PLISKA

STUDIA MATHEMATICA BULGARICA

ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

Pliska Stud. Math. Bulgar. **13** (2000), 155-160

PLISKA
STUDIA MATHEMATICA
BULGARICA

ON SOME SUFFICIENT CONDITIONS FOR HIGH BREAKDOWN POINT OF ML ESTIMATORS* *

Maya Marintcheva

High breakdown point estimators $LME(k)$ and $LTE(k)$ for location and scale are obtained for symmetrical exponentially decreasing density family.

1 Introduction

Let us consider a defined on p-dimensional Euclidean space E^p multivariate density family: $f(x, \mu, S) = \frac{Cp}{\sqrt{det(S)}}\varphi((x-\mu)'S^{-1}(x-\mu))$, with fixed shape function φ, where μ and S denote location and scale correspondingly. Vandev [1] developed a breakdown point technique for the robustified LME and LTE estimators. Their breakdown point is not less than $\frac{n-k}{n}$, i.e. they are $\frac{n-k}{n}$-robust, for k, chosen by the user within some reasonable range of values. Vandev and Neykov [2] studied the connection of the finite sample breakdown point, dimension of the Gaussian distribution and the notion of d-fullness, introduced in [1]. Now following the technique [3], a high breakdown point for LME and LTE is obtained for $\varphi(z) = O(e^{-\alpha z^{\beta}})$; α is a positive constant and β lies between 0 and 1. A contra example in case of $\varphi(z) = 1/z^p$ demonstrates the need of exponential decrease for the theory.

2 Basic Definitions

Definition 1 *Estimators $LME(k)$ and $LTE(k)$ of the unknown parameter θ, for $k > \frac{n}{2}$ are defined as:*

$$LME(k)(x_1, x_2, \ldots, x_n) = arg\min_{\theta}(-\ln f(x_{l(k)}, \theta)),$$

*The author owes a dept of gratitude to Prof. D. Vandev for his remarks and directions

*This paper is partly financed by I-625/96 of Bulgarian Ministry of Education Science and Technologies

$$LTE(k)(x_1, x_2, \dots, x_n) = arg\min_{\theta} \sum_{i=1}^{k} [-\ln f(x_{l(i)}, \theta)],$$

where $f(x_{l(1)}, \theta) \geq f(x_{l(2)}, \theta) \geq \ldots \geq f(x_{l(n)}, \theta)$ *are the ordered density values.*

Definition 2 *The real valued function* $g(z)$ *defined on a topological space* Z *is called subcompact, if its Lebesque sets* $L(M) = \{z : g(z) \leq M\}$ *are compact or empty for every positive constant* M*.*

Definition 3 *A finite set* F *of* n *functions is called d-full, if for any subset of cardinality* $d > 0$ *from* F*, the supremum of all functions in this subset is a subcompact function.* [1]

Theorem 1 *If* $\frac{1}{2}(n+d) \leq k \leq n-d$*, then* $LME(k)$ *and* $LTE(k)$ *are* $(n-k)$*-robust.* [1]

Lemma 1 *(a standard Linear Algebra fact) Let* α_i *are the eigenvalues of* S*, and there exist real constants* α *and* β*, such that* $\alpha \leq \alpha_i \leq \beta$*. Then* $\alpha \leq \|S\| \leq \beta$*.*

Lemma 2 *If* $\lambda_1, \lambda_2, \ldots, \lambda_p$ *are positive real numbers and* $H = \sum_{i=1}^{p} (\lambda_i - \ln \lambda_i)$*, then* $e^{-H} \leq \lambda_i \leq eH/(e-1)$. [3]

3 Results

Lemma:* Let $x_1, x_2, \ldots, x_n$ be a set of independent observations in E^p over a random variable ξ with density function: $f(x, \mu, S) = \frac{Cp}{\sqrt{det(S)}} \varphi((x-\mu)' S^{-1} (x-\mu))$, and let F be the finite set: $F = \{-\ln f(x_1, \mu, S), -\ln f(x_2, \mu, S), \ldots, -\ln f(x_n, \mu, S)\}$. Then:

$$LME(k)(x_1, x_2, \ldots, x_n) = arg\min_{S} (-\ln f(x_{l(k)}, \mu, S)), \text{ and}$$

$$LTE(k)(x_1, x_2, \ldots, x_n) = arg\min_{S} \sum_{i=1}^{k} (-\ln f(x_{l(i)}, \mu, S));$$

both have a breakdown point not less than $\dfrac{n-k}{n}$, for:

$$\frac{1}{2}(n+p+1) \leq k \leq n-p-1 \quad \text{and} \quad \varphi(z) = O(e^{-\alpha z^{\beta}}): \quad \alpha > 0,\ 0 < \beta < 1.$$

Contra – example:

Let choose a function $\varphi(z) = 1/z^p$ that does not satisfy the assumption to be $O(e^{-\alpha z^{\beta}})$. In this case we show that $A = \left\{ S : \max_{i \in \{1,2\ldots,p+1\}} \{-\ln f(x_i, \mu, S)\} \leq K \right\}$ contains matrices S with eigenvalues that can not be separated from the zero point. Therefore A is not a compact set [5], we have not $(p+1)$-fullness and Theorem1 is not applicable.

