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LIMIT THEOREMS FOR BRANCHING PROCESSES WITH RANDOM MIGRATION COMPONENTS*

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1 Introduction

The classical Bienaymé-Galton-Watson (BGW) branching process can be interpreted as mathematical model of population dynamics when the members of an isolated population reproduce themselves independently of each other according to a stochastic law.

Let $\xi = \{\xi_i(\text{n})\}$ be a set of nonnegative integer–valued i.i.d. random variables with p.g.f. $h(s) = Es^{\xi_i(n)}$. Denote $S_n(j) = \sum_{i=1}^{j} \xi_i\,(n)\,, \quad j \geq 1$ and $S_n(0) \equiv 0$. Then the BGW process $\{X_n\}$ is well-defined by the following recurrent formula:

$$X_0 = 1, \quad X_n = S_n(X_{n-1}), \quad n = 1, 2, ... \tag{1}$$

Now, if we interpret $\xi_i(n)$ as number of offspring of the ith member of the $(n-1)$st generation, then X_n is the population size of the nth generation.

Let $h_n(s) = Es^{X_n}$ be the p.g.f. of X_n. Then (1) is equivalent to

$$h_0(s) = s, \qquad h_n(s) = h_{n-1}(h(s)), \quad n = 1, 2, ... \tag{2}$$

It follows from (1) that zero is an absorbing state and the lines of the descendants are independent, i.e. $E(s^{X_n} \mid X_0 = N) = h_n^N(s)$.

It is not difficult to obtain, from (1) (or (2)), that $EX_n = m^n$, where $m = h'(1) = E\xi_i(n)$ is the offspring mean. Obviously, $EX_n \to 0$ if $m < 1$ (subcritical case), $EX_n = 1$ if $m = 1$ (critical case) and $EX_n \to \infty$ if $m > 1$ (supercritical case). Recall that $q = \lim P(X_n = 0) < 1$ for $m > 1$ and $q = 1$ for $m \leq 1$. The classical Kolmogorov and Yaglom limit theorems (see, for example, Athreya and Ney (1972)) for the critical BGW process, i.e. $m = 1$ state that

$$P(X_n > 0) \sim (bn)^{-1};\ \lim P(\frac{X_n}{bn} \leq x \mid X_n > 0) = 1 - e^{-x},\ x \geq 0, \tag{3}$$

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provided that the offspring variance $h''(1) = 2b$ is finite.

In many situations in the applications the population of interest cannot be considered as an isolated population but a random migration i.e. immigration or emigration of population members, takes place instead. In the present paper we shall consider some general models of branching processes that admit two types of emigration and state-dependent immigration. Note that in the models under consideration the lines of descendants are not, in general, independent. It turns out that the population dynamics in this case depends on the relative sizes of the parameters of emigration, immigration and reproduction and the results obtained are quite different from those for the classical BGW process.

2 Models and basic equations

Let $\xi = \{\xi_i(n)\}$, $\eta = \{\eta_1(n), \eta_2(n)\}$ and $I = \{I_n, I_n^0\}$ be three independent sets of integer-valued nonnegative random variables, which are i.i.d. in each set. Define

$$Y_{n=}(S_n(Y_{n-1}) + M_n)^+, \; n = 1, 2, ..., \tag{4}$$

where $P(M_n = -\{S_n(\eta_1(n)) + \eta_2(n)\}) = p_n$, $P(M_n = 0) = q_n$, $P(M_n = \{I_n \; 1(Y_{n-1} > 0) + I_n^0 \; 1(Y_{n-1} = 0)\}) = r_n$, where $p_n + q_n + r_n = 1$, for $n = 1, 2, ...$ and Y_0 is independent of ξ, η and I. As usual $a^+ = \max(a, 0)$.

