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SCALING AND MULTISCALING EXPONENTS IN NETWORKS AND FLOWS

Edward C. Waymire

1 Introduction

The main focus of this paper is on mathematical theory and methods which have a direct bearing on problems involving multiscale phenomena. Modern technology is refining measurement and data collection to spatio-temporal scales on which observed geophysical phenomena are displayed as intrinsically highly variable and intermittant heirarchical structures,e.g. rainfall, turbulence, etc. The heirarchical structure is reflected in the occurence of a natural separation of scales which collectively manifest at some basic unit scale. Thus proper data analysis and inference require a mathematical framework which couples the variability over multiple decades of scale in which basic theoretical benchmarks can be identified and calculated. This continues the main theme of the research in this area of applied probability over the past twenty years. While the problems and methods are relatively new to "mainstream mathematical research", the recent article entitled "Multiscale Science: A Challenge for the Twenty First Century" in SIAM NEWS, v.30(8), Oct. 1997, by James Glimm and Daniel H. Sharp points to a widespread emerging interest in techniques which seek to couple distinct space and time scales.

Specific mathematical problems which this theory seeks to address are largely motivated by applications to river basin hydrology and to fluid turbulence. Two of the most basic and dominant elements of the hydrologic cycle at the river basin scale which arise in the regionalization of floods are precipitation inputs (rainfall, snowmelt) and basin topography (river networks, hillslopes). So the science begins with the quantification of precipitation and topography with a view toward identifying their signature in the resulting flows. A most illusive question regarding the research on the dominant components of precipitation and landforms has been that of putting them together to determine flow structure. Moreover the mathematical formulation and the tools of data analysis have reached a maturity in which a quite feasible approach can be described. Much of the data collected and reported on heirarchical structures in geophysics is in the form of log-log plots of some quantity versus a length scale. This has led to new classes of self-similar landform models and multiplicative cascade models for precipitation as inputs into the network flow equations. This mathematical formulation provides a framework in which connections between the scaling exponents of the extreme flows and those of the precipitation and landform exponents will assist in analyzing and interpreting the data being collected in this area. The corresponding mathematical questions therefore concern various methods of computing and analyzing the multiscale heirarchy in terms of simple idealized models. The prospect of a theory which computes structure functions (or multiscaling exponents) for floods from corresponding structure function calculations on landforms and precipitation defines the frontiers of this research. In the case of fluid turbulence the corresponding objective is to relate the statistical multiscaling structure of the energy dissipation and fluid velocity fields to the physics defined by the Navier-Stokes equations. In particular the statistical cascade structure intrinsic to these equations continues to be an important area for mathematical study.

The organization of the paper is in two parts. As is the case in many areas of application, determining a proper mathematical formulation suitable to both the data and the goals of river basin hydrology is itself a non-trivial activity. So we begin with an attempt to construct a contemporary theoretical formulation which promises to fill this role in section two. In section three a more precise mathematical formulation of the elements involved and resulting mathematical problems are described. This second part consists of three subsections 3.1, 3.2 and 3.3 on multiplicative cascades (T-martingales), on tree networks, and finally on flow extremes. respectively. Various open problems will be noted along the way. For a more comprehensive overview on this topic see Gupta and Waymire (1997).

2 An Overview of Some Motivating Applications

River basin hydrology is the study of the hydrologic cycle on river basins. The big picture is as follows. A river basin is a geographic region containing a branched channel network (tree). Two hills appear on each side of every channel in the network. In this sense, a network partitions a region into an ensemble of hills. The study of the hydrologic cycle on each hillside involves a partition of rainfall into infiltration through the near surface unsaturated soils and surface runoff. The infiltrated water recharges the soil moisture in the unsaturated zone and ground water aquifers. Some of this moisture from the soil surface is reintroduced into the atmosphere via evapotranspiration, and some appears as subsurface runoff in a channel adjoining the hill. In this manner rainfall and/or snowmelt is transformed into surface and subsurface runoff. The water flow on a hillside also erodes sediments. The runoff and sediments are fed into a channel network for their journey towards an ocean. All of these physical processes are highly variable in space and time.

Typical spatial scales of a river basin span about four decades from 100-1,000,000 m. The hydrologic cycle on subbasins larger than a single hillside represents a spatially integrated behavior of several hills along a channel network. An understanding of spatial variability among hillsides and their interactions through a channel network is required for this integration. A mathematical approach to this problem is to introduce a general (localized) difference equation of mass conservation in a channel network-hills system; e.g. (2.1), (2.2) below. Solutions of this equation may be obtained using idealized examples to analyze how key features of spatial variability in rainfall, landforms and runoff are reflected in peak flows. This approach is motivated by the long standing classic problem of flood prediction from ungauged basins (PUB), referring to those basins where gauged runoff data is unavailable and the data base consists of topographic maps, historical precipitation records, remote measurements, etc. Of course the success of such an approach depends critically on proper identification and description of some dominant observable structures. For the problem at hand one seeks a theory which serves to unify multi-scaling structure of rainfall, river networks, and river flows as defined by various empirically observed (multi-)scaling exponents.