*These robust estimators are useful tools for variety of theories including Teletrffic theory.

$$A = \left\{ S : \frac{1}{2}\ln(detS) - \ln\frac{1}{(\max_{i\in\{1,2,\ldots,p+1\}}(x_i-\mu)\prime S^{-1}(x_i-\mu))^p} \le K \right\} =$$
$$= \left\{ S : \frac{1}{2}\ln(detS) + p\ln\max_{i\in\{1,2,\ldots,p+1\}}((x_i-\mu)\prime S^{-1}(x_i-\mu)) \le K \right\}$$

$$\max_{i\in\{1,2,\ldots,p+1\}}((x_i-\mu)\prime S^{-1}(x_i-\mu)) \le \sum_{i=1}^{p+1}((x_i-\mu)\prime S^{-1}(x_i-\mu)) \quad \Rightarrow$$

$$\begin{aligned} A \subset A_1 &= \left\{ S : \frac{1}{2}\ln(detS) + p\ln\sum_{i=1}^{p+1}((x_i-\mu)\prime S^{-1}(x_i-\mu)) \le K \right\} \\ &= \left\{ S : -\frac{1}{2}\ln(detS^{-1}) + p\ln Tr(BS^{-1}) \le K \right\} \\ &= \left\{ S : -\frac{1}{2}\ln(detBS^{-1}) + p\ln Tr(BS^{-1}) \le K_1, \text{ where: } K_1 = K - \frac{1}{2}\ln(detB) \right\} \\ &= \left\{ S : -\ln\sqrt{det(BS^{-1})} + \ln\left(Tr(BS^{-1})\right)^p \le K_1 \right\} = \\ &= \left\{ S : \left(\sum_{i=1}^{p}\lambda_i\right)^p \Big/ \sqrt{\prod_{i=1}^{p}\lambda_i} \le K_2 \right\}. \end{aligned}$$

$K_2 = e^{K_1}$ and $\lambda_i, i \in \{1,2,\ldots,p\}$ are the eigenvalues of BS^{-1}, so we have that: $det(BS^{-1}) = \prod_{i=1}^{p}\lambda_i$ and $Tr(BS^{-1}) = \sum_{i=1}^{p}\lambda_i$. For $\lambda_1 = \ldots = \lambda_p = \lambda$:
$\left(\sum_{i=1}^{p}\lambda_i\right)^p \Big/ \sqrt{\prod_{i=1}^{p}\lambda_i} = \frac{p^p\lambda^p}{\lambda^{\frac{p}{2}}} = p^p\lambda^{\frac{p}{2}}$, which ever can be made smaller than K_2 for $\lambda \to 0$.

4 Proof

Conclusions follow from [1] and [3] if only $(p+1)$-fullness of F is obtained. Considering definitions 1-3 and Theorem 1, it only remains to show that for any constant K:

$$A = \left\{ S : \max_{i\in\{1,2\ldots,p+1\}}\{-\ln f(x_i,\mu,S)\} \le K \right\}$$

is compact or empty. As closeness is easy to obtain [3] we concentrate on proving that A is bounded. It is shown by means of expanding A until a bounded set A_4 is achieved. As $A \subset A_4$, A is bounded too.

$$A=\left\{S:\frac{1}{2}\ln(detS)-\ln\varphi\left(\max_{i\in\{1,2,\ldots,p+1\}}((x_i-\mu)\prime S^{-1}(x_i-\mu))\right)\leq K+\ln C^p=K_1\right\};$$

We need the following inequalities (1),(2) and denotations (3),(4).

$$(1)\qquad \max_{i\in\{1,2,\ldots,p+1\}}((x_i-\mu)\prime S^{-1}(x_i-\mu))\geq\frac{1}{p+1}\sum_{i=1}^{p+1}((x_i-\mu)\prime S^{-1}(x_i-\mu))$$

$$(2)\qquad \sum_{i=1}^{p+1}((x_i-\mu)\prime S^{-1}(x_i-\mu))\geq\sum_{i=1}^{p+1}((x_i-\overline{x})\prime S^{-1}(x_i-\overline{x}))$$

$$(3)\qquad B=\frac{1}{p+1}\sum_{i=1}^{p+1}(x_i-\overline{x})(x_i-\overline{x})\prime$$

$$(4)\qquad Tr(BZ)=\frac{1}{p+1}\sum_{i=1}^{p+1}((x_i-\overline{x})\prime Z(x_i-\overline{x})),\quad Z=S^{-1}.$$

$$A\subset A_1=\left\{S:-\tfrac{1}{2}\ln(detBZ)-\ln\varphi\left(\tfrac{1}{p+1}\sum_{i=1}^{p+1}(x_i-\overline{x})\prime S^{-1}(x_i-\overline{x})\right)\leq K_2\right\},$$

where: $K_2=K_1-\frac{1}{2}\ln detB$.

