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RANDOMIZED PUSH-OUT MECHANISMS IN PRIORITY QUEUING AND THEIR PROBABILITY CHARACTERISTICS

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Abstract. The non-preemptive priority queueing with a finite buffer is considered. A randomized push-out buffer management mechanism that allows to control very efficiently the loss probability of priority packets is introduced. The packet loss probabilities for priority and non-priority traffic are derived with the use of the generating function approach. For the standard non-randomized push-out scheme, the explicit analytic expressions are obtained. A procedure for the numerical calculation of mean queues is also proposed.

1. Introduction

Priority queueing disciplines have a number of important applications in computer networks, for example, in the Differentiated Services architecture for the Internet [7].

Consider the non-preemptive priority queueing system with two classes of packets. Class 1 packets have priority over class 2 packets. The packets of class 1 (2) arrive into the buffer according to the Poisson process with rate λ_1 (λ_2), respectively. The service time has the exponential distribution with the same rate μ for each class. The service times are independent of the arrival processes. The buffer has a finite size N and it is shared by both types of

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customers. If the buffer is full, a new coming customer of class 1 can push out of the buffer a customer of class 2 with the probability α. Note that if $\alpha = 1$ we retrieve the standard non-randomized push-out mechanism.

The infinite buffer priority queueing has been thoroughly studied in [4, 8, 9]. The case of finite buffer priority queueing received considerably less attention. The M/M/C/K type finite buffer non-preemptive priority queueing with non-randomized push-out mechanism is analyzed by Kapadia *et al* [5, 6]. Bondi [1] considers the M/M/1/K type preemptive and non-preemptive priority queueing with the following buffer management schemes: complete partitioning, complete sharing and sharing with minimum allocation. Wagner and Krieger [10] analyze the M/M/1/K type non-preemptive priority queueing with the complete sharing buffer management scheme and with the class-dependent service rates. In [2] Cheng and Akyildiz consider the priority queueing with general service time distributions and a general service discipline function.

Most of the above works use recursive relations to solve steady state Kolmogorov equations. We use a generating function approach, which only requires the solution of a linear system of N equations. As far as we know, the randomized push-out mechanism is analyzed for the first time.

2. The generating functions

Denote by $p(i,n)$ the stationary probability of the event that there are n packets in the queue including i packets of class 1. We also use p_0 for the stationary probability of the event that there are no packets in the system. These probabilities satisfy the following stationary Kolmogorov equations:

$$(\lambda_1 + \lambda_2)p_0 = \mu p(0,0);$$

- $n = 0$

$$(\lambda_1 + \lambda_2 + \mu)p(0,0) = \mu p(1,1) + \mu p(0,1) + (\lambda_1 + \lambda_2)p_0;$$

- $0 < n < N$

$$\begin{aligned}(\lambda_1 + \lambda_2 + \mu)p(0,n) = &\quad \mu p(1,n+1) + \mu p(0,n+1) + \lambda_2 p(0,n-1),\\ (\lambda_1 + \lambda_2 + \mu)p(i,n) = &\quad \mu p(i+1,n+1) + \lambda_1 p(i-1,n-1) + \lambda_2 p(i,n-1),\\ (\lambda_1 + \lambda_2 + \mu)p(i,n) = &\quad \mu p(n+1,n+1) + \lambda_1 p(n-1,n-1);\end{aligned}$$

- $n = N$

$$\begin{array}{rlll} (\alpha\lambda_1+\mu)p(0,N) = & & \lambda_2 p(0,N-1), & \\ (\alpha\lambda_1+\mu)p(i,N) = & \lambda_1 p(i-1,N-1) & +\lambda_2 p(i,N-1) & +\alpha\lambda_1 p(i-1,N), \\ \mu p(N,N) = & \lambda_1 p(N-1,N-1) & & +\alpha\lambda_1 p(N-1,N). \end{array}$$

Next we introduce the generating function for $p(i,n)$ by the index i

$$F_n(x) = \sum_{i=0}^{n} p(i,n)x^i.$$

Using the above given Kolmogorov equations, we obtain the following relations for the generating functions $F_n(x), n = 0,1,...,N$:

