(t) \mid (0,0)\} = M_{22}(t) + M_2(t)[1 - M_2(t)]$,
$J_{12}(t) = Cov\{Y_1(t), Y_2(t) \mid (0,0)\} = M_{12}(t) - M_1(t)M_2(t)$,
$\Lambda_{12}(t) = Corr\{Y_1(t), Y_2(t) \mid (0,0)\} = J_{12}(t)/\sqrt{W_1(t)W_2(t)}$.

We deduce from Theorem 4.3 and the asymptotic behavior for $M_{11}(t)$ that

	$\sim e^{2\alpha_1 t}\{(C_{01} + \gamma C_1)e^{-\alpha_1 t}\widehat{R}_{\alpha_1}(t) + \gamma_2 C_1^2 \widehat{R}_{2\alpha_1}(t)\}$	$\alpha_1 < 0,$
$W_1(t)$	$\sim \gamma a_{11}\{tR(t) - R^*(t)\}/\mu_1$	$\alpha_1 = 0,$
	$\sim e^{2\alpha_1 t}(C_{11} + \gamma_2 C_1^2)\widehat{R}_{2\alpha_1}(t) + \gamma C_1 e^{\alpha_1 t}\widehat{R}_{\alpha_1}(t)$	$\alpha_1 > 0.$

From here it is not difficult to deduce asymptotic expansion for the homogeneous case ($R(t) = rt$)

	$\to r\gamma_2 C_1^2/2(-\alpha_1)$	$\alpha_1 < 0,$
$W_1(t)$	$\sim r\gamma a_{11} t^2/2\ \mu_1$	$\alpha_1 = 0,$
	$\sim re^{2\alpha_1 t}(C_{11} + \gamma_2 C_1^2)/2\alpha_1$	$\alpha_1 > 0.$

By applying Table 6 and Theorem 4.3, we also obtain the asymptotic behavior for $W_2(t)$ (as $t \to \infty$) (see Table 8).

In the case of an homogeneous immigration, the expressions in Table 8 simplify to the following Table 9.

Using Table 4 and Theorem 4.3. one can obtain the corresponding limit behavior for the covariance $J_{12}(t)$ as $t \to \infty$

From Table 10 we deduce the asymptotic behavior of $J_{12}(t)$ in the homogeneous case ($R(t) = rt$) given in the following Table 11.

Finally, we deduce from Tables 11 and Table 9 the asymptotic behavior for the correlation between the numbers of type-1 and type-2 cells in the case of a time-homogeneous immigration given in the following Table 12.

Table 8

Malthus roots	$W_2(t) \sim$
$0 < \alpha_1 > \alpha_2,$	$e^{2\alpha_1 t}\{(\gamma D_{21}\widehat{R}_{2\alpha_1}(t) + \gamma_2 K_2^2 \widehat{R}_{2\alpha_1}(t) + \gamma K_2^2 e^{-\alpha_1 t}\widehat{R}_{\alpha_1}(t)\},$
$0 = \alpha_1 > \alpha_2,$	$\gamma D_{22}[tR(t) - R^*(t)]$
$0 > \alpha_1 > \alpha_2,$	$e^{2\alpha_1 t}\{\gamma[D_{23} + K_2]e^{-\alpha_1 t}\widehat{R}_{\alpha_1}(t) + \gamma_2 K_2^2 \widehat{R}_{2\alpha_1}(t)\},$
$\alpha_1 < \alpha_2 > 0,$	$e^{2\alpha_2 t}\{\widehat{R}_{2\alpha_2}(t)[\gamma D_{24} + \gamma_2 K_3^2] + \gamma K_3 e^{-\alpha_2 t}\widehat{R}_{\alpha_2}(t)\},$
$\alpha_1 < \alpha_2 = 0,$	$\gamma D_{25}[tR(t) - R^*(t)],$
$\alpha_1 < \alpha_2 < 0,$	$e^{2\alpha_2 t}\{\gamma[D_{26} + K_3]e^{-\alpha_2 t}\widehat{R}_{\alpha_2}(t) + \gamma_2 K_3^2 \widehat{R}_{2\alpha_2}(t)\},$
$\alpha_1 = \alpha_2 = 0,$	$\gamma D_{27}\{t^3 R(t) - 3t^2 R^*(t) + 3tR^{**}(t) - R^{***}(t)\}$ $+\gamma^2 K_1^2 [tR(t) - R^*(t)]^2 + \gamma K_1 \overline{R}(t)[1 - \gamma K_1 \overline{R}(t)],$
$\alpha_1 = \alpha_2 < 0,$	$\gamma D_{28} e^{\alpha_1 t}\widehat{R}_{\alpha_1}(t) + \gamma^2 K_1 e^{2\alpha_1 t}[t\widehat{R}_{\alpha_1}(t) - \widehat{R}^*_{\alpha_1}(t)]^2$ $+\gamma_2 K_1^2 e^{2\alpha_1 t}[t^2 \widehat{R}_{2\alpha_1}(t) - 2t\widehat{R}^*_{2\alpha_1}(t) + \widehat{R}^{**}_{2\alpha_1}(t)]$ $+\gamma K_1 e^{\alpha_1 t}\widetilde{R}_{\alpha_1}(t)[1 - e^{\alpha_1 t}\widetilde{R}_{\alpha_1}(t)],$
$\alpha_1 = \alpha_2 > 0,$	$\gamma D_{29} e^{2\alpha_1 t}[t^3 \widehat{R}_{2\alpha_1}(t) - 3t^2 \widehat{R}^*_{2\alpha_1}(t) + 3t\widehat{R}^{**}_{2\alpha_1}(t) - \widehat{R}^{***}_{2\alpha_1}(t)]$ $+\gamma K_1 e^{\alpha_1 t}\{\gamma K_1 [t\widehat{R}_{\alpha_1}(t) - \widehat{R}^*_{\alpha_1}(t)] + \widetilde{R}_{\alpha_1}(t)[1 - e^{\alpha_1 t}\widetilde{R}_{\alpha_1}(t)]\}.$