We choose a constant $k=[(1-\beta)\ln p-\ln\alpha-\ln\beta]/\beta$. Let γ_i for $i\in\{1,2,\ldots,p\}$ be the eigenvalues of BZ and let consider: $\lambda_i=(e^{-k}\gamma_i)^{\frac{1}{\beta}}$ which is equivalent to $\gamma_i={\lambda_i}^{\beta}e^k$. In terms of λ_i we have that:

$$det(BZ)=\prod_{i=1}^{p}\gamma_i=e^{pk}\prod_{i=1}^{p}{\lambda_i}^{\beta},\quad Tr(BZ)=\sum_{i=1}^{p}\gamma_i=e^k\sum_{i=1}^{p}{\lambda_i}^{\beta},$$

and $A_1=\left\{S:\sqrt{det(BZ)}.\varphi(Tr(BZ))\geq L\right\},\ L=-K_2.$

$$A_1=\left\{\lambda_1,\lambda_2,\ldots,\lambda_p:\sqrt{e^{pk}\prod_{i=1}^{p}{\lambda_i}^{\beta}}.\varphi\left(e^k\sum_{i=1}^{p}{\lambda_i}^{\beta}\right)\geq L\right\}$$

$$=\left\{\lambda_1,\lambda_2,\ldots,\lambda_p:\sqrt{\prod_{i=1}^{p}{\lambda_i}^{\beta}}.\varphi\left(e^k\sum_{i=1}^{p}{\lambda_i}^{\beta}\right)\geq L_1,\text{where: }L_1=Le^{\frac{-pk}{2}}\right\}$$

$$=\left\{\lambda_1,\lambda_2,\ldots,\lambda_p:-\ln\sqrt{\prod_{i=1}^{p}{\lambda_i}^{\beta}}-\ln\varphi\left(e^k\sum_{i=1}^{p}{\lambda_i}^{\beta}\right)\leq L_2,\text{where: }L_2=-\ln L_1\right\}$$

$$= \left\{ \lambda_1, \lambda_2, \ldots, \lambda_p : -\frac{1}{2} \ln \prod_{i=1}^{p} {\lambda_i}^{\beta^2} \le \beta \ln \varphi \left(e^k \sum_{i=1}^{p} {\lambda_i}^{\beta} \right) + L_3, \text{where: } L_3 = \beta . L_2 \right\}$$

$$= \left\{ \lambda_1, \lambda_2, \ldots, \lambda_p : \frac{1}{2} \sum_{i=1}^{p} {\lambda_i}^{\beta^2} - \frac{1}{2} \ln \prod_{i=1}^{p} {\lambda_i}^{\beta^2} \le \frac{1}{2} \sum_{i=1}^{p} {\lambda_i}^{\beta^2} + \beta \ln \varphi \left(e^k \sum_{i=1}^{p} {\lambda_i}^{\beta} \right) + L_3 \right\}.$$

Because $\frac{1}{2} \sum_{i=1}^{p} {\lambda_i}^{\beta^2} \le \sum_{i=1}^{p} {\lambda_i}^{\beta^2}$, we enlarge A_1 to A_2 :

$$A_2 = \left\{ \lambda_1, \lambda_2, \ldots, \lambda_p : \frac{1}{2} \sum_{i=1}^{p} {\lambda_i}^{\beta^2} - \frac{1}{2} \ln \prod_{i=1}^{p} {\lambda_i}^{\beta^2} \le \sum_{i=1}^{p} {\lambda_i}^{\beta^2} + \beta \ln \varphi \left(e^k \sum_{i=1}^{p} {\lambda_i}^{\beta} \right) + L_3 \right\}$$

$$= \left\{ \lambda_1, \lambda_2, \ldots, \lambda_p : H \le 2 \left(\sum_{i=1}^{p} {\lambda_i}^{\beta^2} + \beta \ln \varphi \left(e^k \sum_{i=1}^{p} {\lambda_i}^{\beta} \right) \right) + 2L_3 \right\};$$

$$H = \sum_{i=1}^{p} {\lambda_i}^{\beta^2} - \ln \prod_{i=1}^{p} {\lambda_i}^{\beta^2} = \sum_{i=1}^{p} \left({\lambda_i}^{\beta^2} - \ln {\lambda_i}^{\beta^2} \right)$$