- $n = 0$

$$(\lambda_1+\lambda_2+\mu)F_0(x) = \frac{\mu}{x}\left[F_1(x) - p(0,1)\right] + \mu p(0,1) + (\lambda_1+\lambda_2)p_0,$$

- $0 < n < N$

$$(\lambda_1+\lambda_2+\mu)F_n(x) = \frac{\mu}{x}\left[F_{n+1}(x) - p(0,n+1)\right] + \mu p(0,n+1) + (\lambda_1 x+\lambda_2)F_{n-1}(x).$$

In particular, we get the following boundary condition

- $n = N$

$$(1) \qquad (\alpha\lambda_1+\mu)F_N(x) - \alpha\lambda_1 p(N,N)x^N = (\lambda_1 x+\lambda_2)F_{N-1}(x) + \alpha\lambda_1 x F_N(x) - \alpha\lambda_1 x^{N+1}p(N,N).$$

Now we introduce the generating function for $F_n(x)$ by the index n

$$\Phi(x,y) = \sum_{n=0}^{N-1} F_n(x)y^n.$$

The generating function $\Phi(x,y)$ satisfies equation (2) given in Lemma 1 below.

Lemma 1. *The generating function $\Phi(x,y)$ satisfies the following equation*

$$(2) \qquad [(\rho+1)xy - xy^2(\rho_1 x+\rho_2) - 1]\Phi(x,y) = -y^{N+1}x(\rho_1 x+\rho_2)F_{N-1}(x) + y^N F_N(x) + y(x-1)A(y) + (xy-1)\rho p_0,$$

where $\rho_i = \lambda_i/\mu$, $\rho = \rho_1+\rho_2$ and $A(y) = \sum_{n=0}^{N-1} p(0,n+1)y^n$.

The generating function $\Phi(x,y)$ is determined by the next result.

Theorem 1. *The generating function $\Phi(x,y)$ is given by*

$$\begin{aligned}\Phi(x,y) &= \\ &\frac{[1-xy+\alpha\rho_1 xy(x-1)]y^N V_{N-1}(x) + y(x-1)A(y)}{(\rho+1)xy - xy^2(\rho_1 x+\rho_2) - 1} \\ &+\frac{[1-xy]x^N y^N p(N,N) + \rho[xy-1]p_0}{(\rho+1)xy - xy^2(\rho_1 x+\rho_2) - 1},\end{aligned}$$

where

$$\begin{aligned}V_{N-1}(x) &= \sum_{k=0}^{N-1} x^k p(k,N), \\ A(y) &= -\alpha\rho y^{N-1} p(0,N) \\ &+\sum_{k=1}^{N-1}[\rho_2 y^{N-k}\frac{U_{k-1}(t)}{\rho_1^{(k+1)/2}} - \alpha\rho y^{N-k-1}\frac{U_k(t)}{\rho_1^{k/2}} \\ &+\alpha y^{N-k-1}\frac{U_{k-1}(t)}{\rho_1^{(k-1)/2}}]p(k,N) + \rho_2\frac{U_{N-1}(t)}{\rho_1^{(N+1)/2}}p(N,N)\end{aligned}$$

with $t = (\rho+1-\rho_2 y)/(2\rho_1^{1/2})$ and where probabilities $p(k,N), k = 0,...,N$ can be obtained as a solution to the following system of linear equations

- $s = 0$

$$\begin{aligned}&\alpha\rho_1 C^1_{N-1}(t_0)p(N-1,N) \\ &+\left[\rho C^1_{N-1}(t_0) - {\rho_1}^{1/2}C^1_N(t_0)\right]p(N,N) + \rho{\rho_1}^{(N+1)/2}p_0 = 0,\end{aligned}$$

