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# LARGE DEVIATIONS AND BRANCHING PROCESSES

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These lecture notes are devoted to present several uses of Large Deviation asymptotics in Branching Processes.

## Overview

# 1 Introduction

The large deviation techniques are relevant in various topics of the branching processes theory. There are many intuitive reasons for that. Let us just give one.

As supercritical branching processes are connected with exponential growing of populations, and subcritical ones with exponential decreasing of presence probabilities, it is rather natural to find large deviations exponential estimates. In various supercritical spatial branching models, there are two phenomena: the branching structure and the process along a branch. If the latter one is a random walk, then the large deviation asymptotics for sums of i.i.d. random variables plays an important role. At level $n$ and speed $a$ there is a competition between the exponentially small probability $\exp -nh(a)$ (where $h$ is the Cramer transform of the motion) and the exponential number of branches $m^n$. If $h(a) - \log m < 0$ the effective of branches following approximately the speed $a$ is roughly $\exp -n(h(a) - \log m)$. If $h(a) - \log m > 0$ then the probability of finding a branch of speed approximately $a$ is very small, actually $\exp -n(h(a) - \log m)$. This topic is widely developped in Section 5 with two types of results, determination of the propagation front and precise results for populations (range of supercriticality) or probabilities (range of subcriticality).

In section 2, we give a background on large deviations as it is presented now in the literature (see for instance Dembo-Zeitouni), limiting ourselves to the main theorems. In section 3 we recall the notations and various models of branching processes. In section 4 we present some recent results of large deviations in Galton-Watson process (tail behavior of the limit r.v. $W$ and growth rates of the generation size).

An excellent bibliography on Branching Processes is in Vatutin (1993). The survey of large deviations in branching processes given here is by no means complete. I omitted superprocesses by lack of space (time). The density dependent branching processes generate also nice large deviation theorems (Klebaner and Zeitouni (1994), Pierre-Loti-Viaud and Portal (1993)).

# 2 Background on Large Deviations

In this survey we state the main results with different notations, which allows to apply them directly in the subsequent paragraphs.

## 2.1 Cramer-Chernov theorem

Let $\mu$ be a probability on $\mathbb{R}$. The log-Laplace of $\mu$ defined from $\mathbb{R}$ to $[0, \infty]$ by

$$\Lambda_\mu(\theta) = \log \int \exp(\theta x)\ \mu(dx),$$

is a convex function (apply Hölder inequality). Let $\Lambda^*_\mu$ be the Fenchel-Legendre dual of $\Lambda_\mu$ defined for $x \in \mathbb{R}$ by:

$$\Lambda^*_\mu(x) = \sup\{\theta x - \Lambda_\mu(\theta) : \theta \in \mathbb{R}\} \in [0, +\infty].$$

As a supremum of linear functions, $\Lambda_\mu^*$ is lower semi-continuous (l.s.c.) and convex. It is positive (take $\theta = 0$). The main feature of the following Cramer-Chernov theorem is that it needs no moment condition.

**Theorem 2.1** *Let $\mu$ a distribution on $\mathbb{R}$ and for $n \geq 1$, let $\mu_n$ the distribution under $\mu^{\otimes n}$ of $\frac{1}{n}\sum_1^n x_i$.*

*a) For every closed $F \subset \mathbb{R}$,*

$$\overline{\lim}\ \frac{1}{n}\log(\mu_n(F)) \leq -\inf\{\Lambda_\mu^*(x); x \in F\} \tag{2.1}$$

*b) For every open $G \subset \mathbb{R}$,*

$$\underline{\lim}\ \frac{1}{n}\log(\mu_n(G)) \geq -\inf\{\Lambda_\mu^*(x); x \in G\} \tag{2.2}$$

**Remark 2.2** *Actually if $\mu$ have an expectation $\underline{x}$ then we have a Cramer-Chernov upper bound holding for every $n$:*

$$\mu_n(F) \leq 2\exp -n\inf\{\Lambda_\mu^*(x); x \in F\} \tag{2.3}$$

If we assume that $\Lambda_\mu$ is defined everywhere, then $\Lambda$ et $\Lambda^*$ have some nice properties described in the following proposition. Let $[a, b]$ be the convex hull of the support of $\mu$.

**Proposition 2.3** *Under the above conditions,*

*a) $\Lambda \in \mathcal{C}^\infty$.*

*b) If $X$ is a r.v. $\mu$ distributed, then $EX = \Lambda'(0)$, $VarX = \Lambda''(0)$.*

*c) $\Lambda$ is strictly convex.*

*d) $\Lambda^* \geq 0$ with equality only in $EX$.*

*e) $(\Lambda)'$ is one-to-one from $\mathbb{R}$ onto $]a, b[$.*

*f) $]a, b[\subseteq D_{\Lambda^*} \subseteq [a, b]$.*

*g) If $z \in \{a, b\}$ then $z \in D_{\Lambda^*}$ if and only if $\mu(\{z\}) > 0$; in that case $\Lambda^*(z) = -\log\mu(\{z\})$. In particular, if Supp $\mu$ is a finite set then $D_{\Lambda^*} = [a, b]$ and $\Lambda^*$ is continuous on its domain.*

*h) $\Lambda^*$ is differentiable on $]a, b[$ and $|(\Lambda^*)'(z)| \to \infty$ as $z \to a$ or $b$ staying in $]a, b[$.*

*i) $(\Lambda^*)'$ is one-to-one from $]a, b[$ onto $\mathbb{R}$ and its inverse is $(\Lambda)'$.*

*j) $\Lambda^*$ is analytic on $]a, b[$.*

## 2.2 Large Deviation Principle

Let $(E, \mathcal{E})$ be a measurable space provided with a topology. A large deviation principle (LDP) describes the asymptotic behaviour, as $\epsilon \to 0$, of a family of probability measures $(\mu_\epsilon)$ on $(E, \mathcal{E})$ by the means of a so-called rate function.

A function $f$ from $E$ to $[0, +\infty]$ is l.s.c. if for all $\alpha \in [0, +\infty[$ the level set $\Psi_f(\alpha) := \{x : f(x) \leq \alpha\}$ is closed in $E$.

**Definition 2.4** *A rate function $I$ is a l.s.c. mapping rom $E$ to $[0,+\infty]$. A good rate function is a rate function with compact level sets.*

Recall that a l.s.c. function reaches its inf on compact sets. If it is good, it reaches its inf on closed sets.