Now, if we look at Y_n as the size of the nth generation of the population, then by (4) there are three options for the further population evolution due to the migration component M_n: (i) emigration with probability p_n, that is $\eta_1(n)$ families emigrate i.e. totally $S_n(\eta_1(n))$ members live the population (family emigration) and , additionally, $\eta_2(n)$ members randomly selected from different families are eliminated from the further evolution (individual emigration); (ii) no migration with probability q_n, i.e. the reproduction is as in the classical BGW process; (iii) state–dependent immigration with probability r_n, i.e. I_n new members joint the population in the positive states or I_n^0 members appear in the state zero.

If $q_n \equiv 1$ then (4) is equivalent to (1), i.e. $\{Y_n\}$ is a BGW process. The case when $r_n \equiv 1$ and $I_n \equiv I_n^0$ a.s. gives the classical BGW process with immigration. The process $\{Y_n\}$ with $r_n \equiv 1$ and $I_n = 0$ a.s. was studied for the first time by Foster (1971) and Pakes (1971). The case $p_n \equiv 1$ i.e. process with net emigration was considered by Vatutin (1977a) when $\eta_1(n) \equiv 1$ and $\eta_2(n) \equiv 0$ a.s. (one family emigrates only), by Kaverin (1990) when $\eta_2(n) \equiv 0$ a.s. (family emigration only) and by Grey (1988) when $\eta_1(n) \equiv 0$ a.s. (individual emigration only).

Processes with time homogeneous migration i.e. when in (4) $p_n \equiv p$, $q_n \equiv q$ and $r_n \equiv r$ were investigated by Yanev and Mitov (1980, 1983), Nagaev and Han (1980), Han (1980), and Badalbaev and Yakubov (1995). Models with time non-homogeneous migration i.e. when p_n, q_n and r_n are not constants w.r.t. n were studied by Yanev and Mitov (1985) and Badalbaev and Rahimov (1993) among others (see also Rahimov (1995) and the references within).

Note that the state zero is a reflecting barrier for $\{Y_n\}$, which is a regenerative branching process. We say that $\tau = \tau(t)$ is the length of a life-period of $\{Y_n\}$ started at $t \geq 0$ if $Y_{t-1} = 0,\ Y_{t+n} > 0$ for $0 \leq n < \tau$ and $Y_{t+\tau} = 0$. Let us define a stopped–at–zero process $\{Z_n\}$ by

$$Z_n \stackrel{\mathrm{d}}{=} Y_{t+n}\ 1\,\{Z_{n-1} > 0\},\ n = 1, 2, ...;\ Z_0 \stackrel{\mathrm{d}}{=} Y_t > 0, \tag{5}$$

where the equality is in distribution and $1\{C\}$ stands for the indicator random variable of C.

The process $\{Z_n\}$ can be considered as the positive part of $\{Y_n\}$ between two successive points of regeneration (or during on one life-period). Note that, the state zero is an absorbing slate for $\{Z_n\}$ and we have $P(Z_n > 0) = P(\tau > n)$, say.

The process $\{Z_n\}$ with $r_n \equiv 1$, i.e. stopped–at–zero branching processes with immigration only, was considered by Zubkov (1972), Vatutin (1977b), Seneta and Tavaré (1983) and Ivanoff and Seneta (1985).

In the sequel, $h(s) = Es^{\xi}$ is the offspring p.g.f., $H(s_1, s_2) = Es_1^{I_n} s_2^{I_n^0}$ is the p.g.f. of the emigration components, $g(s) = H(s, 1)$ is the p.g.f. of the immigration in the positive states and $g_0(s) = H(1, s)$ is the p.g.f. of the immigration in state zero.