Consider a *drainage network* represented by a finite binary rooted tree graph τ. The *size* of τ, denoted $||\tau||$, is defined by the total number n of vertices (sources and junctions) $v \in \tau$, excluding the root. The root is a distinguished vertex, denoted $\emptyset$, representing the *network outlet.* The root may be used to direct the tree and we identify edges and vertices according to the convention that to each edge e there is a unique vertex $v = \underline{e}$ nearest to the root; the opposite vertex connected by e to $\underline{e}$ is denoted $\overline{e}$. We fix a time scale Δ representing a typical time for flow to traverse an arbitrary edge e in the network and choose time units such that $\Delta = 1$. Let $r(e,t)$ represent the surface and subsurface runoff intensity rate (in units of length per unit time) from the hillsides which arrives at an edge e during times t-1 to t. If $|a(e)|$ denotes the area of hillsides draining an edge e then $R(e,t) = r(e,t)|a(e)|$ represents the volumetric flux from the hillside into the edge e during time t-1 to t. Water is then drained from edges $f = (\overline{f}, \underline{f})$ into the uniquely determined edge $e \in \tau$, defined by $\overline{e} = \underline{f}$. River discharges are represented by a space-time random field $q(e,t)$ assigned to edges $e \in \tau$ at times $t \geq 0$, representing the volume of flow across the edge $(\overline{e}, \underline{e})$, or past the vertex $\overline{e}$, per unit time satisfying the *mass balance equation*

$$\Delta S(e,t) = -q(e,t)\Delta t + \sum_{f:\underline{f}=\overline{e}} q(f,t-1)\Delta t + R(e,t)\Delta t \quad t \geq 1, \tag{2.1}$$

where the left-hand side $\Delta S(e,t) = S(e,t) - S(e,t-1), e \in \tau, t \geq 0$, represents the change in total volume of runoff stored per unit time in the edge e in time increment $\Delta t = 1$; (storage simply refers to the difference in input and output per unit time). The runoff intensity term $R(e,t) = \tilde{R}(\theta(e,t))$ is determined from a hillslope water balance equation of the form

$$\Delta\theta(e,t) = I_r(e,t)\Delta t - E_T(\theta(e,t))\Delta t - \tilde{R}(\theta(e,t))\Delta t, \tag{2.2}$$

where $\theta(e,t)$ is hillside storage, $I_r(e,t)$ is precipitation intensity, and E_T is the evapotranspiration rate in time (t-1,t). The main known results concern the very special case $\Delta S = \Delta\theta = 0$ and $E_T = 0$, so that $I_r = R$, while extensions to the more general framework defined by (2.1) and (2.2) are largely open.

A main quantity of interest here is the *peak flow* (*flood values*) $Q(e), e \in \tau$, defined for an interval of time $[0, T]$ by the maximum "instantaneous flows,"i.e.

$$Q(e) := \max_{0 \leq t \leq T} q(e, t). \tag{2.3}$$

In particular one is interested in the scaling structure of $Q(e)$ as a function of the size of the subtree $||\tau(\overline{e})||$. Data analysis in geographically homogeneous regions is conducted by plots of log-drainage area $\log |A|$ against the log statistical moments $\log E|Q^j(A)|$ of the annual (T = 1yr) peak flows $Q(A)$ at a gauged outlet of a sub-basin of area $|A|$ for fixed j; see Gupta, Mesa, Dawdy (1994). An interesting characteristic exhibited by many data sets is that (i) such plots are log-log linear for fixed j; and (ii) the slopes of the lines $s(j)$ are linear as a function of j; for example in the cases of peak flows generated from stratiform (spatially uniform) rainfall, as for example in the northwestern United States, and in snowmelt generated runoff. For other types of rainfall generated floods the slopes $s(j)$ are nonlinear; see Dawdy and Gupta (1995). If the function $s(j) = \theta j$ is linear in j then the field may be said to be *(weakly) simple scaling*, while if $s(j)$ is nonlinear in j, necessarly either concave or convex in j, then the field is said to be *(weakly) multiscaling*. To fix a mathematical example illustrating such structure consider $R(A) = R_0 \exp\{B_{-\log A}\}$, where B_t is standard Brownian motion starting at zero. As above, denote the slopes of the lines in these log-loglinear plots for each j by the function $s(j)$, referred to as a *structure function exponent* for floods. The spatial extrapolation of this "multiplicative structure"is the random cascade model originating in Kolmogorov's statistical theory of turbulence. In the case of turbulence one imagines the introduction of kinetic energy via some mechanism (e.g. large scale stirring of the fluid), which is then redistributed to lower scales by the splitting off of eddies by some random proportions.

A similar formulation based on observed scaling properties is also possible for rainfall distributions. The hierarchical structure of spatial rainfall fields takes the form of clusters of high intensity rain cells embedded in clusters of lower intensity regions, called small mesoscale areas (SMSA), which are in turn embedded in rainbands of identifiably lower intensity, called large mesoscale areas (LMSA), embedded in a still larger scale synoptic rain area of lower rainrate. This structure is supported by radar and raingage observations. While this structure is the supposed consequence of combined effects of vertical and horizontal motions, the precise dynamics of rainfall formation are not available. As a rule of thumb, the (possibly artificial) scales of these regions decrease by successive factors of $\frac{1}{10}$ from the synoptic scale through LMSA, SMSA, and down to a cell, while the corresponding rainrates nearly double at each level until the scale of a cell where this rule generally breaks down; supercells are possible where the rainrate may be larger than the SMSA by several orders of magnitude.