**Definition 2.5** *The family $(\mu_\epsilon)$ satisfies a LDP of rate function $I$ if*
*a) For every closed set $F \in \mathcal{E}$,*

$$\limsup \epsilon \log \mu_\epsilon(F) \ \le -\inf\{I(x); x \in F\}$$

*b) For every open set $G \in \mathcal{E}$,*

$$\liminf_\epsilon \epsilon \log \mu_\epsilon(G) \ \ge -\inf\{I(x); x \in F\}.$$

Since $\mu_\epsilon$ is a probability, it is needed that $\inf\{I(x); x \in E\} = 0$ due to the upper bound. When $I$ is good, $I = 0$ has at least one solution.

**Remark.** *The rate function associated with a LDP satisfied by $(\mu_\epsilon)$ on a metric space is unique.*

## 2.3 Gärtner-Ellis theorem

We want to weaken assumptions about independence and dimension.

Let $(Y_n)$ be a sequence of real r.v. defined on $\{(\Omega_n, \mathcal{F}_n, P_n); n = 1, 2, ..\}$ and let $(a_n)$ a sequence of positive numbers tending to infinity. Let

$$\Lambda_n(\theta) := \frac{1}{a_n} \log E_n\{\exp < \theta, Y_n >\}$$

**Assumption GE** For all $\theta \in \mathbb{R}^d$, the limit of $\Lambda_n(\theta)$, denoted $\Lambda(\theta)$ exists in $[-\infty, +\infty]$. Moreover $0 \in \text{int } D_\Lambda$.

For instance if $a_n = n$ and $Y_n = S_n$, with $S_n$ a sum of i.i.d. r.v. we have $\Lambda_n \equiv \Lambda$.

Under GE, it is easy to see that $\Lambda$ take its values in $]-\infty, +\infty]$ and is convex. Its dual $\Lambda^*$ is l.s.c. and convex. Since $0 \in \text{int } D_\Lambda$ it is good and $\inf\{\Lambda^*(x); x \in \mathbb{R}^d\} = 0$.

**Definition 2.6** *$y \in \mathbb{R}^d$ is an exposed point of $\Lambda^*$ if there exists some $\theta \in \mathbb{R}^d$ such that for all $x \neq y$ we have*

$$< \theta, y > -\Lambda^*(y) > \ < \theta, x > -\Lambda^*(x).$$

*Such a $\theta$ is called exposed hyperplane, it belongs to the subdifferential of $\Lambda^*$ in $y$.*

**Definition 2.7** *A convex function $\Lambda : \mathbb{R}^d \to ]-\infty, +\infty]$ is called essentially smooth if*

*a) int $D_\Lambda \neq \emptyset$*

*b) $\Lambda$ is differentiable on int $D_\Lambda$*

*c) $\Lambda$ is steep i.e. $\lim \mid \nabla\Lambda(\theta_n) \mid = +\infty$ for all sequence $\theta_n \in$ int $D_\Lambda$ tending to a border point of int $D_\Lambda$.*

**Theorem 2.8** *Assume (GE). Let $\mu_n$ be the distribution of $\dfrac{Y_n}{a_n}$ sur $\mathbb{R}^d$,*

*a) For all closed set $F \in \mathbb{R}^d$,*

$$\overline{\lim}\ \frac{1}{a_n} \log \mu_n(F) \leq -\inf\{\Lambda^*(x); x \in F\}$$

*b) For all open set $G \in \mathbb{R}^d$,*

$$\underline{\lim} \frac{1}{a_n} \log \mu_n(G) \geq -\inf\{\Lambda^*(x); x \in G \cap \mathcal{F}\}$$

*where $\mathcal{F}$ is the set of exposed points of $\Lambda^*$ whose exposed hyperplane lies in int $D_\Lambda$.*

*c) If $\Lambda$ is essentially smooth, l.s.c., then we have a LDP of good rate function $\Lambda^*$.*

**Definition 2.9** *The sequence $\{Y_n\}$ is said to be exponentially tight if for each $\alpha > 0$ there exists a compact set $K_\alpha$ such that*

$$\limsup \frac{1}{a_n} \log P_n(Y_n \in (K_\alpha)^c) \leq -\alpha.$$

## 2.4 Varadhan theorem

We now study integrals of functions.

Let $(Y_\epsilon)$ be a family of r.v. with values in a regular topological space $X$, and $(\mu_\epsilon)$ the corresponding distributions. We consider the following properties:

(LDP) *$(\mu_\epsilon)$ satisfies a LDP and its rate function $I$ is good.*

(LIM) *For all $f \in \mathcal{C}_b$ the following limit exists:*

$$\Lambda_f = \lim_\epsilon\ \epsilon \log \int e^{f(x)/\epsilon}\ d\mu^\epsilon(x).$$

**Theorem 2.10** *a) $(LDP) \Rightarrow (LIM)$ with*

$$(*) \qquad \Lambda_f = \sup f - I.$$

*b) (LIM) + exponential tightness implies (LDP) with*

$$(**) \qquad I(x) = \sup\{f(x) - \Lambda_f\ ;\ f \in \mathcal{C}_b\}$$

*and (*) holds.*

Part a) is Varadhan's theorem and b) is Bryc's theorem.

We can weaken the conditions. Assume that $(\mu_\epsilon)$ satisfies a LDP and that $f$ is continuous. Assume moreover that the following uniform exponential integrability holds:

$$\lim_{M\to\infty} \limsup_{\epsilon\to 0} \epsilon \log E\big[e^{f(Y_\epsilon)/\epsilon} 1_{\{f(Y_\epsilon)\geq M\}}\big] = -\infty$$

then LIM is satisfied.

For instance, if

$$\limsup_{\epsilon\to 0} \epsilon \log E\big[e^{\gamma f(Y_\epsilon)}\big] \ < \ \infty$$

for some $\gamma > 1$, then we have the uniform exponential integrability.

## 2.5 Schilder's theorem

Let $B_t$ be a standard brownian motion in $\mathbf{R}$ , $B_\epsilon(t) := \epsilon B_t$ and let $(\mu_\epsilon)$ the family of corresponding distributions in $\mathcal{C}_0([0,1])$. Let $H_1 \subset \mathcal{C}_0([0,1])$ be the Cameron Martin space of absolutely continuous functions $\varphi$ whose derivative $\dot\varphi$ is square-integrable.

**Theorem 2.11** *The family $(\mu_\epsilon)$ satisfies a LDP of speed $\epsilon^{-2}$ and good rate function $I$ defined on $\mathcal{C}_0([0,1])$ by:*

$$I(\varphi) = \int_0^1 [\dot\varphi(t)]^2 \ dt, \quad \varphi \in H_1$$

$$I(\varphi) = +\infty$$

*in other cases.*

There are various proofs of this result. The classical proof (Varadhan) uses for the lower bound a change of probability (Girsanov), and for the upper bound a discretization by polygonal lines. We can also proof this result as an application of a Cramer's theorem in infinite dimension.