- $0 < s < N$

$$\begin{aligned}&\sum_{k=0}^{s-1}\left[\rho\frac{C^{s-k}_{N-s-1}(t_0){\rho_1}^{k+1}}{(-\rho_2)^{k+1}} - {\rho_1}^{3/2}(1+\alpha\rho)\frac{C^{s-k}_{N-s}(t_0){\rho_1}^k}{(-\rho_2)^{k+1}}\right. \\ &\left.+\rho_1\alpha\frac{C^{s-k+1}_{N-s-1}(t_0){\rho_1}^k}{(-\rho_2)^k}\right]p(N-1-k,N) + \alpha{\rho_1}^{s+1}\frac{C^1_{N-s-1}(t_0)}{(-\rho_2)^s}p(N-1-s,N) \\ &+\left[\rho C^{s+1}_{N-s-1}(t_0) - {\rho_1}^{3/2}C^{s+1}_{N-s}(t_0)\right]p(N,N) = 0,\end{aligned}$$

- $s = N$

$$-{\rho_1}^{3/2}(1+\alpha\rho)\sum_{k=0}^{N-1}\frac{C_0^{N-k}(t_0){\rho_1}^k}{(-\rho_2)^{k+1}}p(N-1-k,N)-{\rho_1}^{1/2}C_0^{N+1}(t_0)p(N,N)=0$$

with $U_n(x)$ *and* $C_n^{\nu}(x)$ *denoting the Chebyshev polynomials of the second kind and the Gegenbauer polynomials* [3]*, respectively, and*

$$p_0 = (1-\rho)/(1-\rho^{N+2}), \qquad t_0 = (\rho+1)/(2\rho_1^{1/2}).$$

The proof is carried out into Section 4.

3. The loss probabilities

Once we know the value of $p(N, N)$, we can derive the loss probabilities of class 1 and class 2 packets.

Theorem 2. *The loss probabilities of class 1 and class 2 packets are given by the following formulae*

$$P_{loss}^{(1)} = p(N,N) + (1-\alpha)[P_N - p(N,N)], \tag{3}$$

$$P_{loss}^{(2)} = P_N + \alpha\frac{\rho_1}{\rho_2}[P_N - p(N,N)], \tag{4}$$

where

$$P_N = \frac{1-\rho}{1-\rho^{N+2}}\rho^{N+1}.$$

Proof: A priority packet can be lost either when the whole buffer is filled only with priority packets or when there are some packets of class 2 but with probability $1-\alpha$ the push-out mechanism is not enabled. The probability of the first event is $p(N, N)$ and the probability of the second event is $\sum_{k=0}^{N-1} p(k, N) = P_N - p(N, N)$. Thus, we obtain formula (3).

The stream of lost packets of class 2 consists of the stream of packets with rate $\lambda_2 P_N$ lost when the buffer is full and the stream of packets with rate $\alpha\lambda_1(P_n - p(N, N))$ pushed out by packets of class 1. Since the system is ergodic, we obtain formula (4).

Note that if $\alpha = 0$ (no push-out), the loss probabilities or two classes coincide and are equal to P_N. Furthermore, due to the fact that the service time distribution is the same for the two classes, the expressions for p_0, $F_N(1)$ and $\Phi(1,1)$ could be obtained immediately by elementary considerations.

In the particular case of the non-randomized push-out mechanism, that is, when $\alpha = 1$, we can calculate the loss probabilities explicitly.

Theorem 3. *The loss probabilities of class 1 and class 2 packets in the case of non-randomized push-out mechanism are given by*

$$P_{loss}^{(1)} = \rho\rho_1^N \frac{(1-\rho_1)(1-\rho^{N+1})}{(1-\rho_1^{N+1})(1-\rho^{N+2})}, \tag{5}$$

$$P_{loss}^{(2)} = P_N + \frac{\rho_1}{\rho_2}[P_N - P_{loss}^{(1)}]. \tag{6}$$

In the case of non-randomized push-out mechanism ($\alpha = 1$), the equation for the generating function (2) takes the form

$$\begin{aligned}(7)\quad [(\rho+1)xy - xy^2(\rho_1 x + \rho_2) - 1]\Phi(x,y) &= y^N[1 - xy + \rho_1 x(x-1)y]F_N(x) \\ &+ y(x-1)A(y) + \rho_1(1-x)x^{N+1}y^{N+1}p(N,N) + (xy-1)\rho p_0.\end{aligned}$$