One of the earliest studies of the heirarchical variability of rainfall was that of LeCam (1961) based on cluster point processes and random measures of the type also occurring in the study of the clustering of galaxies, earthquake aftershock sequences, population growth, etc. The development of new mathematical methods, namely characteristic functional theory for random measures and generalized random fields, to represent space-time rainfall phenomena in LeCam (1961) was also motivated by the hydrologic application being addressed in this theory. Namely, quoting from LeCam (1961): *"The problems encountered by this group included the evaluation of probabilities of excessive discharges, the evaluation of probabilities of excessive droughts, as well as the development of optimal management procedures for the big and small hydroelectric reservoirs. As the studies progressed, the need for a mathematically tractable description of the random structure of stream flow became more and more imperative. To obtain such a description it was found necessary to start with a description of the random structure of rainfall."* This problem continues to be a basic motivation for research in surface water hydrology. Of course the high intensity localized small scale structure cannot apriori be ignored in connection with floods.

To determine scaling structure in spatial rainfall consider the behavior of rainfall moments over regions $\lambda\Delta$ of area λ^2 where simple scaling would imply

$$\log \mathbf{E}R^h(\lambda\Delta) = h\theta \log \lambda + c_h. \tag{2.4}$$

In particular simple scaling translates into the two properties:
i. log-log linearity between a specified moment and length scale.
ii. a linear change in slope $s(h) = \theta h$ of the line as a function of moment order.
The analysis of spatial rainfall data again leads to the very interesting observations that property (i) is preserved but the slope function $s(h)$ in (ii) is nonlinear; see Gupta and Waymire (1993), Over and Gupta (1994).

Finally we turn to the heirarchical structure of the landforms (river networks). A natural *scale of resolution* associated with river networks was introduced by Horton (1945) and later refined by Strahler (1957) according to the following algorithm: First the vertices of either degree one or two will be called "non-branching". Those of degree one are called *leaves.* All leaves and adjacent paths of adjacent non-branching vertices are assigned order one. The orders of all other vertices (or associated edges) are recursively defined as the maximum of orders of the offspring vertices when these are not all equal, else it is the common order of the offspring incremented by one. A contiguous path of edges of equal order is called a *stream* of the said order. The order of the root ϕ defines the *order of the network* τ and is denoted $\omega(\tau)$. This scheme provides an "order or scale of resolution" in which the given tree is regarded to be at the finest scale of resolution and the next level of coarsening is obtained by removing the order 1 streams. The next level of "non-branching" vertices in the pruned tree are assigned order 2. The next level of coarsening is obtained by pruning off the (lowest) order 2 streams, etc. As an aside it may be of independent interest to note that the algorithm for network order described above has been shown to provide a natural optimization parameter in binary arithmetic register allocation problems for certain classes of "arithmetic flow" in computer science; e.g. Ershov (1958), Flajolet and Prodinger (1986).

Modern technology has made available the topographic data of the entire United States down to 90 meter resolution (30 meter in some places) in the form of Digital Elevaton Maps from which river network data is readily available at these resolutions. Given the notion of order described above, a finite tree graph for which the number T_{ij} of order j subtrees supported by a degree two vertex of an order i stream, called a *stream* *order generator,* is *(a)* the same for each order i stream in the network and *(b)* a function of i, j only through $i-j,\ j \leq i$, is called *topologically self-similar*; ie. the matrix of stream order generators is Toeplitz. Of course in actual river network data analysis one computes the sample average $\overline{T}_{ij}$ of the number of order j subtrees supported by the various streams of order i in the network. The available data on river network morphology includes average channel lengths, meander statistics, sub-basin areas, elevations drops, etc. in great detail over several decades of scale. It is an empirical certainty that symmetries in the form of self-similarity and recursive embedding are the natural order in regions absent of geological controls. An interesting mathematical example of this phenomena, though perhaps not widely known, is that of the critical binary Galton-Watson branching process where one may check that $\mathbf{E}T_{i,j} = ab^{i-j}$ with $a = .5, b = 2$. Natural river network data, though of this general form, consistently departs from these computed values for a, b.

In summary, the general problem of analysing the extremes of flows (floods) in a region involves a mathematical framework which will acomodate descriptions and relationships between the inputs (e.g.rainfall, snowmelt) and the landform storage and routing of flows (e.g. surface and subsurface charging, evapotranspiration losses, network routing). Space-time precipitation and channel network structure are viewed as given elements of the hydrologic cycle in a basin which possess natural heirarchical scaling structure. The broad question is then to determine how properties of rainfall and landform topography are reflected in the the flows from the basin.

As noted above, fluid turbulence is another flow which is characteristically highly intermittant and variable in both space and time, and amenable to multiplicative models of the type of interest to this theory. Since the first half of this century the basic Navier-Stokes equations of fluid mechanics and Kolmogorov's statistical cascade theory have stood side by side, sharing certain scaling and dimensional consistency but not otherwise mathematically related. In recent years there have been a number of new attempts to "compute"multiscaling exponents directly from Navier-Stokes equations and related physical flow equations; see e.g. Constantin and Fefferman (1994), Constantin (1994), She and Leveque (1993), She and Waymire (1995), Dubrulle (1994), Frisch(1997). It is worth emphasising that unlike the case of rainfall phenomena which is largely limited to a purely statistical description, i.e. there do not seem to be equations for rain, classical fluid mechanics provides a framework in which one has a physics of incompressible viscous fluids in the form of equations, albeit also phenomonological to certain extents. Moreover, the recent paper by LeJan and Sznitman (1997) shows in no uncertain mathematical terms that a *statistical cascade structure* (with Markovian dependent multipliers) is as intrinsic to solutions of unrestricted Navier-Stokes equations for three-dimensional turbulence as a Brownian motion is to the solution of the heat equation! In view of these developments and the remarkably excellent agreement between experimental data, numerical simulations of Navier-Stokes equations, and the log-Poisson exponents, the outstanding problem of understanding the extent to which the scaling exponents corresponding to the log-Poisson cascade statistics can be *derived* as the appropriate correction to Kolmogorov's log-Normal hypothesis is both intensely interesting and a potentially achievable goal of probability and pde's.