This can be extended to diffusion processes with small parameter

$$dx_t^\epsilon = b(x_t^\epsilon)dt + \sqrt{\epsilon}\ \sigma(x_t^\epsilon)\ dw_t \ , \quad \sigma > 0.$$

This is the Freidlin-Wentzell theory (see Dembo-Zeitouni p.187-212). The action functional is then

$$I(\varphi) = \int_0^1 \frac{[\dot\varphi(t) - b(\varphi(t)]^2}{\sigma^2(\varphi(t))}\ dt.$$

# 3 Branching processes

## 3.1 Trees

We give here some notations and background on branching processes (see Neveu (1986), Chauvin (1986), Lyons (1998)). If $(p_k)_{k\in\mathbb{N}}$ is a probability distribution, the process $Z_n$ is defined by $Z_0 = 1$ (the ancestor) and this ancestor give birth to $k$ descendents with probability $p_k$. These descendents have offspring independently with the same probability distribution. Formally, let $\mathbb{U}$ be the space of finite sequences $u = j_1...j_n$ of strictly positives integers. The sequence $\emptyset$ belongs to $\mathbb{U}$. So

$$\mathbb{U} = \{\emptyset\} \cup \bigcup_{n\geq 1} \mathbb{N}^{*n}$$

The length of $u \in \mathbb{U}$ is denoted by $|u|$; the concatenation of $u$ and $v$ is denoted by $uv$. A tree $\omega$ is a subset of $\mathbb{U}$ satisfying the following conditions:

(i) $\emptyset \in \omega$
(ii) $uv \in \omega \Rightarrow u \in \omega$
(iii) $u \in \omega \Rightarrow \exists N_u(\omega) \in \mathbb{N}$ such that $uj \in \omega, j \geq 1 \Leftrightarrow 1 \leq j \leq N_u(\omega)$

Let $N$ instead of $N_\emptyset$. The set $\Omega$ of trees is provided with the $\sigma$-field $\mathcal{F}$ generated by $\Omega_u, u \in \mathbb{U}$ where $\Omega_u = \{\omega \in \Omega, u \in \omega\}$. So $N_u$ are r.v. defined on $\Omega_u$. For every integer $n$ the $n$-th generation is

$$z_n(\omega) = \{u \in \omega, |u| = n\}$$

and

$$Z_n(\omega) = Card\ z_n(\omega)$$

(with $Z_1 = N$).

The set of trees $\Omega$ is now provided with a filtration $\mathcal{F}_n$ generated by $z_m, 0 \leq m \leq n$ and with (measurable) shifts

$$\begin{array}{rccl} T_u: & \Omega & \to & \Omega \\ & \omega & \mapsto & T_u(\omega) \quad = \{v, v \in U, uv \in \omega\} \end{array}$$

so that for all $n$ et $k$,

$$Z_{n+k} = \sum_{u \in z_n} Z_k \circ T_u \tag{1.1}$$

We now make $(\Omega, \mathcal{F})$ a probability space.

**Theorem 3.1** *Let $p = (p_k)_{k\geq 0}$ a probability distribution. There exists a unique probability $P$ on $(\Omega, \mathcal{F})$ such that*

*(i) the distribution of $N$ is $p$.*
*(ii) conditionally on $\{N = j\}$, the r.v. $T_1, T_2, ..., T_j$ are independent and $P$-distributed.*

*Consequently, for all $n \geq 0$, conditionally on $\mathcal{F}_n$, the r.v. $\{T_u, u \in z_n\}$ are independent and $P$-distributed. This means that for all measurable positive functions $f_u, u \in U$*

$$E^{\mathcal{F}_n}(\prod_{u \in z_n} f_u \circ T_u) = \prod_{u \in z_n} E(f_u) \tag{3.1}$$

*The sequence $(Z_n)_{n \geq 0}$ is called a Galton-Watson process of offspring distribution $p$ and initial state $Z_0 = 1$. The formula (3.1) is called the branching property.*

The classical tool is the generating function $f(s) = \sum s^k p_k$.

## 3.2 Marked trees

The idea is to mark each node of a tree with a mark.

Branching random walk

We mark a node $u$ with a point process $(X_{u1}, ..., X_{u\nu_u})$ representing the positions of the children of $u$ relatively to the position of $u$.

Branching diffusion

We mark a node $u$ with a trajectory.

Multitype processes

We mark a node $u$ with an element of a finite set of types $E = \{1, ..., r\}$.

We can of course mix the models to obtain the multitype branching random walks or multitype branching diffusions. In these lecture, marks are supposed to fit with the branching property and markovian along branches (see Rouault 1987).

In each model, at the end of the construction each node $u$ of a random tree is provided with a label $Y_u$ in some set $S$. Important tools are then the generating functional $M$ acting on $\varphi$, function of $S$ in $[0, 1]$ by:

$$M(\varphi)(x) = E_x \prod_{u \in z_1} \varphi(Y_u). \tag{3.2}$$

and the additive functional

$$S(\varphi)(x) = E_x \sum_{u \in z_1} \varphi(Y_u). \tag{3.3}$$

# 4 Large Deviations and Galton-Watson Processes

We present here two types of applications of large deviations in supercritical Galton-Watson process processes. The first one is a large deviation theorem for the Lotka-Nagaev estimator of the mean. The second one is particularly nice. It uses the "self-similarity" of the r.v. $W$ limit of the process to give tail behaviours of the distribution of $W$ (in 0 and in $\infty$). Both need some properties of the iterated generating functions.

## 4.1 Asymptotic behaviour of the generating function

When $p_0 = 0$ and $p_1 \neq 0$ then

$$p_1^{-n} f_n(s) \to Q(s) \tag{4.1}$$

for $0 \leq s < 1$.

When there is a finite maximum family size $d$, it is known that

$$\frac{\log f_n(s)}{d^n} \to \log B_r(s) \tag{4.2}$$

for $s > 1$ where $B_r$ is the right Böttcher function (Kuczma Choczewski and Ger 1990).

When there is a minimum $k \geq 2$ family size, it is known that

$$\frac{\log f_n(s)}{k^n} \to \log B_l(s) \tag{4.3}$$

for $0 \leq s < 1$ where $B_r$ is the left Böttcher function.

The three limit functions, $Q,\ \log B_r$ and $\log B_l$ satisfy a Schröder equation:

$$F \circ f = \lambda F \tag{4.4}$$

with $\lambda = p_1,\ d$ and $k$ respectively.

## 4.2 Growth rates

In a supercritical Galton-Watson process with $p_0 = 0$ the classical Lotka-Nagaev estimator of the mean $m$ is $\frac{Z_{n+1}}{Z_n}$ (Dion 1991). We present here large deviations of this estimator essentially due to Athreya.