Setting $x = 1$ in (7), and then reducing it by the term $(y-1)$, we get

$$(1-\rho y)\Phi(1,y) = \rho p_0 - y^N F_N(1).$$

Then in the above equation we take subsequently $y = 1$ and $y = 1/\rho$ to obtain

$$(1-\rho)\Phi(1,1) = \rho p_0 - F_N(1) \tag{8}$$

and

$$0 = \rho p_0 - \frac{1}{\rho^N} F_N(1). \tag{9}$$

Solving equations (8) and (9) together with the normalization condition

$$\Phi(1,1) + p_0 + F_N(1) = 1,$$

we obtain the following expressions for p_0, $F_N(1)$ and $\Phi(1,1)$:

$$p_0 = \frac{1-\rho}{1-\rho^{N+2}}, \qquad F_N(1) = \frac{1-\rho}{1-\rho^{N+2}}\rho^{N+1}, \qquad \Phi(1,1) = \frac{1-\rho^{N+1}}{1-\rho^{N+2}}\rho.$$

Next we take $y = 1$ in equation (7) and then reduce it by the term $(x-1)$

$$(1-\rho_1 x)\Phi(x,1) = -(1-\rho_1 x)F_N(x) + A(1) - \rho_1 x^{N+1} p(N,N) + \rho p_0.$$

We now set subsequently $x = 1$ and $x = 1/\rho_1$ in the above equation. This results in the following two equations:

$$(1-\rho_1)\Phi(1,1) = -(1-\rho_1)F_N(1) + A(1) - \rho_1 p(N,N) + \rho p_0, \tag{10}$$

$$0 = A(1) - \frac{1}{\rho_1^N} p(N,N) + \rho p_0. \tag{11}$$

Solving equations (10) and (11), we obtain

$$p(N,N) = \frac{(1-\rho_1)(1-\rho^{N+1})}{(1-\rho_1^{N+1})(1-\rho^{N+2})}\rho\rho_1^N.$$

The loss probability of class 1 packets $P_{loss}^{(1)}$ is given by $p(N,N)$. Then, we note that the stream of lost packets of class 2 consists of the stream of packets with rate $\lambda_2 F_N(1)$ lost when the buffer is full and the stream of packets with rate $\lambda_1(F_N(1) - p(N,N))$ pushed out by packets of class 1. Hence, using the ergodicity property of the system, we obtain formula (6) for $P_{loss}^{(2)}$.

4. Proof of Theorem 1

By substituting boundary condition (1) into equation (2) for the generating function $\Phi(x,y)$, we get

$$[(\rho+1)xy - xy^2(\rho_1 x + \rho_2) - 1]\Phi(x,y) = [1 - xy + \alpha\rho_1 xy(x-1)]y^N V_{N-1}(x)$$
$$+[1-xy]x^N y^N p(N,N) + y(x-1)A(y) + \rho[xy-1]p_0, \tag{12}$$

where $V_{N-1}(x) = \sum_{i=0}^{N-1} x^i p(i,N)$, and hence the expression for $\Phi(x,y)$.

Next, we set $z := xy$ and rewrite equation (12) as follows:

$$[(\rho_1+\rho_2+1)z - \rho_1 z^2 - \rho_2 yz - 1]\Phi\left(\frac{z}{y}, y\right) = [(1-z)y + \rho_1\alpha(z-y)z]y^{N-1}V_{N-1}\left(\frac{z}{y}\right)$$
$$+(z-y)A(y) + (1-z)z^N p(N,N) + \rho(z-1)p_0.$$

Let us now consider the analyticity condition for the generating function $\Phi(z/y, y)$. Namely, the following two conditions have to be satisfied simultaneously

$$(\rho_1+\rho_2+1)z - \rho_1 z^2 - \rho_2 yz - 1 = 0,$$

$$[(1-z)y+\rho_1\alpha(z-y)z]y^{N-1}V_{N-1}\left(\frac{z}{y}\right)$$
$$+(z-y)A(y)+(1-z)z^N P(N,N)+\rho(z-1)p_0=0.$$