3 Mathematical Elements

In this section of the theory we shall more precisely describe aspects of the mathematical framework which acomodate the phenomena and problems discussed in section 2. There are a number of mathematical and statistical problems which are key to the continued development of multiscaling theory which will be identified along the way.

3.1 Multiplicative Cascades

Let us begin with a class of random measures (mass or energy distributions) on $X \subseteq R^d$ which may be described as follows. Let μ be an arbitrary Borel measure on X (eg. Lebesgue measure on $X = [0,1]^d$) and consider the sequence of random measures $\mu_n(dx) = Q_n(x)\mu(dx)$, where the "random densities"$\{Q_n(x) : x \in X, n = 0,1,2,\ldots\}$ comprise a sequence of a.s. non-negative random Borel functions. μ_n represents the spatial distribution of "stuff"at the length scale $\lambda = b^{-n}, b > 1$. There are various natural *conservation laws* which may be imposed. An important example is the conservation condition on averages of the form $E[Q_{n+1}(x)|Q_k, k \leq n] = Q_n(x), x \in X, n \geq 1$ (ie. a martingale property). Motivated by the cascade picture in statistical turbulence, this particular class of conservation laws was introduced by Jean-Pierre Kahane (1987) and termed a T-martingale property (where here T is the metric space X). One observes essentially from the martingale convergence theorem that for a continuous bounded function f on X, the sequence of random variables $\{\int_X f d\mu_n\}$ will a.s. converge. By this one obtains a limit measure μ_∞ (in the sense of vague convergence). One also writes $\mu_\infty = Q_\infty\mu$ to denote this random transformation from μ to μ_∞, although it is generally not the case that μ_∞ has a density with respect to μ, i.e. there is *no* corresponding $Q_\infty(x)$. An important special case is obtained by taking $X = [0,1]^d$. For simplicity of the exposition consider the case $d = 1$. For *i.i.d.* mean one non-negative random variables $W_\gamma : \gamma \in X^* = \cup_{n=0}^{\infty}\{0,1,\ldots,b-1\}^n$, referred to as the *cascade generators*, let

$$Q_n(x) = \prod_{i=0}^{n} W_{\gamma|i}, \quad \text{for } x \in J_n(\gamma) = [\sum_{i=1}^{n}\gamma_i b^{-i}, \ \sum_{i=1}^{n}\gamma_i b^{-i} + b^{-n}), \tag{3.1}$$

where $\gamma|0 = \emptyset$, $\gamma|i = (\gamma_1,\ldots,\gamma_i)$. Then $\{Q_n(x)\}$ defines a homogeneous independent multiplicative cascade $\mu_\infty = Q_\infty\mu$ for a given finite Borel measure μ on X. It is a simple matter to check by the SLLN that for each fixed $x \in X$, $Q_n(x) \to 0$ as $n \to \infty$ with probability one unless $P(W_\gamma = 1) = 1$. Thus one may expect mass distributions $\mu_\infty = Q_\infty\mu$ which are nondegenerate to be thinly supported. Problems considered in this theory generally involve the determination of reasonable criteria for nondegeneracy of μ_∞ and, in those cases of nondegeneracy, to compute both the fine and large scale structure for more general *dependent* cascades.

Criterion for nondegeneracy were originally obtained by Kahane and Peyriere (1976) via an analysis of the simple distributional recursion

$$Z_\infty =^d b^{-1}\sum_{j=0}^{b-1} W_j Z_\infty(j), \tag{3.2}$$

where $Z_\infty(j)$ are *i.i.d.* as the total mass $Z_\infty = \lambda_\infty(X)$ and independent of the generators. Distributional fixed points of this recursion are also of interest in other connections; eg, see Durrett, Liggett (1983), Holley, Liggett (1981) and Rösler (1994). However these are largely analytic methods which exploit independence in such a strong way as not to apply to correlated models. In Waymire and Williams (1994) an entirely new approach was announced which, in addition to providing a simple new probabilistic rather than analytic approach, applies to a large class of correlated cascades; eg. Markov, exchangeable generators along paths. The key ideas behind this approach are the introduction of the following three tools: (i) size-bias change of measure, (ii) a percolation method, and (iii) cascade weighting systems as explained in Waymire and Williams (1995,1996). It is now very clear that these are precisely the right tools for analysing cascades. Further evidence for the power of this approach provided by the application of size-bias theory to limit theorems of classical branching processes previously studied by heavy analytic machinery in Lyons, Pemantle, Peres (1995) and Kurtz, Lyons, Pemantle, and Peres (1997). In the case of turbulence the remarkable logPoisson correction to Kolmogorov's logNormal hypothesis on the generator distribution for fully developed turbulence in the inertial range originated with physical arguments on the size-bias moments of She and Levesque (1993). There is no doubt that the percolation methods, weight systems, and size biasing will continue to play an important role in the analysis of cascades.