**Theorem 4.1** *Let $p_0 = 0, p_1 > 0, p_j \neq 1$ for $j \geq 1$ and $m > 1$. Let $E(Z_1^{2r+\delta}) < \infty$ for some $r \geq 1$ and $\delta > 0$ such that $p_1 m^r > 1$. Then*

$$\lim_{n\to\infty} p_1^{-n} P\big(\mid \frac{Z_{n+1}}{Z_n} - m \mid > \epsilon\big) = g(\epsilon) < \infty$$

This last result leads to a LDP for $\frac{Z_{n+1}}{Z_n} - m$ at rate $n^{-1}$ with rate function

$$I(x) = -\log p_1 \text{ if } x \neq 0\ ,\ I(0) = 0$$

This is indeed a rate function, but it is not *good*.

**Theorem 4.2** *In the minimum family size case, let $\Lambda(\theta) = \log f(e^\theta)$. If there is some $\theta > 0$ in dom $\Lambda$, then the family $\frac{Z_{n+1}}{Z_n}$ satisfies a LDP at speed $k^n$ with good rate function* $\log B_l(e^{-\Lambda^*(.)})$

Proof. The upper bound is proved by Athreya (th. 3). We give it for the sake of completeness. Let $F$ be a closed set. Conditioning on $Z_n$ gives

$$P\big(\frac{Z_{n+1}}{Z_n} \in F\big) = \sum_j P(Z_n = j)P\big(\frac{S_j}{j} \in F\big) \tag{4.5}$$

where $S_k$ is the sum of $k$ independent r.v. distributed as $Z_1$. The Chernov upper bound (2.3) yields

$$P\big(|\ \frac{Z_{n+1}}{Z_n} \in F\big) \leq 2f_n(\exp -\Lambda^*(F))$$

and the upper bound follows. For the lower bound, let us first remark that in (4.5) the sum actually begins at $j = k^n$. Let $\alpha > 0$. From the Chernov lower bound (2.2), there exists $N$ such that for $j > N$

$$P\big(\frac{S_j}{j} \in F\big) \geq \exp -[\Lambda^*(G) + \alpha]$$

This yields for $k^n > N$,

$$P\big(\frac{Z_{n+1}}{Z_n} \in G\big) \geq f_n(\exp -\Lambda^*(G) - \alpha)$$

and then

$$\liminf k^{-n} \log P\big(\frac{Z_{n+1}}{Z_n} \in G\big) \geq \log B_l \circ (\exp -\Lambda^*(G) - \alpha)$$

Since it holds for all $\alpha$ the lower bound holds. □

These results may be extended to the multitype case (see Athreya and Vidyashankar 1995).

## 4.3 Tail Behaviour

We study here the tail behaviour of the r.v. $W := \lim_n \frac{Z_n}{m^n}$ associated to a supercritical Galton-Watson process . Let $\Phi$ the Laplace transform of $W$ defined by $\Phi(s) = Ee^{sW}$. It satisfies the equation:

$$f \circ \Phi(s) = \Phi(ms). \tag{4.6}$$

**Theorem 4.3** *1) In the finite maximum size case (f is a polynomial of degree d), let*

$$\gamma = \frac{\log d}{\log m}(> 1) \quad \textit{and} \quad \frac{1}{\gamma} + \frac{1}{\gamma^*} = 1.$$

*a) As $s \to \infty$*

$$\log \Phi(s) = \log B_r \circ \Phi(s) + O(1) \tag{4.7}$$

*and the function $H$ defined by $H(s) = s^{-\gamma} \log B_r \circ \Phi(s)$, is multiplicatively periodic of period $m$, positive and continuous.*

*b) As* $x \to \infty$

$$-\log P(W \geq x) = (\log B_r \circ \Phi)^*(x) + o(x^{\gamma^*})$$

*and* $(\log B_r \circ \Phi)^*(x) = x^{\gamma^*} H^\dagger(x)$ *where* $H^\dagger$ *is multiplicatively periodic with period* $m^{\gamma-1}$.

*c) As* $x \to \infty$

$$\begin{aligned}
\liminf \frac{-\log P(W \geq x)}{x^{\gamma^*}} &= \frac{1}{\gamma^*(\gamma\tau)^{\frac{1}{\gamma-1}}} \\
\limsup \frac{-\log P(W \geq x)}{x^{\gamma^*}} &= \frac{1}{\gamma^*(\gamma\underline{\tau})^{\frac{1}{\gamma-1}}}
\end{aligned}$$

*where* $\underline{\tau} = \min_{x \leq m} H(x)$ *and* $\underline{\tau} = \max_{x \leq m} H(x)$

*2) In the minimum size case, let*

$$\delta = \frac{\log k}{\log m} (< 1) \quad \text{and} \quad \frac{1}{\delta} - \frac{1}{\delta^*} = 1.$$

*As* $x \to 0$

$$-\log P(W \leq x) = (\log B_l \circ \Phi)^*(x) + o(x^{-\delta^*})$$

*and* $(\log B_l \circ \Phi)^*(x) = x^{\delta^*} L^\dagger(x)$ *where* $L^\dagger$ *is multiplicatively periodic with period* $m^{1-\delta}$.

1) a) is due to Harris (1948).

1) b) and 2) are from Biggins and Bingham (1993). These results can be proved by application of the Gärtner-Ellis theorem.

Actually from (4.6) we have

$$\Phi(m^n s) = f_n \circ \Phi(s) \tag{4.8}$$

and from (4.2) we deduce that $\Lambda_n(\theta) := \frac{1}{m^{n\gamma}} \log E \exp \theta m^n W$ if $\theta > 0$, has the limit $\Lambda(\theta) = \log B_r \circ \Phi(\theta)$. So we can apply a one-sided version of the Gärtner-Ellis theorem yielding a one-sided LDP for the family of distributions of $(m^{n(1-\gamma)}W)$ at speed $m^{-n\gamma}$ with rate function $(\log B_r \circ \Phi)^*$.

c) is a consequence of b) and results of Liu (see p.63).

To prove 2), let $L$ defined for $s \geq 0$ by:

$$L(s) = -s^{-\delta} \log B_l \circ \Phi(-s).$$

It is straightforward from (4.4) and (4.6) to see that $L$ is multiplicatively $m$-periodic, continuous and positive, and that

$$\frac{\log \Phi(-m^n s)}{k^n} \to -s^\delta L(s).$$

From this we deduce that $\Lambda_n(\theta) := \frac{1}{m^{n\delta}} \log E \exp \theta m^n W$ if $\theta < 0$, has the limit $\Lambda(\theta) = \log B_l \circ \Phi(\theta)$. Again a one-sided version of the Gärtner-Ellis theorem yields a LDP for the family of distributions of $(m^{n(1-\delta)}W)$ at speed $m^{-n\delta}$ with rate function $(\log B_l \circ \Phi)^*$.