Once again A_2 enlarges to A_3 according to: $0 \le r \le 1 : \sum_{i=1}^{p} {y_i}^{r} \le \left(\sum_{i=1}^{p} y_i \right)^{r} \frac{1}{p^{r-1}}$ [4], substituted for $y_i = {\lambda_i}^{\beta}, i \in \{1, 2, \ldots, p\}$ and $r = \beta$:

$$\sum_{i=1}^{p} {\lambda_i}^{\beta^2} \le \left(\sum_{i=1}^{p} {\lambda_i}^{\beta} \right)^{\beta} \frac{1}{p^{\beta-1}} \tag{5}$$

$$A_3 = \left\{ \lambda_1, \lambda_2, \ldots, \lambda_p : H \le 2 \left[\left(\sum_{i=1}^{p} {\lambda_i}^{\beta} \right)^{\beta} \frac{1}{p^{\beta-1}} + \beta \ln \varphi \left(e^k . \sum_{i=1}^{p} {\lambda_i}^{\beta} \right) \right] + L_3 \right\}$$

Now we remember that: $\varphi(z) = O(e^{-\alpha z^{\beta}})$, $\varphi(z) \le A e^{-\alpha z^{\beta}} \iff \ln \varphi(z) \le \ln A - \alpha z^{\beta}$, for any constant $A > 0$. For $z = e^k \sum_{i=1}^{p} {\lambda_i}^{\beta} : \ln \varphi(e^k \sum_{i=1}^{p} {\lambda_i}^{\beta}) \le \ln A - \alpha . e^{k\beta} \left(\sum_{i=1}^{p} {\lambda_i}^{\beta} \right)^{\beta}$ and A_3 goes into A_4, where:

$$A_4 = \left\{ \lambda_1, \lambda_2, \ldots, \lambda_p : H \le 2 \left[\left(\sum_{i=1}^{p} {\lambda_i}^{\beta} \right)^{\beta} \frac{1}{p^{\beta-1}} + \beta \ln A - \alpha \beta e^{k\beta} \left(\sum_{i=1}^{p} {\lambda_i}^{\beta} \right)^{\beta} \right] + 2L_3 \right\}$$

$$= \left\{ \lambda_1, \lambda_2, \ldots, \lambda_p : H \le 2 \left(\sum_{i=1}^{p} {\lambda_i}^{\beta} \right)^{\beta} (p^{1-\beta} - \alpha \beta e^{k\beta}) + L_4, \text{ where: } L_4 = 2\beta \ln A + 2L_3 \right\}$$

$$= \{ \lambda_1, \lambda_2, \ldots, \lambda_p : H \le L_4 \}.$$

Because of the special choice of $k:\quad k=[(1-\beta)\ln p-\ln\alpha-\ln\beta]/\beta,\quad$ we have that:

$$p^{1-\beta}-\alpha\beta e^{k\beta}=p^{1-\beta}-\alpha\beta e^{\beta[(1-\beta)\ln p-\ln\alpha-\ln\beta]/\beta}=\quad p^{1-\beta}-\alpha\beta(p^{1-\beta})/(\alpha\beta)=0.$$

Finally $A\subset A_1\subset A_2\subset A_3\subset A_4$ is obtained. As from Lemma 2 and $H\le L_4$ we have that: $e^{-L_4}\le\lambda_i\le\frac{eL_4}{e-1}$, when apply Lemma 1, we obtain that A is bounded.

Bibliography

[1] D. L. Vandev. A Note on the Breakdown Point of the Least Median and Least Trimmed Estimators. *Statistics and Probability Letters* **16** (1993), 117-119.

[2] D. L. Vandev, N. M. Neykov. Robust Maximum Likelihood in the Gaussian Case. In: New Directions in Statistical Data Analysis and Robustnes, 1993, 259-264.

[3] D. L. Vandev, M. Z. Marintcheva. On High Breakdown Point Estimators of Scale and Location in the Multidimensional Case. SDA'95, September, Varna, 1995, 23-27.

[4] A. Marshall, J. Olkin. Inequalities: Theory of Majorization and Its Application, Moskwa, MIR, 1983, 107-10 (in Russian).

[5] G. P. Klimov. Applied Mathematical Statistics, Sofia, Science and Art, 1975 (in Bulgarian).

Dept Telecommunication
Institute of Mathematics
Bulgarian Academy of Sciences
Acad. G.Bontchev str., bl. 8
1113 Sofia, Bulgaria
e-mail: `telecom@math.bas.bg`