A simple but illustrative choice for the generator distribution of a homogeneous independent cascade is the "zero-nonzero" model defined by Bernoulli generators:

$$W = \begin{cases} \frac{1}{p} & \text{with probability } p \\ 0 & \text{with probabiility } q = 1-p. \end{cases} \tag{3.3}$$

For this model, with uniform initial measure μ, the evolution of non-zero mass is a Galton-Watson branching process Z_n with mean offspring $m = bp$ and $\mu_n(X) = Z_n(\frac{1}{p})^n b^{-n} = \frac{Z_n}{m^n}$, so that non-degeneracy occurs iff $m = bp > 1$. In spite of its simplicity, the importance of this case both mathematically and for the applications cannot be overstated. It relates two basic critical parameters: the nonextinction parameter and the critical Hausdorff carrying dimension (exponent). The percolation method derived in Waymire and Williams (1995) reduces the general problem of computing carrying dimension to a nondegeneracy problem by composition with "zero-nonzero" (Bernoulli) models. Composition with an independent such model provides a percolation method in the spirit of work by Lyons (1990), since non-degeneracy occurs here if $b^{\dim(\mu)}p > 1$, and degeneracy occurs if $b^{\dim(\mu)}p < 1$. In the case of independent generators one also has degeneracy at criticality, however this need not more generally be the case for correlated generators, and depend on the specific models.

The fine scale structure of measures carried on thin sets is a topic of interest both in the mathematical and the physical sciences. The spectrum of singularities has proven to be an important quantity for models in both rainfall and statistical turbulence as it identifies the scaling exponents $s(h)$ of Section 1 via a Legendre transform. To develop a perspective consider that expected value computations in the case of independent random cascades are easily obtained as follows. Let $\Delta_\lambda(i), i = 1, 2, \ldots$ denote a partition of a region X of d-dimensional space into cells at the length scale λ. Then

$$\log_b E[\sum_i \mu_\infty^h(\Delta_\lambda(i))] = -d\chi_b(h)\log(\lambda) + \log E\mu_\infty^h(X), \tag{3.5}$$

where

$$\chi_b(h) = \log_b \left[EW^h\right] - (h-1) \tag{3.6}$$

and

$$\frac{\log_b Prob[\mu_\infty(\Delta_\lambda) > \lambda^{d\alpha}]}{\log \lambda} \to -d\chi_b^*(1-\alpha), \quad \lambda \to 0 \tag{3.7}$$

where

$$\chi_b^*(a) = \sup_h [ah - \chi_b(h)] \tag{3.8}$$

is the Legendre transform of $\chi_b(h)$. Note from (3.5) that the structure function exponent $s(h)$ for random cascades is given by $-d\chi_b(h)$. Of course applications to turbulence and rainfall data limits the data to single sample realizations and it is necessary to "drop the expectations"for the above results to be useful. This was achieved by Holley and Waymire (1992) under suitable bounds on the cascade generators in the case of independent cascades. In particular this means that the slope function $s(h)$ furnishes an empirical estimate of the moment generating function of the cascade generators! The significance is obvious since the distribution of the generators comprise the apriori unknowns. Moreover, Troutman and Vecchia (1997) have recently obtained Normality of the asymptotic sampling distribution for h suitably small. While simulations show that the sampling distribution is in general non-Gaussian for larger values of h, the precise determination is unsolved. Another important related open problem concerns the identification of suitable topologies to quantify the stability of $Q_\infty\mu$ with respect to the perturbations in the distribution of the generators. In view of the criticality of survival, one expects abrupt changes at some parameters but the overall instability picture is incomplete.

There is another very natural resolution of scales which may be viewed in terms of a dimension disintegration of the form obtained by Cutler (1986) and Kahane and Katznelson (1990) for Borel measures μ :

$$\mu(\cdot) = \int \mu_\beta(\cdot)\nu(d\beta), \tag{3.9}$$

where the corresponding dimension spectral measure of μ_β is a Dirac point mass; i.e., the components μ_β are unidimensional. In Waymire and Williams (1996,1997) size-biasing, the percolation method and weighting systems are used to compute the dimension disintegrations for correlated cascades $\mu = \mu_\infty$ which include certain classes of Markov and exchangeable generators.

An entirely new and interesting class of dependent cascades is obtained when "local correlations"are also permitted among the generators in the vector $(W_{\gamma * i}\ :\ i =$

$0, 1, \ldots b-1$). An interesting feature of local correlations is that they lead to an essential change in the conservation law and a violation of the $T-martingale$ property. However, all is not lost. In particular, while the sequence $\int_X f d\mu_n$ fails to be a bounded martingale for all bounded continuous functions, it is a bounded martingale for a dense subset of such functions. This has led us to introduce the notion of a *graded T-martingale* as the proper framework for local correlations. While certain fundamental aspects of T-martingale theory go over without change to graded T-martingales, Kahane's T-martingale decomposition is lost. This presents interesting new challenges for the computation of fine scale structure since, for example, our proof of the percolation theorem uses Kahane's T-martingale decomposition theorem. From the point of view of intended applications, an interesting way in which local correlations are introduced statistically is by conditioning on non-degeneracy as is done by the very act of observation! The implications of conditioning on data analysis and parameter estimation represents another important problem for this theory.

In addition to purely spatial distributions, it is important to consider dissipative temporal evolutions. On the mathematical side, for example the theory of superprocesses provides a Markovian time evolution of random measures in which mass is re-distributed infinitesimally by critical birth-death branching and spatial diffusion; see Dawson, Perkins (1991). In the present framework one may consider a cascade time evolution obtained by replacing the generator $W_{x|n}$ by a temporal stochastic process $\{W_{x|n}(t) : t \geq 0\}$. This leads to a cascade evolution $Q_\infty(t)\mu = \mu_\infty(t)$. Of particular mathematical interest is the case in which $\log W_{x|n}(t)$ is a process with independent increments. In this case one may show that the log-infinite divisibility of the generator makes the process $\mu_\infty(t)$ Markov by an application the cascade composition theorem in Waymire and Williams (1995).y Two important classes of log-infinitely divisible distributions for our considerations are the log-Poisson and the the Bernoulli generator (the latter being log-infinitely divisible when viewed as a probability distribution over the extended real numbers). There is some applied literature in which movies of storms are produced corresponding to temporal birth-death Markov processes for the Bernoulli generators. It is of interest to determine the corresponding evolutionary properties of the cascade in the space of measures.