The first condition can be rewritten as

$$\rho_2(y-z)z=(1-z)(\rho z-1),$$

which gives

$$y-z=\frac{(1-z)(\rho z-1)}{\rho_2 z}.$$

We substitute the above expression for $y-z$ into the first two terms of the second analyticity condition and then reduce it by $1-z$, to get

$$\left(y-\frac{\rho_1}{\rho_2}\alpha(\rho z-1)\right)y^{N-1}V_{N-1}\left(\frac{z}{y}\right) \tag{13}$$
$$-\frac{\rho z-1}{\rho_2 z}A(y)+z^N p(N,N)-(\rho_1+\rho_2)p_0=0.$$

Next we denote by a and b the roots of the following quadratic equation with respect to the variable z

$$(\rho_1+\rho_2+1)z-\rho_1 z^2-\rho_2 yz-1=0.$$

Now we substitute subsequently the roots a and b into (13), which allows us to eliminate $A(y)$

$$\frac{\rho b-1}{b}\left(y-\frac{\rho_1}{\rho_2}\alpha(\rho a-1)\right)y^{N-1}V_{N-1}\left(\frac{a}{y}\right)$$
$$-\frac{\rho a-1}{a}\left(y-\frac{\rho_1}{\rho_2}\alpha(\rho b-1)\right)y^{N-1}V_{N-1}\left(\frac{b}{y}\right)+\left(\frac{\rho b-1}{b}a^N-\frac{\rho a-1}{a}b^N\right)p(N,N)$$
$$-\rho\left(\frac{\rho b-1}{b}-\frac{\rho a-1}{a}\right)p_0=0.$$

Taking into account the properties of roots of the quadratic equation

$$ab=1/\rho_1,\quad (\rho a-1)(\rho b-1)=\frac{\rho_2}{\rho_1}(\rho y-1),$$

we have

$$((\rho-\rho_1 a)y-q(\rho y-1)\rho_1 a)y^{N-1}V_{N-1}\left(\frac{a}{y}\right)-((\rho-\rho_1 b)y-q(\rho y-1)\rho_1 b)y^{N-1}V_{N-1}\left(\frac{a}{y}\right)$$

$$+(\rho(a^N - b^N) - \rho_1(a^{N+1} - b^{N+1}))p(N,N) + \rho\rho_1(a-b)p_0 = 0,$$

$$\rho y^N \left(V_{N-1}\left(\frac{a}{y}\right) - V_{N-1}\left(\frac{b}{y}\right)\right) - \rho_1(y + q(\rho y - 1))\left(aV_{N-1}\left(\frac{a}{y}\right) - \right.$$

$$\left. bV_{N-1}\left(\frac{b}{y}\right)\right) y^{N-1} + (\rho(a^N - b^N) - \rho_1(a^{N+1} - b^{N+1}))p(N,N) + \rho\rho_1(a-b)p_0 = 0,$$

$$\rho y \sum_{i=1}^{N-1} v_i(a^i - b^i)y^{N-1-i} - \rho_1(y + q(\rho y - 1)) \sum_{i=0}^{N-1} v_i(a^{i+1} - b^{i+1})y^{N-1-i}$$

$$+(\rho(a^N - b^N) - \rho_1(a^{N+1} - b^{N+1}))p(N,N) + \rho\rho_1(a-b)p_0 = 0. \tag{14}$$

By denoting $\cos\varphi = (\rho + 1 - \rho_2 y)/(2{\rho_1}^{1/2})$, the roots a and b can be written in the form

$$a = \frac{\exp(i\varphi)}{{\rho_1}^{1/2}}, \quad b = \frac{\exp(-i\varphi)}{{\rho_1}^{1/2}}.$$