¿From now on we assume that $p_n \equiv p,\ q_n \equiv q$ and $r_n \equiv q$. Let us introduce some more notations

$$\delta(s) = pH(h^{-1}(s), s^{-1}) + q + rg(s),\ \Delta(s) = 1 - \delta(s) - r(1 - g_0(s)),$$

$$\zeta_1(n) = \eta_1(n) - Y_{n-1},\ \zeta_2(n) = \eta_2(n) - S_n(-\zeta_1(n)),$$

$$A_n = \{\zeta_1(n) \geq 0\}, \qquad B_n = \{\zeta_2(n) \geq 0,\ \zeta_1(n) \leq 0\},$$

$$W_n(s) = E(1 - h^{-\zeta_1(n)}(s)s^{-\eta_2(n)})1\{A_n\} + E\left(1 - s^{-\zeta_2(n)}\right)1\{B_n\}.$$

For the p.g.f. $\Psi_n(s) = Es^{Y_n}$ of the process $\{Y_n\}$ we have (see Yanev and Yanev (1997)) the equation

$$\Psi_{n+1}(s) = \delta(s)\Psi_n\left(h(s)\right) + \Delta(s)\Psi_n(0) + pW_n(s)\ . \tag{6}$$

Let $\Phi_n\left(s\right) = Es^{Z_n}$ and $\zeta_1\left(n\right) = \eta_1\left(n\right) - Z_{n-1}$. Then, similarly to (6), one can obtain

$$\Phi_{n+1} = \delta\left(s\right)\Phi_n\left(h\left(s\right)\right) + \left(1 - \delta\left(s\right)\right)\Phi_n\left(0\right) + pW_n\left(s\right) \tag{7}$$

The asymptotic results presented in the next section are obtained by careful study of equations (6) and (7).

3 Limit theorems for critical branching processes with random migration

We shall consider the critical case i.e. $m = 1$ with finite offspring variance i.e. $2b < \infty$. We need also some conditions on the migration component. For the emigration components $\eta_1(n)$ and $\eta_2(n)$ we assume that they are bounded random variables i.e. there exist integers N_1 and N_2 such that

$$\eta_1(n) \le N_1, \qquad \eta_2(n) \le N_2 \ . \tag{8}$$

For the immigration components I_n and I_n^0 we only assume that their first moments are finite. Under these assumptions we obtain an additional critical and recurrence parameter θ, say, which plays a vital role in the asymptotic behavior of the processes $\{Y_n\}$ and $\{Z_n\}$. It is given by

$$\theta = \frac{2E(M_n \mid Y_{n-1} > 0)}{Var\xi_1(n)} = \frac{rEI_n - pE(\eta_1(n) + \eta_2(n))}{b} \ .$$

For our results we need slightly stronger moment conditions on the reproduction and immigration

$$\begin{aligned} &EI_1 \log(1 + I_1) < \infty \text{ if } \theta = 0; \\ &EI_1 \log(1 + I_1) < \infty \quad E\xi_1(1)^2 \log(1 + \xi_1(1)) < \infty \text{ if } 0 < \theta \le 1. \end{aligned} \tag{9}$$

Now, we are in a position to state our first result for the length τ of a life–period of the regenerative branching process $\{Y_n\}$.

Theorem 1 *Assume (8) and (9). Then as* $n \to \infty$,

(i) $P(\tau > n) \sim c(\theta) > 0$ *if* $\theta > 1; \sim c(1)/\log n$ *if* $\theta = 1; \sim c(\theta)/n^{1-\theta}$ *if* $0 \le \theta < 1; = o(1/n)$ *if* $\theta < 0$;

(ii) $E\tau = \infty$ *if* $\theta \ge 0; = M(\theta) < \infty$ *if* $\theta < 0$;

(iii) $P(Y_n = 0) \sim D(\theta)$ *if* $\theta < 0; \sim D(0)/\log n$ *if* $\theta = 0; \sim D(\theta)/n^{\theta}$ *if* $0 < \theta \le 1/2$, $\sum_{k=0}^{n} P(Y_k = 0) \sim K(\theta) n^{1-\theta}$ *if* $1/2 < \theta < 1; \sim K(1) \log n$ *if* $\theta = 1; \sim K(\theta)$ *if* $1/2 < \theta < 1; \sim K_1 \log n$ *if* $\theta = 1; \sim K(\theta)$ *if* $\theta > 1$.

All constants can be exactly calculated. Theorem 1 is proved in Yanev and Yanev (1995).