We close this subsection by mentioning the broader mathematical relevance of the T-martingale results for other applications. For example, one may note that by judicously ignoring certain correlations in spin glass and random polymer models, the partition function may be represented as the total mass of an independent cascade having lognormal generators; see Collet and Koukiou (1992), Derrida (1991) and references therein. So it is also of more general interest to consider correlated cascades in this connection. There is also overlap with the branching random walk theory of Kingman (1975) and Biggins (1976).

3.2 Tree Networks

We begin with a well-studied stochastic model which in the hydrology and geomorphology literature serves as a frame for viewing both agreements and departures with various empirical observations. This model was introduced into geomorphology by Shreve (1967) and is referred to as the *random model* in which all binary rooted trees of size n are assigned equal probability. For our purposes we view this model as a Bieneyme-Galton-Watson branching process conditioned on total progeny. We choose to focus on this model for definiteness since this is where the most complete set of precise mathematical results are known. Alternative models will be introduced along the way, and finally a broad class of Gibbsian models will be cited that contains all of these as special cases depending on a choice of parameters.

Let $\mathbf{T}$ be the space of *labelled tree graphs rooted* at $\emptyset$. An element τ of $\mathbf{T}$ may be coded as a *set* of finite sequences of positive integers $< i_1, i_2, \ldots, i_n > \in \tau$ such that: $(i)\emptyset \in \tau$ is coded as the empty sequence; (ii)If $< i_1, \ldots, i_k > \in \tau$then $< i_1, \ldots i_j > \in T \,\forall\, 1 \leq j \leq k$: (iii)If $< i_1, i_2, \ldots, i_n > \in \tau$ then $< i_1, \ldots i_{n-1}, j > \in \tau \,\forall\, 1 \leq j \leq i_n$. If $< i_1, \ldots, i_n > \in \tau$ then $< i_1, \ldots, i_{n-1} > \in \tau$ is referred to as the *parent* vertex to $< i_1, \ldots i_n >$. A pair of vertices are connected by an edge (adjacent) if and only if one of them is parent to the other. In this way edges are identified with the (unique) non-parental or *descendant* vertex as in the previous section. This specifies the graph structure of τ and makes τ a rooted connected graph without cycles. $\mathbf{T}$ may be viewed as a metric space with metric $\rho(\tau, \gamma) = (\sup\{n : \gamma|n = \tau|n\})^{-1}$, and $\tau|n = \{< i_1, \ldots, i_k > \in \tau : k \leq n\}$. The countable dense subset $\mathbf{T}_0$ of finite labelled tree graphs rooted at $\emptyset$ makes $\mathbf{T}$ a Polish space. This fact is useful for the construction of stochastic network models as probability distributions on $\mathbf{T}$.

The *Bieneyme-Galton-Watson probability distributions (BGW)* for a single progenitor and given offspring distribution $p_k, k = 0, 1, \ldots$ is a probability on the Borel sigma field of $\mathbf{T}$ for which the probability assigned to a ball $B(\tau, \frac{1}{N}), \tau \in \mathbf{T}_0, N \in \{1, 2, \ldots\}$ is

$$P(B(\tau, \frac{1}{N})) = \prod_{v \in \tau|(N-1)} p_{l(v)}, \tag{3.10}$$

where $l(v) = \#\{j :< v, j > \in \tau|N\}$. The *weighted BGW model* refers to a random field $\{W(e)\}$ of positive weights independent of τ.

Observe that in the flow equation (2.1) in the case of unit instantaneously applied runoff defined by $R(e, 0) = 1, R(e, t) \equiv 0, t \geq 1$, one obtains, ignoring storage terms, i.e. $S(e, t) \equiv 0$, that

$$q(e, t) = Z_{\overline{e}}(t - 1), t = 1, 2, \ldots, \tag{3.11}$$

where $\{Z_v(t) : t = 1, 2, \ldots\}$ is the number of edges located t generations above the vertex v in the subtree $\tau(v)$ of τ consisting of edges $e \in \tau$ at height $|e| > |v|$ and on a directed path connected to v. Hydrologists and geomorphologists refer to the function $t \to Z_v(t), t = 1, 2, \ldots,$ as the *local width function* at v. In the case that network height is measured in units of distance after transforming distance to time via a constant velocity u, i.e. $t = \frac{|e|}{u}$, the width function provides a *unit hydrograph* kernel for the basin based on simple landform considerations. The structure of the width function depends on the underlying network model for τ.