Then equation (14) can be rewritten as

$$\rho y \sum_{i=1}^{N-1} v_i U_{i-1}(t) \frac{y^{N-1-i}}{{\rho_1}^{i/2}} - \rho_1(y + q(\rho y - 1)) \sum_{i=0}^{N-1} v_i U_i(t) \frac{y^{N-1-i}}{{\rho_1}^{(i+1)/2}}$$

$$+ \left(\rho U_{N-1}(t) \frac{1}{{\rho_1}^{(N)/2}} - \rho_1 U_N(t) \frac{1}{{\rho_1}^{(N+1)/2}}\right) p(N,N) + \rho{\rho_1}^{1/2} p_0 = 0, \tag{15}$$

where $t := \cos\varphi = (\rho+1-\rho_2 y)/(2{\rho_1}^{1/2})$ and $U_s(t)$ are the Chebyshev polynomials of the second kind [3]

$$U_s(\cos\varphi) = \frac{\sin(s+1)\varphi}{\sin\varphi}.$$

The Taylor series for the function $U_s(t)$ with respect to y, being actually a polynomial in this case, has the following form

$$U_s(t(y)) = \sum_{s=0}^{s} \frac{U_s^{(i)}(t_0)}{i!}(-1)^i \frac{\rho_2^i y^i}{2^i {\rho_1}^{i/2}}$$

with $t_0 = (\rho+1)(2{\rho_1}^{1/2})$. By changing the order of summation in the expressions

$$\sum_{i=1}^{N-1} v_i U_{i-1}(t) \frac{y^{N-1-i}}{{\rho_1}^{i/2}} = \sum_{l=0}^{N-2} y^l \sum_{k=0}^{l} v_{N-1-k} \frac{U_{N-k-2}^{(l-k)}(t_0)(-\rho_2)^{l-k}}{(l-k)!2^{l-k}{\rho_1}^{(N-1-2k+l)/2}},$$

$$\sum_{i=0}^{N-1} v_i U_i(t) \frac{y^{N-1-i}}{{\rho_1}^{(i+1)/2}} = \sum_{l=0}^{N-1} y^l \sum_{k=0}^{l} v_{N-1-k} \frac{U_{N-k-1}^{(l-k)}(t_0)(-\rho_2)^{l-k}}{(l-k)!2^{l-k}{\rho_1}^{(N-2k+l)/2}},$$

we rewrite equation (15) as follows:

$$\rho \sum_{s=1}^{N-1} y^s \sum_{k=0}^{s-1} v_{N-1-k} \frac{U_{N-k-2}^{(s-k-1)}(t_0)(-\rho_2)^{s-k-1}}{(s-k-1)!2^{s-k-1}{\rho_1}^{(N-2-2k+s)/2}}$$

$$-\rho_1(1+\alpha\rho) \sum_{s=1}^{N} y^s \sum_{k=0}^{s-1} v_{N-1-k} \frac{U_{N-k-1}^{(s-k-1)}(t_0)(-\rho_2)^{s-k-1}}{(s-k-1)!2^{s-k-1}{\rho_1}^{(N-2k+s-1)/2}}$$

$$+\rho_1\alpha \sum_{s=0}^{N-1} y^s \sum_{k=0}^{s} v_{N-1-k} \frac{U_{N-k-1}^{(s-k)}(t_0)(-\rho_2)^{s-k}}{(s-k)!2^{s-k}{\rho_1}^{(N-2k+s)/2}} +$$

$$\left(\rho \sum_{s=0}^{N-1} y^s \frac{U_{N-1}^{(s)}(t_0)(-\rho_2)^s}{(s)!2^s {\rho_1}^{(N+s)/2}} - \rho_1 \sum_{s=0}^{N} y^s \frac{U_{N}^{(s)}(t_0)(-\rho_2)^s}{(s)!2^s {\rho_1}^{(N+s+1)/2}}\right) p(N,N) + \rho{\rho_1}^{1/2} p_0 = 0.$$