Corollary 1 *The process* $\{Y_n\}$ *is an aperiodic and irreducible Markov chain which is also: (i) non-recurrent for* $\theta > 1$*; (ii) null-recurrent for* $0 \le \theta \le 1$*; (iii) positive-recurrent for* $\theta < 0$.

Comment 1. Let us consider the stopped–at–zero process with migration component, $\{Z_n\}$. Note that, if $\theta > 1$ then the probability of non-extinction $P(Z_n > 0) = P(\tau > n)$ has positive limit here in the critical case $m = 1$, whereas in the classical BGW process such positive limit exists only in the supercritical (!) case $m > 1$. If $0 < \theta < 1$ then $P(Z_n > 0)$ tends to zero but slower than in the critical BGW process. Finally, when $\theta < 0$ the convergence to zero is faster than that in the critical BGW process. Only when $\theta = 0$ we have a result similar to Kolmogorov's (3) for BGW process.

The asymptotics of the mean and the variance of $\{Y_n\}$ is given in the following theorem, which is proved in Yanev and Yanev (1996).

Theorem 2 *Assume (8) and (9). Then as $n \to \infty$,*

$$EY_n \begin{cases} \sim b\theta n & \text{if } \theta > 0, \\ \sim bn/\log n & \text{if } \theta = 0, \\ = o(n) & \text{if } \theta < 0, \end{cases} \qquad VarY_n \begin{cases} \sim b^2\theta(1+\theta)n^2 & \text{if } \theta > 0, \\ \sim bn^2/\log n & \text{if } \theta = 0, \\ = o(n^2) & \text{if } \theta < 0. \end{cases}$$

Comment 2. If $\theta < 0$ then in the particular case when $\eta_1(n) \equiv 1,\ \eta_2(n) \equiv 0$ a.s., we have obtained under some additional moment conditions that $P(\tau > n) \sim A(\theta)/n^{1-\theta}$, $E\tau = EI_0/(-\theta b)$ and $EY_n \sim B(\theta)n^{1+\theta}$, if $\theta \in (-1;0)$; $\sim B(-1)\log n$, if $\theta = -1$; $\sim B(\theta)$, if $\theta < -1$. We conjecture the same asymptotics in the general case.

Asymptotic behavior of the critical process $\{Y_n\}$ changes significantly with the range of the parameter θ, as it is shown in next theorem.

Theorem 3 *Assume (8) and (9).*

(i) If $\theta < 0$ then there exist $\{v_k\}$, with $\sum_{k=0}^{\infty} v_k = 1$, such that $\lim_{n\to\infty} P(Y_n = k) = v_k$, The p.g.f. $V(s) = \sum_{k=0}^{\infty} v_k s^k$ of the limiting distribution is the unique solution of the functional equation $V(s) = V(h(s))\delta(s) + V(0)\Delta(s) + p\sum_{k=0}^{\infty} W_k(s)$, where $V(0) = (-\theta b)/(-\theta b + rEI_1^0 + paW)$, with $W = \sum_{k=0}^{\infty} W_k^{'}(1) < \infty$ and $a = r(1 - g_0(0))$.

(ii) If $\theta = 0$ then $\lim_{n\to\infty} P(\log Y_n/\log n \le x) = x \in [0,1]$.

(iii)If $\theta > 0$ then $\lim_{n\to\infty} P(Y_n/bn \le x) = \int_0^x y^{\theta-1}e^{-y}dy/\Gamma(\theta), \quad x \ge 0$.

The proof of Theorem 3 can be found in Yanev and Yanev (1996).

Comment 3. If $\theta < 0$ we have that the mean of the emigration is greater than the mean of the immigration i.e. the emigration dominates immigration. In this case, there exists a stationary distribution for the process with migration as in the subcritical (!) BGW branching process with immigration. In the case $\theta > 0$ (immigration dominates emigration) the limit distribution is Gamma, the same as in the critical BGW process with immigration only. If $\theta = 0$ (balanced migration) we have $Y_n \sim n^U$ (in distribution), where U is uniformly distributed in $[0,1]$. This result is similar to one for BGW processes allowing immigration in the state zero only, obtained by Foster (1971) .