**Remark.** *For the tree martingale issued from more general branching processes there are also tail results (see Liu (1996)). For instance, for the cascade process (see Waymire (1998)) the rate of increasing of* $-\log P_r(Z_\infty > x)$ *is* $O(x^{\gamma^*})$ *with* $\gamma^* = \frac{\log b\|W\|_\infty}{\log N}$.

# 5 Large Deviations in Spatial Processes

As shortly presented in the introduction the competition between small probabilities and large populations provides with three problems:

the speeds of propagation (borders separating supercritical and subcritical ranges)
the growth rates (of population) in the supercritical range
the evaluation of presence probabilities in the subcritical range.

## 5.1 Rate of Propagation

A) Branching Random Walk

Marks of nodes are point processes. We suppose these points processes i.i.d. of generating functional $\psi$ so that

$$M(\varphi)(x) = \psi(\varphi(. - x)), \tag{5.1}$$

(see (3.2)) and of intensity measure $\mu$ so that

$$S(\varphi(x)) = \int \varphi(y - x) d\mu(x) \tag{5.2}$$

(see (3.3)). The offspring mean is $m = \int d\mu(x)$. We assume $m > 1$ (the underlying Galton-Watson is supercritical). Let $\Lambda$ be the log-Laplace of $\mu$. We assume that $0 \in \operatorname{intdom} \Lambda$. We assume also for the sake of simplicity that $\mu$ is zero-mean.

For $n > 0$ we denote $\mathcal{F}_n$ the sigma field generated by the marks of nodes of length smaller than $n$. Let for $u \in \omega$, $S_u = X_{u_1} + X_{u_1 u_2} + \cdots + X_{u_1 u_2 \ldots u_n}$. If $\delta_a$ is the Dirac mass in $a \in \mathbb{R}$, let $Z_0 = \delta_0$ and for $n > 0$

$$Z_n = \sum_{u \in z_n} \delta_{S_u} \quad ,$$

so that if $B$ is a Borel set of $\mathbb{R}$, we have:

$$Z_n(B) = \text{card } \{u \in z_n : S_u \in B\}.$$

For all $\theta \in \mathbb{R}$ and $n > 0$, we denote

$$W_n(\theta) = \sum_{u \in z_n} \exp[\theta S_u - n\Lambda(\theta)].$$

This is a $\mathcal{F}_n$ - martingale positive. Let $W(\theta)$ be its a.s. limit. From our assumptions, it turns out that $\{x : \Lambda^*(x) \leq 0\}$ is a compact set $[c^-, c^+]$.

**Theorem 5.1** *If*

$$M_n^+ = \max\{S_u; u \in z_n\}\ ,\ M_n^- = \min\{S_u; u \in z_n\}$$

*then a.s. as* $n \to +\infty$*:* $\lim M_n^+/n = c^+$ *and* $\lim M_n^-/n = c^-$.

Proof.

We will only treat the case of the maximum.

a) Let us show first that a.s. $\limsup M_n/n \leq c^+$.

For all $a$ we have: $\{Z_n([na, +\infty[) \geq 1\} = \{M_n \geq na\}$ and then $P\{M_n \geq na\} \leq EZ_n([na, +\infty[)$ by Markov inequality. For all $f$ measurable positive,

$$\begin{aligned} Z_n(f) &= \int f(x)\, dZ_n(x) = \sum_{u \in z_n} f(S_u) \\ &= \sum_{u \in z_{n-1}} \sum_{v \in z_1^u} f(S_u + X_{uv}). \end{aligned}$$

Hence

$$E[Z_n(f) \mid \mathcal{F}_{n-1}] = Z_{n-1} \star \mu(f).$$

This yields:

$$E\big(Z_n(g)\big) = \mu^{\star n}(g).$$

In particular $EZ_n([na, +\infty[) = \mu^{\star n}([na, +\infty[)$ and from (2.3)

$$\mu^{\star n}([na, +\infty[) \leq \exp -n\Lambda^*(a) \text{ if } a > 0.$$

Moreover if $\Lambda^*(a) > 0$, then $\sum P(Z_n([na, +\infty[) \geq 1) < \infty$ and by Borel-Cantelli there exists a.s. $n_0$ such that $Z_n([na, +\infty[) = 0$ for all $n \geq n_0$. This gives a.s. $\limsup M_n/n \leq c^+$.

b) The proof of the lower bound is an easy consequence of the next result. □

**Proposition 5.2** *If* $a > 0$ *et* $\Lambda^*(a) < 0$ *then a.s.*

$$\lim \frac{1}{n} \log Z_n([na, +\infty[) = -\Lambda^*(a).$$

Proof. The upper bound is proved using Borel-Cantelli and argument from a). The lower bound can be proved using an auxiliary branching process (Biggins 1976) or using the following martingale argument (Chauvin, 1986, see also Neveu 1988). □

**Lemma 5.3** *1) For* $\theta \in \mathbb{R}$, $W_n(\theta), n > 0$ *is a* $\mathcal{F}_n$ *- martingale positive. Let* $W(\theta)$ *be its a.s. limit.*

*2) For all* $a$ *such that* $\Lambda^*(a) < 0$, *let* $\theta = \Lambda'(a)$. *Then* $W_n(\theta) \to W(\theta)$ *in* $\mathbb{L}^1$ *and* $P(W(\theta) = 0) = 0$.

*3) If* $\Lambda^*(a) > 0$, *then* $W_n(\theta) \to 0$ *a.s. when* $n \to \infty$.

Suppose this lemma proved. We have

$$W_n(\theta) = \Gamma_n + \sum_{u \in z_n :\, S_u \in [n(a-\epsilon), n(a+\epsilon)]} \exp[\theta S_u - n\Lambda(\theta)].$$

In the last sum we can bound above $S_u$ by $n(a+\epsilon)$ ($\theta > 0$) and keep only those $u$ such that $S_u \in [n(a-\epsilon), +\infty[$. This yields

$$W_n(\theta) \le Z_n([n(a-\epsilon), +\infty[) \exp[n(a+\epsilon)\theta - n\Lambda(\theta)] \; + \Gamma_n.$$

If

$$d\mu_\theta(x) = \exp \theta x - \Lambda(\theta) d\mu(x)$$

then

$$\begin{aligned} E\Gamma_n &= \int_{x \notin [n(a-\epsilon), n(a+\epsilon)]} \exp[\theta x - n\Lambda(\theta)] d\mu^{\star n}(x) \\ &= 1 - \mu_\theta([n(a-\epsilon), n(a+\epsilon)] \end{aligned}$$

From (2.1) this series converges and then a.s. $\Gamma_n \to 0$. From the lemma,

$$\underline{\lim}_n Z_n([n(a-\epsilon), +\infty[) \exp[n\theta(a+\epsilon) - n\Lambda(\theta)] \ge W(\theta) > 0$$

a.s.. For all $a > 0$ such that $\Lambda^*(a) < 0$ and all $\epsilon > 0$, we then have:

$$\underline{\lim}_n \frac{1}{n} \log Z_n([n(a-\epsilon), +\infty[) \ge -\Lambda^*(a) - \epsilon\theta.$$

It means that for any $a > 0$ such that $\Lambda^*(a) < 0$ and any $\epsilon$ small enough,

$$\underline{\lim}_n \frac{1}{n} \log Z_n([na, +\infty[) \ge -\Lambda^*(a+\epsilon) - \epsilon\Lambda'(a+\epsilon).$$

Then let $\epsilon \to 0$.