Table 9

Malthus roots	$W_2(t) \sim$
$0 < \alpha_1 > \alpha_2,$	$e^{2\alpha_1 t} r(\gamma D_{21} + \gamma_2 K_2^2)/2\alpha_1,$
$0 = \alpha_1 > \alpha_2,$	$t^2 r\gamma D_{22}/2,$
$0 > \alpha_1 > \alpha_2,$	$r\{2\gamma D_{23} + (2\gamma + \gamma_2)K_2\}/2(-\alpha_1),$
$\alpha_1 < \alpha_2 > 0,$	$e^{2\alpha_2 t} r\{2\gamma D_{24} + \gamma_2 K_3^2\}/2\alpha_2,$
$\alpha_1 < \alpha_2 = 0,$	$t^2 r\gamma D_{25}/2,$
$\alpha_1 < \alpha_2 < 0,$	$r\{2\gamma D_{26} + \gamma_2\ K_3\}/2(-\alpha_2),$
$\alpha_1 = \alpha_2 = 0,$	$t^4 r\gamma D_{27}/4,$
$\alpha_1 = \alpha_2 < 0,$	$r\{4\alpha_1^2 \gamma D_{28} + 4(-\alpha_1)\gamma K_1 + \gamma_2 K_1^2\}/4(-\alpha_1^3),$
$\alpha_1 = \alpha_2 > 0,$	$t^3 e^{2\alpha_1 t} r\gamma D_{29}/2\alpha_1.$

Table 10

Malthus roots	$J_{12}(t) \sim$
$0 < \alpha_1 > \alpha_2,$	$e^{2\alpha_1 t}(\gamma D_{11} + \gamma_2 C_1 K_2)\widehat{R}_{2\alpha_1}(t),$
$0 = \alpha_1 > \alpha_2,$	$\gamma D_{12}[tR(t) - R^*(t)],$
$0 > \alpha_1 > \alpha_2,$	$e^{2\alpha_1 t}[\gamma D_{13} e^{-\alpha_1 t}\widehat{R}_{\alpha_1}(t) + \gamma_2 \widehat{R}_{2\alpha_1}(t)],$
$\alpha_1 < \alpha_2 > 0,$	$e^{(\alpha_1+\alpha_2)t}\{\widehat{R}_{\alpha_1+\alpha_2}(t)[\gamma D_{14} + \gamma_2 C_1 K_3] + \gamma^2 C_1 K_3 \widehat{R}^2_{\alpha_1}(t)\},$
$\alpha_1 < \alpha_2 = 0,$	$e^{\alpha_1 t}\gamma D_{15}[t\widehat{R}_{\alpha_1}(t) - \widehat{R}^*_{\alpha_1}(t)],$
$\alpha_1 < \alpha_2 < 0,$	$e^{(\alpha_1+\alpha_2)t}[\gamma D_{16} e^{-\alpha_2 t}\widehat{R}_{\alpha_1}(t) + C_1 K_3 \gamma_2 \widehat{R}_{\alpha_1+\alpha_2}(t)],$
$\alpha_1 = \alpha_2 = 0,$	$\gamma D_{17}\{t^2 R(t) + R^{**}(t) - 2tR^*(t)\} - \gamma^2 K_1 R(t)\overline{R}(t),$
$\alpha_1 = \alpha_2 < 0,$	$e^{2\alpha_1 t}\{\gamma D_{18} e^{-\alpha_1 t}\widehat{R}_{\alpha_1}(t) + \gamma_2 C_1 K_1[t\widehat{R}_{2\alpha_1}(t) - \widehat{R}^*_{2\alpha_1}(t)]\}$ $+\gamma^2 C_1 K_1 \widehat{R}_{\alpha_1}(t)[t\widehat{R}_{\alpha_1}(t) - \widehat{R}^*_{\alpha_1}(t) - \widetilde{R}_{\alpha_1}(t)],$
$\alpha_1 = \alpha_2 > 0,$	$e^{2\alpha_1 t}\{(\gamma D_{19} + \gamma_2 C_1 K_1)[t\widehat{R}_{2\alpha_1}(t) - \widehat{R}^*_{2\alpha_1}(t)]$ $+\gamma^2 C_1 K_1 \widehat{R}_{\alpha_1}(t)[t\widehat{R}_{\alpha_1}(t) + \widehat{R}^*_{\alpha_1}(t) - \widetilde{R}_{\alpha_1}(t)]\}.$