Next we use the relation between the derivatives of the Chebyshev polynomials and Gegenbauer polynomials [3, v.2, p.186]

$$U_n^{(m)}(x) = 2^m m! C_{n-m}^{m+1}(x)$$

to get

$$\rho \sum_{s=1}^{N-1} y^s \sum_{k=0}^{s-1} v_{N-1-k} \frac{C_{N-s-1}^{s-k}(t_0)(-\rho_2)^{s-k-1}}{{\rho_1}^{(N-2-2k+s)/2}}$$

$$-\rho_1(1+\alpha\rho) \sum_{s=1}^{N} y^s \sum_{k=0}^{s-1} v_{N-1-k} \frac{C_{N-s}^{s-k}(t_0)(-\rho_2)^{s-k-1}}{{\rho_1}^{(N-2k+s-1)/2}}$$

$$+\rho_1\alpha \sum_{s=0}^{N-1} y^s \sum_{k=0}^{s} v_{N-1-k} \frac{C_{N-s-1}^{s-k+1}(t_0)(-\rho_2)^{s-k}}{{\rho_1}^{(N-2k+s)/2}} +$$

$$\left(\rho \sum_{s=0}^{N-1} y^s \frac{C_{N-s-1}^{s+1}(t_0)(-\rho_2)^s}{{\rho_1}^{(N+s)/2}} - \rho_1 \sum_{s=0}^{N} y^s \frac{C_{N-s}^{s+1}(t_0)(-\rho_2)^s}{{\rho_1}^{(N+s+1)/2}}\right) p(N,N) + \rho{\rho_1}^{1/2} p_0 = 0.$$

Collecting the terms with the same power of y, we obtain the required system of equations:

- $s = 0$

$$\alpha\rho_1 C_{N-1}^1(t_0) v_{N-1} + \left[\rho C_{N-1}^1(t_0) - {\rho_1}^{1/2} C_N^1(t_0)\right] p(N,N) + \rho{\rho_1}^{(N+1)/2} p_0 = 0,$$

- $0 < s < N$

$$\rho \sum_{k=0}^{s-1} \frac{C_{N-s-1}^{s-k}(t_0){\rho_1}^{k+1}}{(-\rho_2)^{k+1}} v_{N-1-k} - {\rho_1}^{3/2}(1+\alpha\rho) \sum_{k=0}^{s-1} \frac{C_{N-s}^{s-k}(t_0){\rho_1}^{k}}{(-\rho_2)^{k+1}} v_{N-1-k}$$
$$+\alpha\rho_1 \sum_{k=0}^{s} \frac{C_{N-s-1}^{s-k+1}(t_0){\rho_1}^{k}}{(-\rho_2)^{k}} v_{N-1-k}$$
$$+ \left[\rho C_{N-s-1}^{s+1}(t_0) - {\rho_1}^{1/2} C_{N-s}^{s+1}(t_0)\right] p(N,N) = 0,$$

- $s = N$

$$-{\rho_1}^{3/2}(1+\alpha\rho) \sum_{k=0}^{N-1} \frac{C_0^{N-k}(t_0){\rho_1}^{k}}{(-\rho_2)^{k+1}} v_{N-1-k} - {\rho_1}^{1/2} C_0^{N+1}(t_0) p(N,N) = 0,$$

or, equivalently,

- $s = 0$

$$\alpha\rho_1 C_{N-1}^{1}(t_0) v_{N-1} + \left[\rho C_{N-1}^{1}(t_0) - {\rho_1}^{1/2} C_N^{1}(t_0)\right] p(N,N) + \rho{\rho_1}^{(N+1)/2} p_0 = 0,$$

- $0 < s < N$

$$\sum_{k=0}^{s-1} \left[\rho \frac{C_{N-s-1}^{s-k}(t_0){\rho_1}^{k+1}}{(-\rho_2)^{k+1}} - {\rho_1}^{3/2}(1+\alpha\rho) \frac{C_{N-s}^{s-k}(t_0){\rho_1}^{k}}{(-\rho_2)^{k+1}} \right.$$
$$\left. +\rho_1\alpha \frac{C_{N-s-1}^{s-k+1}(t_0){\rho_1}^{k}}{(-\rho_2)^{k}}\right] v_{N-1-k} + \alpha \frac{C_{N-s-1}^{1}(t_0){\rho_1}^{s+1}}{(-\rho_2)^{s}} v_{N-1-s}$$
$$+ \left[\rho C_{N-s-1}^{s+1}(t_0) - {\rho_1}^{1/2} C_{N-s}^{s+1}(t_0)\right] p(N,N) = 0,$$