Troutman and Karlinger (1984) used generating function methods to compute the expected width function in the case of weighted BGW random trees for weight distributions whose moment generating function exists in a neighborhood of the origin. Let

$$\mu_n(h) = E[Z(h)|\nu = n]; \tag{3.12}$$

i.e. the best width function predictor from total progeny counts in the sense of least squares. Let K_n be the normalization constant defined by

$$K_n = \int_0^\infty \mu_n(h)dh. \tag{3.13}$$

and define a probability measure F_n with density $K_n^{-1}\mu_n(h)$, suitably scaled. That is,

$$\frac{dF_n(h)}{dh} = a_n K_n^{-1}\mu_n(a_n h), \;\; h \geq 0 \tag{3.14}$$

where a_n is positive scale parameter. If we take $a_n = \sqrt{n}$ in (3.14), then $F_n \Rightarrow F$ where $F'(h) = \frac{h}{2}e^{-\frac{h^2}{4}}$, is a *Rayleigh density*. Recently an extension of this result has been obtained in Ossiander, Waymire, Zhang (1997) which, in addition to the expected value calculation, under nearly best possible conditions on the tails of the weight distribution also provides the sample path fluctuation law of the width function as that of an occupation time for a Brownian excursion. This also provides a weak form of a conjecture of Aldous (Conjecture IV, 1991). The more difficult problem of proving weak convergence of the local time processes was recently solved in the case of unit weights by Dromta and Gittenberger(1996) using generating function methods and by Kersting (personal communication) using random walk transformations, but the local extension is open for weighted trees.

Using generating function methods these calculations have also been made to allow for certain *nonhomogeneous* weight distributions which depend on location through size of the drainage network in Waymire (1992). In particular suppose that the weights are indpependently distributed such that the weight at a vertex of a subtree of size m has an exponential distribution with mean $e^{-(m-1)\theta}\mu, \theta > 0$. Then $F_n \Rightarrow F$ where F is uniquely determined by its moments given by

$$\int_0^\infty h^k F(dh) = \frac{\mu^k e^{k\theta}\hat{r}(\frac{1}{4}e^{-(k+1)\theta})k!}{\hat{r}(\frac{1}{4}e^{-\theta}\sqrt{\prod_{j=1}^k (1-e^{-j\theta})}},$$

where $r(s) = \frac{1}{2}(1-\sqrt{1-4s})$. The decay of the mean roughly corresponds to the concave shape of river basins in which the weights represent elevation drops and the larger the drainage area supported by a vertex the closer to the outlet and the smaller the elevation drop in an edge. An important special case for which there are no precise results is that in which the exponentially decaying mean is replaced by a power law decay of the from $m^{-\theta}\mu$. In fact data analysis suggests this power law form (Gupta and Waymire,1989), but closed form asymptotics remain open for this case.

Another functional of interest in river network analysis is the *main channel length* defined here as the height of the tree; for alternative definitions see Troutman and Karlinger (1993). Since clearly this is a continuous functional one gets weak convergence to the height of the Brownian excursion immediately from the above theorem. Apart from the log factor, the conditions in Ossiander etal (1997) on the tails of the weight distribution are nearly best possible. However in the special case of this particular functional comparison to the results of Durrett, Kesten, and Waymire (1991) suggest that slightly sharper results are indeed possible. Kesten (1994) has also recently calculated the asymptotic distribution conditioned on both total progeny n and the unweighted height k_n, such that $\frac{k_n^2}{n}$ is bounded away from 0 and ∞ and the weights have finite fourth moment. Kesten shows that the centered weighted height rescaled by $n^{-\frac{1}{4}}$ is asymptotically Gaussian. As Kesten notes, this shows that most of the fluctuation in the weighted height is due to fluctuations in the unweighted height in this case.

The interest in main channel length is inspired by work of Hack (1957) on the behavior of the length of the main channel as a function of basin size. Hack's law is a set of empirical observations which reports the growth of the main channel to be $O(Area)^\alpha$ with $\alpha \approx .6$. A model which has been shown to naturally exhibit this behavior is the *coalescing random walk* Scheidegger (1967), Ngyuen (1990). The coalescing random walk, on the other hand, lacks space filling properties of real networks.

Let us now turn to stream order properties of the random model. As noted at the end of section 2, the expected BGW critical tree is self-similar in the sense of Toepltz stream order generators. A natural extension to a form of *stochastic self-similarity* may be introduced as follows. Define a map π on the subset $\mathbf{T}_0$ of finite trees in $\mathbf{T}$ by $\pi(\{\emptyset\}) = \emptyset$, else $\pi(\tau)$ is the tree graph obtained by pruning the lowest order streams from τ. Also define $\overline{\tau}$ as the tree graph obtained by identifying adjacent vertices of degrees one or two with a single vertex. Then the order $\omega(\tau)$ of the tree may be expressed as

$$\omega(\tau) = \inf\{n : \pi^{(n-1)}(\tau) = \{\emptyset\}\} \tag{3.15}$$

We refer to the invariance under the composite map $\overline{\pi}$ of the distribution of a finite random tree τ, conditional on $Z_0 > 0$, as *stochastic self-similarity.* Recent computations of Burd and Waymire (1997) show that a BGW model is stochastically self-similar if and only if it has a critical binary offspring distribution. Perhaps because actual river network evolution is more properly the result of coalescence than branching, the relevance of BGW models appears to be limited to at most Shreve's simple random model. In fact, while computations for branching models are greatly aided by the Markov property, basic structural deficiencies often occur with such models, e.g. in connection with Hack's law described earlier. The construction of other stochastically self-similar models is being considered from the point of view of other invariant evolutions in the space of trees with $\emptyset$ as an absorbing state as described above.