Table 11

M. Roots	$J_{12}(t) \sim$
$0 < \alpha_1 > \alpha_2$	$e^{2\alpha_1 t} r(\gamma D_{11} + \gamma_2 C_1 K_2)/2\alpha_1,$
$0 = \alpha_1 > \alpha_2$	$t^2 r\gamma D_{12}/2,$
$0 > \alpha_1 > \alpha_2$	$r\{\gamma D_{13} + C_1 K_2 \gamma_2(-\alpha_1)\}/2\alpha_1^2,$
$\alpha_1 < \alpha_2 > 0$	$\alpha_1 > 0 : e^{(\alpha_1+\alpha_2)t} r[\gamma D_{14} + \gamma_2 C_1 K_3]/(\alpha_1 + \alpha_2)$ $\alpha_1 = 0 : te^{\alpha_2 t} r\gamma^2 C_1 K_3/\alpha_2$ $\alpha_1 < 0 :$ $\alpha_1 + \alpha_2 > 0 : \frac{e^{(\alpha_1+\alpha_2)t} r[\gamma D_{14} + \gamma_2 C_1 K_3]}{\alpha_1+\alpha_2}$ $\alpha_1 + \alpha_2 = 0 : tr[\gamma D_{14} + \gamma_2 C_1 K_3]$ $\alpha_1 + \alpha_2 < 0 : \frac{r[\gamma D_{14} + \gamma_2 C_1 K_3]}{-(\alpha_1+\alpha_2)}$
$\alpha_1 < \alpha_2 = 0$	$r\gamma D_{15}/\alpha_1^2,$
$\alpha_1 < \alpha_2 < 0$	$r\{\gamma D_{16}/(-\alpha_1) - C_1 K_3 \gamma_2/(\alpha_1 + \alpha_2)\},$
$\alpha_1 = \alpha_2 = 0$	$r\gamma D_{17} t^3/3,$
$\alpha_1 = \alpha_2 < 0$	$r\{4\gamma D_{18}(-\alpha_1) + \gamma_2 C_1 K_1\}/4\alpha_1^2,$
$\alpha_1 = \alpha_2 > 0$	$te^{2\alpha_1 t} r(\gamma D_{19} + \gamma_2 C_1 K_1)/2\alpha_1.$