PROOF.[ of 5.3] See Lyons (1997). □

**Definition 5.4** *We assume* $0 \in intdom\Lambda$. *We say that* $a \in \mathbb{R}$ *is in the range of supercriticality if* $a \in \Lambda'(int\ dom\Lambda)$ *and if* $\Lambda^*(a) < 0$. *Let* $\theta$ *such that* $\Lambda'(\theta) = a$. *We say that* $a$ *is in the range of subcriticality if* $a \in \Lambda'(int\ dom\Lambda)$ *and if* $\Lambda^*(a) > 0$.

B) Malthusian phenomena

A classical model for growing populations is the Bellman-Harris process. Each individual has a (random) lifetime of distribution $G$ and at its deathtime give birth to a random number (of mean $m$) of chidren. The population alive at time $t$ comes from different generations. For large $t$, the empirical mean of lifetimes on a given branch of the genealogic tree is approximately $\int sdG(s)$ from the LLN. Branches corresponding to means less than $\int sdG(s)$ are less probable, but had time to give more offsprings. There is a compensation and asymptotically the number of alive individual is near $e^{\alpha t}W$. The Malthusian parameter $\alpha$ is the unique positive solution of

$$\int_0^\infty e^{-\alpha t}\ mdG(t) = 1$$

In general this very classical result (Athreya-Ney 1971) is proved using the renewal theory. Here we give the "large deviations heuristics" to allow a generalization.

Let $X_1, \ldots, X_n, \ldots$ independent and $G$-distributed r.v.. For every $\beta > 0$, the contribution of generation number $[t/\beta]$ to the mean population alive at time $t$ is roughly (for $t$ large)

$$\begin{aligned} m^{[t/\beta]} P(X_1 + \ldots + X_{[t/\beta]} \simeq t) &= m^{[t/\beta]} \exp\big(-\frac{t}{\beta}\Lambda_G^*(\beta) + o(1))\big) \\ &= \exp\big(-\frac{t}{\beta}(\Lambda_G^*(\beta) - \log m + o(1))\big). \end{aligned}$$

When $\beta$ takes different values, we get exponentials of different orders. The larger one is overwhelming. It is precisely $\exp \alpha' t$ where

$$\alpha' = -\inf_{\beta>0}\big(-\frac{1}{\beta}(\Lambda_G^*(\beta) - \log m)\big)$$

This infimum is reached in $\beta_0 = \int_0^\infty se^{-\alpha s}\ mdG(s)$ and $\alpha' = \alpha$. The generations whose number is $\frac{t}{\beta_0} + o(t)$ are overwhelming (look at the means and apply a LLN). This $\beta_0$ is the mean age at childbearing, well known in demography. (see Jagers p.214).

This heuristics allows us to extend it to the so called Crump-Mode-Jagers branching process with random walk. We mark every individual $y$ with a lifelength $\lambda_y$, a birthtime $T_y$ and a (fixed) position $X_y$. Its immediate offspring is described by a point process $\xi_y$ on $\mathbb{R}^+ \times \mathbb{R}$. Each point $yk$ of $\xi_y$ corresponds to a child of $y$, the first coordinate is the age of $y$ when $yk$ was born and the second coordinate is the increment of position from $y$ to $yk$.. Here we assume that given the tree, all the $(\lambda_y, \xi_y)$ are i.i.d.. The jumps along a genealogical branch are $\mathbb{R}^+ \times \mathbb{R}$ valued. The asymptotic scaling of Bellman-Harris' model ($t \to \infty$) leads us to introduce a small parameter $\epsilon$ and perform a scaling of rate $\epsilon$ in $\mathbb{R}^+ \times \mathbb{R}$. In other words, from a reference model, we build a family of $\epsilon$ indexed processes , multiplying the jumps (in $\mathbb{R}^+ \times \mathbb{R}$) and the lifetimes by $\epsilon$. Let $\zeta_t^\epsilon(A)$ the size of the poulation alive at $t$ and located in $A$, Borel set of $\mathbb{R}$. Let $\mu$ be the intensity of the point process, $\Lambda$ its log-Laplace, assumed to be finite everywhere and $\Lambda^*$ its Cramer transform.

Let $\alpha$ 1-homogeneous from $\mathbb{R}^+ \times \mathbb{R}$ to $\mathbb{R}^+$, by

$$\alpha(t, x) = -\inf_{\beta>0} \frac{1}{\beta}\Lambda^*(t\beta, x\beta).$$

We showed (Laredo-Rouault 1983), under some assumptions, that there exists a cone $\mathcal{S}$ such that for every interval $I$ of $\mathbb{R}$

1) If $t \times I \in \mathbb{R} - \hat{\mathcal{S}}$ then a.s. $\zeta_t^\epsilon(I) = 0$ for $\epsilon$ small enough.
2) If $t \times I \cap \mathcal{S}$ is non empty, then as $\epsilon \to 0$,

$$\epsilon \log \zeta_t^\epsilon(I) \to \sup_{x \in I} \alpha(x,t)$$

in probability.

Function $\alpha$ plays the role of a local Malthusian parameter, and relation ? becomes

$$\Lambda(-\frac{\partial \alpha}{\partial t}, -\frac{\partial \alpha}{\partial x}) = 0.$$

## 5.2 Range of Supercriticality

We improve here the result of theorem 5.1 , giving sharp estimates of the population near $na$ where $a$ is in the range of supercriticality.

Single type case

From the classical result of Bahadur-Rao (see Dembo-Zeitouni p.95), we get:

$$EZ_n(na + I_\delta) = \mu^{\star n}(na + I_\delta) = \frac{\exp -n\Lambda^*(a)}{\sqrt{2\pi\Lambda"(\theta)}} \frac{e^{\theta\delta} - e^{-\theta\delta}}{\theta}(1 + o(1)).$$

where $I_\delta = [-\delta, +\delta]$. The following result is due to Biggins (1979).