- $s = N$

$$-{\rho_1}^{3/2}(1+\alpha\rho) \sum_{k=0}^{N-1} \frac{C_0^{N-k}(t_0){\rho_1}^{k}}{(-\rho_2)^{k+1}} v_{N-1-k} - {\rho_1}^{1/2} C_0^{N+1}(t_0) p(N,N) = 0.$$

Finally, to obtain an expression for $A(y)$ in terms of $p(k,N), k=0,...,N$ and Chebyshev polynomials, we again substitute subsequently the roots a and b into (13) and subtract one equation from another

$$y^N \sum_{k=0}^{N-1} \frac{a^k-b^k}{y^k} p(k,N) - \frac{\rho_1}{\rho_2}\alpha\rho y^{N-1} \sum_{k=0}^{N-1} \frac{a^{k+1}-b^{k+1}}{y^k} p(k,N)$$

$$+\frac{\rho_1}{\rho_2}\alpha y^{N-1} \sum_{k=0}^{N-1} \frac{a^k-b^k}{y^k} p(k,N) + (a^N-b^N)p(N,N) - \frac{\rho_1}{\rho_2}A(y)(a-b) = 0.$$

As above, taking into account that

$$\frac{a^k-b^k}{a-b} = \frac{U_{k-1}(t)}{\rho_1^{(k-1)/2}},$$

we can express $A(y)$ in terms of $p(k,N), k=0,...,N$ and the Chebyshev polynomials of the second type.

5. On the numerical calculation of mean queue values

Now we consider the calculation of mean queues. The total mean queue of preemptive and non-preepmptive priority packets is given by

$$\bar{n} = \sum_{n=0}^{N} n \sum_{i=0}^{n} p(i,n) = \Phi_y'(1,1) + NV_{N-1}(1) + Np(N,N).$$

The mean queue of preemptive priority packets is equal to

$$\bar{i} = \sum_{n=0}^{N} \sum_{i=0}^{n} ip(i,n) = \Phi_x'(1,1) + V_{N-1}'(1) + Np(N,N).$$

The mean queue of non-preepmptive priority packets is $\bar{n} - \bar{i}$.

We have also $V_{N-1}'(1) = \sum_{i=0}^{N-1} ip(i,N)$.

To derive the unknown functions, we have to do the following steps:

- Substitute $x=1$ into the expression for the generating function(12) and reduce it by $(y-1)$, hence

$$(\rho y - 1)\Phi(1,y) = y^N V_{N-1}(1) + y^N p(N,N) - \rho p_0$$

Differentiating by y and substituting $y = 1$, we have:

$$(\rho - 1)\Phi(1,1) = V_{N-1}(1) + p(N,N) - \rho p_0,$$

$$\rho\Phi(1,1) + (\rho - 1)\Phi'_y(1,1) = NV_{N-1}(1) + Np(N,N)$$

- Substitute $y = 1$ into the expression for the generating function(12) and reduce it by $(x - 1)$, hence

$$(\rho_1 x - 1)\Phi(x,1) = (1 - \alpha\rho_1 x)V_{N-1}(x) + x^N p(N,N) - A(1) - \rho p_0$$

Differentiating by x and substituting $x = 1$, we get:

$$(\rho_1 - 1)\Phi'_x(1,1) + \rho_1\Phi(1,1) = -\alpha\rho_1 V_{N-1}(1) + (1 - \alpha\rho_1)V'_{N-1}(1) + Np(N,N)$$

As $\bar{n} = \Phi'_y(1,1) + NV_{N-1}(1) + Np(N,N)$, $\bar{i} = \Phi'_x(1,1) + V'_{N-1}(1) + Np(N,N)$ and the values $p(i,N)$ are determined by the solution of the system of equations from Theorem 1, the mean queue values are easily determined numerically.

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