Let us now consider the stopped–at–zero process $\{Z_n\}$ in the more general case when the initial distribution of the number of ancestors Z_0 may have not finite mean but belongs to the normal domain of attraction of a stable low with parameter $\rho \in (0,1]$. The following result is proved in Yanev and Yanev (1997).

Theorem 4 *Let $Q(s) = Es^{Z_0} = 1 - (1-s)^\rho L(1/(1-s))$, $0 < \rho \le 1$, where $L(x)$ is a slowly varying function as $x \to \infty$. Assume (8) and (9) and one of the following two conditions: $\{Q^{'}(1) = EZ_0 = \infty$ and $-\infty < \theta < \infty\}$ or $\{Q^{'}(1) = EZ_0 < \infty$ and $\theta > 0\}$. Then*

(i) $P(Z_n > 0) \sim k(\theta,\rho)/n^\rho$ if $\theta + \rho < 1$; $\sim k(\theta,\rho)/n^{1-\theta}$, if $\theta + \rho \ge 1$, or $\{Q'(1) < \infty$ and $\theta \in [\,0,1]\}$; $k(\theta,\rho)$ if $\theta > 1$;

(ii) If $\theta + \rho < 1$, then $\varphi_n(s) = E(\exp\{-\lambda Z_n/bn\} \mid Z_n > 0) \to \varphi(\lambda)$, where $\varphi(\lambda) = 1 - C\lambda^\rho/(1+\lambda)^{\theta+\rho} - \theta\lambda\int_0^1 (1+x)^{-\rho}(1+x\lambda)^{-\theta-1}dx$ and $C = \Gamma(1-\theta)\Gamma(1-\rho)\ /\ \Gamma(1-\theta-\rho)$;

(iii) If $\theta + \rho \ge 1$, or both $Q'(1) < \infty$ and $\theta > 0$ hold, then $\varphi_n(s) \to (1+\lambda)^{-(\theta\vee 1)}$.

Corollary 2 *If $\theta+\rho \geq 1$, or both $Q'(1)<\infty$ and $\theta>0$ hold, then*

$$\lim_{n\to\infty} P(\frac{Z_n}{bn} \leq x \mid Z_n>0) = \begin{array}{ll} \int_0^x y^{\theta-1}e^{-y}dy/\Gamma(\theta) & \textit{if } \theta>1, \\ 1-e^{-x} & \textit{if } \theta \leq 1. \end{array}$$

If $\theta<0$ then

$$\varphi(\lambda)=1+\theta\lambda^{\rho}B_{1/(1+\lambda)}(-\theta,1-\rho)/(1+\lambda)^{\theta+\rho},$$

where $B_x(\alpha,\beta)=\int_0^x u^{\alpha-1}(1-u)^{\beta-1}du$ is the uncomplete Beta function.

Comment 4. Since $P(Z_n>0)=P(\tau>n)$, Theorem 4(i) extends the result of Theorem 1(i). It follows from (i) that the initial distribution has dominated influence on the asymptotics provided that $\theta+\rho<1$, otherwise the migration component determines the asymptotic behavior of the process.

4 Conclusions

The results presented here reveal new effects, in comparison with the classical Kolmogorov and Yaglom's results (3), in the asymptotic behavior of branching processes due to the migration component. One can distinguish three different modes of asymptotics for the critical process with migration, $\{Z_n\}$: (i) critical–supercritical when $(m=1,\theta>1)$, (ii) critical–critical when $(m=1, 0\leq\theta\leq 1)$, and (iii) critical–subcritical when $(m=1,\theta<0)$. In the first case, when $\theta>1$, we have $P(Z_n\to\infty)>0$ (non-extinction), whereas in the other two cases, when $\theta\leq 1$, we have $P(Z_n\to\infty)=1$ (extinction).

One direction for future research is to study the asymptotic behavior of the branching processes with migration component in the multitype setting.

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