**Theorem 5.5** *If $a$ is in the range of supercriticality and if the moment condition*

$$E[W_1(\theta) \log_+^\epsilon W_1(\theta)] < \infty \textit{ for some } \epsilon > 5/2$$

*holds, then a.s.*

$$\frac{Z_n(na + I_\delta)}{EZ_n(na + I_\delta)} \to W(\theta)$$

Multitype case

The result is due to Bramson, Ney and Tao (1992). It is stated in the case of a lattice random walk.

Let $Z_{ij}^n([x])$ be the number of type $j$ particles at the point "integer part of $x$" at time $n$, descended from a type $i$ ancestor at time 0. Let $M = (m_{ij})$ the matrix of intensity measures of this process. We suppose it irreducible and aperiodic. For $\theta \in \mathbb{R}$, let

$$\hat{Z}_{ij}^n(\theta) = \sum_{x \in I} e^{\theta x} Z_{ij}^n(x)$$

and

$$\hat{M}(\theta) = (\hat{m}_{ij}(\theta) = \sum_{x \in I} e^{\theta x} m_{ij}(x) \ ; \ i, j \in S)$$

Let $\lambda(\theta)$ be the Perron-Frobenius root of $\hat{M}(\theta)$ and $\Lambda(\theta) = \log \lambda(\theta)$. Let also $u(\theta)$ and $v(\theta)$ the normalized right and left eigenvectors associated with $\lambda(\theta)$. It is easy to see that

$$W_i^n(\theta) = \frac{\sum_j \hat{Z}_{ij}^n(\theta) v_j(\theta)}{\lambda(\theta)^n v_i(\theta)} \tag{5.3}$$

is a positive martingale. Let $W_i(\theta)$ be its limit. We assume $\lambda(0) > 1$ which means that the underlying multitype Galton-Watson process is supercritical. We assume that $a$ is in the range of supercriticality. Moreover we assume the moment condition:

$$E(\hat{Z}_{ij}(\theta))^p < \infty \text{ for some } p > 1 \text{ for } i, j = 1, ..., d.$$

**Theorem 5.6** *Under the above assumptions, for $i, j = 1, ..., d$ a.s.*

$$\lim \sqrt{2\pi n}\ \sigma_i(\theta) \exp\{\theta([an] - an)\} \exp n\Lambda^*(a) Z_{ij}^n([an]) = u_j(\theta) v_i(\theta) W_i(\theta)$$

*where $\sigma_i(\theta)$ is an appropriate second moment parameter.*

**Corollary 5.7** *For $i,\ j,\ k = 1, ..., d$, a.s.*

$$\frac{Z_{ij}^n([an])}{Z_{jk}^n([an])} \to \frac{u_j(\theta) v_i(\theta)}{u_k(\theta)} v_j(\theta).$$

## 5.3 Range of Subcriticality

We want here estimates of probability of presence in the range of subcriticality. Let us first recall the classical result for the subcritical Galton-Watson process (Athreya Ney 1972). We have, as $n \to \infty$,

$$\frac{P(Z_n > 0)}{EZ_n} \to C$$

and

$$C \neq 0 \ \Leftrightarrow\ \sum (k \log k) p_k < \infty$$

Let us recall the main ideas of the proof to extend it further. If $f$ is the g.f. of the offspring (with mean $m < 1$), then the sequence

$$u_n = P(Z_n > 0) = 1 - f_n(0)$$

satisfies

$$\begin{aligned} u_{n+1} &= 1 - f(1 - u_n) \\ u_0 &= 1 \end{aligned} \tag{5.4}$$

We compare it with the sequence

$$v_n = EZ_n = m^n$$

which satisfies obviously

(5.5)
$$\begin{aligned} v_{n+1} &= m v_n \\ v_0 &= 1 \end{aligned}$$

We introduce the classical auxiliary function:

$$r(s) = m - \frac{1 - f(1-s)}{s}$$

which is increasing from 0 and we deduce:

(5.6)
$$u_n = v_n \prod_{j=0}^{n-1} [1 - \frac{r(u_j)}{m}]$$

and of course

$$u_n \le v_n$$

It is known that for all $\lambda < 1$

$$\sum_j r(O(\lambda^j)) < \infty \iff \sum (k \log k) p_k < \infty$$

A) In the Branching Random Walk We may follow the same route. Let

$$u_n(x) = P_x(Z_n(I_\delta) \neq 0)$$

and

$$v_n(x) = E_x Z_n(I_\delta)$$

Let us remark that since $I_\delta$ is symmetric $u_n(x) = P_{-x}(Z_n(I_\delta) \neq 0)$ and since the motion is homogeneous $u_n(x) = P(Z_n(x + I_\delta) \neq 0)$, and $v_n(x) = EZ_n(x + I_\delta)$. With the branching property, we can prove:

(5.7)
$$\begin{aligned} u_{n+1} &= 1 - \Psi[1 - u_n] \\ u_0 &= 1_{I_\delta} \end{aligned}$$

and

(5.8)
$$\begin{aligned} v_{n+1} &= v_n \star \mu \\ v_0 &= 1_{I_\delta} \end{aligned}$$

(So-called M-equation and S-equation of Ikeda, Nagasawa and Watanabe).

**Theorem 5.8** *If a is in the range of subcriticality and if*

$$EW_1(\theta) \log_+^{1+\epsilon} W_1(\theta) < \infty \text{ and } EW_1(0) \log_+^{1+\epsilon} W_1(0) < \infty$$

*then*

$$\frac{P(Z_n(nx + I_\delta) \neq 0)}{EZ_n(nx + I_\delta)} \to C > 0.$$

This result is in Rouault (1993). The proof is built on the following representation formulas:

$$\begin{aligned} u_n(x) &= m^n E_x\{u_0(S_n) \\ \prod_{k=1}^n [1 &- \int_{\mathcal{M}} \int_0^1 [1 - \exp \int_{\mathbb{R}} \log 1 - \lambda u_{n-k}(S_{k-1} + y) Z(dy)] d\lambda P^{!X_k}(dZ)]\} \\ (5.9)\ \ v_n(x) &= m^n E_x u_0(S_n) \end{aligned}$$

where $\{S_n\}$ is a random walk $m^{-1}\mu$-distributed.