Table 12

M. Roots	$\Lambda_{12}(t) \sim$
$0 < \alpha_1 > \alpha_2$	$(\gamma D_{11} + \gamma_2 C_1 K_2)/\sqrt{(\gamma D_{21} + \gamma_2 K_2^2)(C_{11} + \gamma_2 C_1^2)}$
$0 = \alpha_1 > \alpha_2$	$\sqrt{a_2 C_2/\mu_1(-\alpha_2)}$
$0 > \alpha_1 > \alpha_2$	$\{\gamma D_{13} + C_1 K_2 \gamma_2(-\alpha_1)\}/C_1\sqrt{\gamma_2\{2\gamma D_{23} + (2\gamma + \gamma_2)K_2\}}$
$\alpha_1 < \alpha_2 > 0$	$\alpha_1 > 0 : \dfrac{2[\gamma D_{14} + \gamma_2 C_1 K_3]\sqrt{\alpha_1 \alpha_2}}{(\alpha_1 + \alpha_2)\sqrt{(2\gamma D_{24} + \gamma_2 K_3^2)(C_{11} + \gamma_2 C_1^2)}}$ $\alpha_1 = 0 : 2\gamma^2 C_1 K_3 \sqrt{\mu_1}/\sqrt{\alpha_2 \gamma a_{11}\{2\gamma D_{24} + \gamma_2 K_3^2\}}$ $\alpha_1 < 0 :$ $\alpha_1 + \alpha_2 > 0 : \dfrac{e^{\alpha_1 t}[\gamma D_{14} + \gamma_2 C_1 K_3]2\sqrt{(-\alpha_1)\alpha_2}}{(\alpha_1 + \alpha_2)C_1\sqrt{\gamma_2(2\gamma D_{24} + \gamma_2 K_3^2)}} \to 0$ $\alpha_1 + \alpha_2 = 0 : \dfrac{te^{-\alpha_2 t}[\gamma D_{14} + \gamma_2 C_1 K_3]2\sqrt{(-\alpha_1)\alpha_2}}{C_1\sqrt{\gamma_2(2\gamma D_{24} + \gamma_2 K_3^2)}} \to 0$ $\alpha_1 + \alpha_2 < 0 : \dfrac{te^{-\alpha_2 t}[\gamma D_{14} + \gamma_2 C_1 K_3]2\sqrt{(-\alpha_1)\alpha_2}}{-(\alpha_1 + \alpha_2)C_1\sqrt{\gamma_2(2\gamma D_{24} + \gamma_2 K_3^2)}} \to 0$
$\alpha_1 < \alpha_2 = 0$	$t^{-1} D_{15}\sqrt{\gamma(-\alpha_1)}/\alpha_1^2 C_1 \sqrt{\gamma_2 D_{25}} \to 0$
$\alpha_1 < \alpha_2 < 0$	$\dfrac{2\sqrt{\alpha_1 \alpha_2}\{\gamma D_{16}(-\alpha_1 - \alpha_2) - \alpha_1 \gamma_2 C_1 K_3\}}{\alpha_1(\alpha_1 + \alpha_2)C_1\sqrt{\gamma_2\{2\gamma D_{26} + \gamma_2\ K_3\}}}$
$\alpha_1 = \alpha_2 = 0$	$\sqrt{2/3}$
$\alpha_1 = \alpha_2 < 0$	$\dfrac{4\gamma D_{18}(-\alpha_1) + \gamma_2 C_1 K_1}{\gamma C_1\sqrt{2\{4\alpha_1^2 \gamma D_{28} + 4(-\alpha_1)\gamma K_1 + \gamma_2 K_1^2\}}}$
$\alpha_1 = \alpha_2 > 0$	$t^{-1/2}(\gamma D_{19} + \gamma_2 C_1 K_1)/\sqrt{\gamma D_{29}(C_{11} + \gamma_2 C_1^2)} \to 0$

6. Conclusions

The main conclusion of this work is that the asymptotic behavior of the second order moments and correlations can be quite diverse for reducible two-type branching processes with immigration. Based on Tables 1–12, we can distinguish 9 different cases depending on the values of the Malthus parameters α_1 and α_2. For time-inhomogeneous immigration, the asymptotic behavior of the process is also governed by the rate of immigration $R(t)$.

In the time-homogeneous case $R(t) = rt$ we identified a number of additional cases. As indicated in Table 3, the asymptotic behavior of the correlation $\rho_{12}(t)$ when $\alpha_1 < \alpha_2 > 0$ admits three, quite distinct, subcases which additionally depend on whether $\alpha_1 > 0$, or $\alpha_1 = 0$, or $\alpha_1 < 0$. The situation is similar for $M_{12}(t)$ (Table 5). In Tables 11 and 12 the subcase $\alpha_1 < \alpha_2 > 0, \alpha_1 < 0$ yields also three additional subcases depending on whether $\alpha_1+\alpha_2 > 0$, or $\alpha_1+\alpha_2 = 0$, or $\alpha_1 + \alpha_2 < 0$.

In the case where $\alpha_1 = \alpha_2 = 0$, it is interesting to point out that the convergence of the correlations to constants that do not dependent on any model parameters. Specifically, we have $\rho_{12}(t) = Corr\{Z_1(t), Z_2(t) \mid (1, 0)\} \to \sqrt{3/4}$ (Table 3) and $\Lambda_{12}(t) = Corr\{Y_1(t), Y_2(t) \mid (0, 0)\} \to \sqrt{2/3}$ (Table 12).

Finally, estimation theory relying solely on the means of the process may lead to problem of non-identifiability of some model parameters. Estimator using higher order moments may resolve the problem (Chen and Hyrien [4]). The application of these asymptotic results for statistical purposes will be presented in another paper.

Dedication. This article is dedicated to the memory of our friends and collaborators Dr. Andrei Yakovlev and Dr. Alexander Zorin. We will always remember our friendship, and continue to develop the ideas that germinated during stimulating discussions with both of them.

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