Let us now turn to classes of deterministic tree networks which are self-similar in the sense of Toeplitz stream order generators. Recursive replacement trees may be constructed as *iterated function systems* (IFS) in the plane as defined in Falconer (1985), for example. Using the IFS-generator one obtains a sequence of sets approximating the limiting invariant set by iterating the process of replacing each line segment by a similar copy of the generator. While the planar similitudes are defined by the IFS-generator only to within reflection, the orientations may be prescribed by showing the first iteration $s(G)$. We are interested in trees, referred to as *recursive replacement trees*, for which the IFS-generator G is itself a finite b-ary tree graph embedded in the plane, say with n vertices of degree $b+1$. In particular, this generator defines an IFS in the plane consisting of $N = nb+1$ similarity maps with common similarity ratio $\alpha = \frac{1}{n+1}$. The stream order generators of recursive replacement trees are then given by

$$T_1 = (b-1)(n-1), \quad T_k = (b-1)n^2(n+1)^{k-2}, \; k \geq 2. \tag{3.17}$$

where $T_k \equiv T_{j+k,j}$. Peckham (1995) has obtained a number of results on the statistics of these trees by generating functions methods. However, his method requires an ansatz in which he guesses the general form of the solution and then solves for the parameters. While this approach is not entirely rigorous, the results appear to be correct under some further conditions. For example, his generating function calculation of the dimension of the recursive replacement tree agrees with calculations using more standard geometric measure theory from Falconer (1985), namely $dim = \frac{\log nb+1}{\log n+1}$.

The width function asymptotics are known for two classes of special recursive replacement trees, the so called *Peano trees* and *uniform b-ary trees*. The Peano tree is represented by a class of self-similar trees with branching number b = 3 and generators $\{T_1 = 0, T_k = 2^{k-1} : k = 2, 3, \ldots\}$. In particular, for the Peano tree, the width function converges weakly to a continuous singular probability measure on $[0,1]$ as $m \to \infty$, namely the induced infinite product measure $(\frac{1}{4}\delta_0 + \frac{3}{4}\delta_1)^{\mathbf{N}}$ under the map $\phi(x) = (x_1, x_2, \ldots), x = \sum_{i=1}^{\infty} x_i 2^{-i}, x \in [0,1], x_i \in \{0,1\}$; see Ossiander etal (1997) for details. The b-ary uniform trees are defined by generators $\{T_k(b) = (b-1)2^{k-1}, k = 1, 2, \ldots\}$. Note that these are the expected stream order generators for the critical Bieneyme-Galton-Watson binary branching process. In this case the width function converges weakly to the uniform distribution on $[0,1]$ as $m \to \infty$; see Ossiander etal (1997). The calculation of the width function asymptotics for the broader class of recursive replacement trees than the above two examples is open.

The full generality in which problems of the above type occur involves (i.) the more general versions of (2.1) and (2.2) than represented by the width function; and (ii.) more general classes of tree distributions for river networks. Perhaps the most general class of models one would consider in the context of river networks is that of the so-called two parameter Gibbsian network models; see Troutman and Karlinger (1997) for an overview. This class contains the above random model and the coalescing random walk model, among others, for certain parameter choices. It is therefore natural to consider the robustness of the results cited above within this broader class.

3.3 Network Flow Extremes

As noted earlier, in the simplest hydrologic context one imagines uniformly distributed runoff over the network τ_n and traveling at a constant unit velocity $u = 1$. Then the instantaneous unit runoff hydrograph at the outlet (∅) in time t is represented by the width function $Z_n(t)$ suitably normalized. The (conditional) expected value is the best approximation in the mean square sense given the size of the network. This simple idealization illustrates the use of some basic results for prediction problems based on the width functions which can be computed from *maps* of river basins. For example, using Shreve's random model one obtains a Rayleigh density for the predicted hydrograph for large networks with constant velocity.

To most simply illustrate the objective of the theory consider the case of uniform rainfall over a square partitioned by the Peano network described in Section 3.2. As discussed in Section 3.2, the width function for the Peano network can be viewed as a Binomial cascade with parameters $p_0 = 1/4, p_1 = 3/4$. Taking unit velocity for the flow one sees immediately the scaling exponent for the peak flows from the definition of dimension as follows. Since $\lambda^{-dimD} \propto$ the number of cells which cover the *contributing set* D at the scale λ, which in turn is $Q(\lambda)$ normalized by λ^2, i.e. $Q(\lambda)\lambda^{-2}$, one has for uniform rain and constant velocity that $Q(\lambda)\lambda^{-2} \sim \lambda^{-dimD}$, where as computed in Section 3.2, $dimD = \frac{\log 3}{\log 2}$. Alternatively, the peak flow at the network outlet $Q(\lambda_n) = \max_t q^{(n)}(t)$ at the nth scale of resolution (and length $\lambda_n = 2^{-n}$) is $(\frac{3}{4})^n$. Thus one has

$$\log Q(\lambda_n) = n \log(\frac{3}{4}) = (2 - \frac{\log 3}{\log 2}) \log \lambda_n. \tag{3.18}$$

Gupta etal (1997) provide calculations for the structure function of peak flows in the case of a rainfalll model having Bernoulli (zero/nonzero) generators composed with the Peano network landform based on an approximation in which peak flows at different length scales λ_n are replaced by the cascade mass distributed over the contributing set D at these scales. This approximation is supported by numerical simulations for large values of the rainfall probability p. The computed structure function of the peak flow exponent combines the network dimension and rainfall exponent. These preliminary calculations illustrate the nature of the broad theoretical objective in PUB. In fact, a significant part of the PUB problem is reduced to this and the corresponding mathematical problem for general cascades on the Peano network.

In spite of the overwhelming complexity inherent in the problem of understanding the transformations between rainfall, landforms and floods, there is promise that a mathematical approach will be possible that will provide a framework for analysing, interpreting, and possibly even predicting data on flows from information about local landforms (e.g. in the form of topographic maps), and regional precipitation records.

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Department of Mathematics
Oregon State University
Corvallis, OR 97331 USA