<u>B) In the homogeneous branching brownian motion</u>

Here recursion formulas become p.d.e.. Let

$$u(t,x) := P_x(Z_t(]-\infty,0]) > 0) = P_0(Z_t([x,+\infty[> 0)$$

(by symmetry) and

$$v(t,x) = E_x Z_t(]-\infty,0]) = E_0 Z_t([x,+\infty[)$$

Using a continuous version of the branching property, we get the M-equation, which is the famous Kolmogorov-Petrovski-Piscounov equation (KPP):

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \alpha[1 - u - f(1-u)] \\ u(0,.) &= 1_{]-\infty,0]}. \end{aligned} \tag{5.10}$$

The S-equation is:

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + \alpha(m-1)u \\ v(0,.) &= 1_{]-\infty,0]} \end{aligned} \tag{5.11}$$

We can use the representation with an auxiliary brownian motion:

$$v(t,x) = E 1_{]-\infty,0]}(x + W_t) \exp \alpha(m-1)t$$

and by the Feynmann-Kac formula:

$$u(t,x) = E\Big[1_{]-\infty,0]}(x + W_t) \exp \alpha(m-1)t - \int_0^t \alpha\ r(u(t-s, W_s))ds\Big]$$

**Theorem 5.9** *For $c > c_0 = \sqrt{2\alpha(m-1)}$ and under the $(k \log k)$ assumption, we have*

$$\lim_{\substack{x\to\infty, t\to\infty \\ x/t \to c}} \frac{u(t,x)}{v(t,x)} > 0$$

*and*

$$v(t,x) \sim \frac{e^{-\frac{t}{2}(c^2 - c_0^2)}}{c\sqrt{2\pi t}}$$

The rates of propagation are $\pm c_0 = \pm\sqrt{2\alpha(m-1)}$ and $\mid c > c_0$ defines the range of subcriticality.

C) In the inhomogeneous BBM

The model consists in a brownian motion with a small variance ($\epsilon^2$), a binary splitting (to simplify) and a branching rate $\frac{c(.)}{\epsilon^2}$ with $c \in \mathcal{C}^3(\mathbb{R}, \mathbb{R}^+)$. We study $u^\epsilon(t,x) = P_x^\epsilon\{Z_t(]-\infty, 0]) \geq 1\}$ and $v^\epsilon(t,x) = E_x^\epsilon\{Z_t(]-\infty, 0])$. The M-equation is:

$$(5.12) \qquad \begin{aligned} u_t^\epsilon &= \frac{\epsilon^2}{2} u_{xx}^\epsilon + \frac{c(x)}{\epsilon^2} u^\epsilon (1-u^\epsilon) \\ u^\epsilon(0,.) &= 1_{\mathbb{R}^-} \end{aligned}$$

and the S-equation

$$(5.13) \qquad {}_t^\epsilon = \frac{\epsilon^2}{2} u_{xx}^\epsilon + \frac{c(x)}{\epsilon^2} u^\epsilon.$$

We use the representations:

$$\begin{aligned} u^\epsilon(t,x) &= E\big[g(x+\epsilon B_t) \exp \frac{1}{\epsilon^2} \int_0^t c(x+\epsilon B_s)[1-u^\epsilon(t-s, x+\epsilon B_s)]ds\big] \\ v^\epsilon(t,x) &= E\big[g(x+\epsilon B_t) \exp \frac{1}{\epsilon^2} \int_0^t c(x+\epsilon B_s)ds. \end{aligned}$$

**Theorem 5.10**

$$\epsilon^2 \log u^\epsilon(t,x) \to V^*(t,x)$$

*where*

$$V^*(t,x) = \inf_\tau \sup\{\int_0^{t\wedge\tau} [c(\phi_s) - \frac{1}{2}\dot{\phi}_s^2]ds; \phi_0 = x,\ \phi_t \in Supp\ g\}$$

*and the supremum is over all stopping times.*

The range of subcriticality is $\{(t,x) : V^*(t,x) < 0\}$. There exists also another formula for $V^*$:

$$V^*(t,x) = \sup\{\inf_{0 \leq a \leq t} \int_0^a [c(\phi_s) - \frac{1}{2}\dot{\phi}_s^2]ds\ ;\ \phi_0 = x,\ \phi_t \in \text{Supp } g\}$$

We present now a sharp (i.e. non logarithmic) asymptotic formula. We assume $c$ at most linear at infinity, and

*(H1) The sup in $V$ is reached in a unique path $\phi$.*
*(H2) $\phi$ is a nondegenerate maximum.*

That means that the operator $A$ on $H^0 = \{\phi \in H^1 : \phi_0 = 0\}$ given by

$$< Ah, h >= \int_0^t c''(\phi_s) h_s^2 ds$$

has its spectrum stricly bounded above by 1. We may define

$$C(\phi) = E \exp \frac{1}{2} \int_0^t c"(\phi_s) B_s^2 ds = \det(I - A)^{-\frac{1}{2}}.$$

For the bridge we define $C_0(\phi)$.

**Theorem 5.11** *Under H1 and H2,*
*a) If $\phi_t = 0$ and $p =: -\dot{\phi}_t > 0$*

$$v^\epsilon(t,x) = \epsilon \exp \frac{V(t,x)}{\epsilon^2} p^{-1} (2\pi t)^{-1/2} C_0(\phi)(1 + o(1))$$

*If $\phi_t < 0$*

$$v^\epsilon(t,x) = \exp \frac{V(t,x)}{\epsilon^2} C(\phi)(1 + o(1)).$$

Let us give other assumptions:

*(H3) The set $\{(\phi, a) : \phi_t \leq 0, \phi_0 = x$ and*

$$\int_0^a [c(\phi_s) - \frac{1}{2}\dot{\phi}_s^2] ds = \inf_{0 \leq b \leq t} \int_0^b [c(\phi_s) - \frac{1}{2}\dot{\phi}_s^2] ds = V^*(t,x)\}$$

*is a singleton.*

This assumption implies in particular:
a) $V(t,x) = V^*(t,x)$
b) There exists one unique $\phi$ reaching the sup in $V^*$
c) The optimal path stays always ahead of the nonlinear front.

Under H1+H2+H3 we have

$$\phi_t = 0,\ p > 0 \text{ et } p \geq [2c(0)]^{1/2} \text{ et } V = V^*.$$

We need

*(H4)* $p > [2c(0)]^{1/2}$.

**Theorem 5.12** *If $V^*(t,x) < 0$, under assumptions H1,H2,H3,H4, we have:*

$$u^\epsilon(t,x) = \epsilon \exp \frac{V(t,x)}{\epsilon^2} p^{-1} (2\pi t)^{-1/2} C_0(\phi) C_1(p)(1 + o(1))$$

*where $C_1(p)$ depends only on $p$.*

There are two extensions of this result. If the brownian motion is replaced by a non-degenerate diffusion (without drift to simplify) (see Rouques 1997), the functional is obtained by changing $\int \frac{1}{2}\dot{\phi}_s^2 ds$ in $\int \frac{1}{2}\frac{\dot{\phi}_s^2}{\sigma^2(\phi_s)}]ds$. The brownian $\mathbb{R}^d$-case with a convenient initial condition is studied in Cohen-Rossignol 1997. In both cases the authors get new constants.

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