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AN INVARIANT THEORY OF SURFACES IN THE FOUR-DIMENSIONAL EUCLIDEAN OR MINKOWSKI SPACE

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Abstract. The present article is a survey of some of our recent results on the theory of two-dimensional surfaces in the four-dimensional Euclidean or Minkowski space. We present our approach to the theory of surfaces in Euclidean or Minkowski 4-space, which is based on the introduction of an invariant linear map of Weingarten-type in the tangent plane at any point of the surface under consideration. This invariant map allows us to introduce principal lines and an invariant moving frame field at each point of the surface. Writing derivative formulas of Frenet-type for this frame field, we obtain a system of invariant functions, which determine the surface up to a motion.

We formulate the fundamental theorems for the general classes of surfaces in Euclidean or Minkowski 4-space in terms of the invariant functions.

We show that the basic geometric classes of surfaces, determined by conditions on their invariants, can be interpreted in terms of the properties of two geometric figures: the tangent indicatrix and the normal curvature ellipse.

We apply our theory to some special classes of surfaces in Euclidean or Minkowski 4-space.

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1. Introduction. Local invariants of surfaces in the four-dimensional Euclidean space $\mathbb{R}^4$ were studied by Eisenhart [9], Kommerell [21], Moore and Wilson [30], Schouten and Struik [32], Spivak [33], Wong [35], Little [23], and others. Their study was based on a special configuration, namely a point and an ellipse lying in the normal space (the ellipse of normal curvature). This configuration leads to a theory of axial principal directions, along which the vector-valued second fundamental form points in the direction of the major and the minor axes of the curvature ellipse. In higher dimensions there is also a similar configuration consisting of a point and a Veronese manifold. This configuration determines second order scalar invariants and generates principal axes "in general" [23]. Points where the construction of principal axes fails are regarded as singularities of the field of axes. Geometric singularities for immersions in Riemannian manifolds are considered in [1]. Special types of tangent vector fields on a surface in $\mathbb{R}^4$ were defined in terms of the properties of the normal curvature ellipse and families of lines determined by such tangent vector fields were studied in [16, 26].

The basic feature of our approach to the theory of 2-dimensional surfaces in the Euclidean space $\mathbb{R}^4$ or the Minkowski space $\mathbb{R}^4_1$ is the introduction of an invariant linear map of Weingarten-type in the tangent plane at any point of the surface.

Studying surfaces in the Euclidean space $\mathbb{R}^4$, in [10] we introduce a linear map of Weingarten-type, which plays a similar role in the theory of surfaces in $\mathbb{R}^4$ as the Weingarten map in the theory of surfaces in $\mathbb{R}^3$. We give a geometric interpretation of the second fundamental form and the Weingarten map of the surface in [11]. In [12] we find a geometrically determined moving frame field at each point of the surface and writing derivative formulas of Frenet-type for this frame field, we obtain eight invariant functions and prove a fundamental theorem of Bonnet-type, stating that these eight invariants under some natural conditions determine the surface up to a motion in $\mathbb{R}^4$.

Following our approach to the surfaces in $\mathbb{R}^4$, in [14] we develop the theory of spacelike surfaces in $\mathbb{R}^4_1$ in a similar way. We consider spacelike surfaces in $\mathbb{R}^4_1$ whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector. Using a geometrically determined moving frame of Frenet-type on such a surface and the corresponding derivative formulas, we obtain eight invariant functions and prove a fundamental theorem, stating that any spacelike surface whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector is determined up to a motion in $\mathbb{R}^4_1$ by its eight invariants satisfying some natural conditions.

In [15] we develop the invariant theory of spacelike surfaces in $\mathbb{R}^4_1$ whose mean curvature vector at any point is a lightlike vector, the so called marginally trapped surfaces. We use the principal lines on a marginally trapped surface to find a geometrically determined moving frame field at each point of such a surface and obtain seven invariant functions which determine the surface up to a motion in $\mathbb{R}^4_1$.

We apply our theory to some special classes of surfaces in $\mathbb{R}^4$ or $\mathbb{R}^4_1$. We consider rotational surfaces with two-dimensional axis in $\mathbb{R}^4$ or $\mathbb{R}^4_1$ and find their invariants. We find all Chen rotational surfaces of elliptic or hyperbolic type in $\mathbb{R}^4_1$. We also consider general rotational surfaces with plane meridians (in the sense of C. Moore) in $\mathbb{R}^4$ and construct in a similar way general rotational surfaces in $\mathbb{R}^4_1$.

We construct a family of surfaces lying on a rotational hypersurface in the Euclidean space $\mathbb{R}^4$, which we call meridian surfaces. We describe the meridian surfaces with constant Gauss curvature and those with constant mean curvature. We use the same idea to construct a special family of two-dimensional spacelike surfaces lying on rotational hypersurfaces in $\mathbb{R}^4_1$. We consider a rotational hypersurface with timelike axis and a rotational hypersurface with spacelike axis to construct two types of meridian surfaces in $\mathbb{R}^4_1$. We find all meridian surfaces in $\mathbb{R}^4_1$ which are marginally trapped. The meridian surfaces are a new source of examples of surfaces in $\mathbb{R}^4$ and $\mathbb{R}^4_1$.

2. Linear map of Weingarten-type. Let $\mathbb{R}^4$ be the four-dimensional Euclidean space endowed with the metric $\langle , \rangle$ and M^2 be a surface in $\mathbb{R}^4$. Denote by ∇' and ∇ the Levi Civita connections on $\mathbb{R}^4$ and M^2, respectively. Let x and y denote vector fields tangent to M and let ξ be a normal vector field. Then the formulas of Gauss and Weingarten give decompositions of the vector fields $\nabla'_x y$ and $\nabla'_x \xi$ into tangent and normal components:

$$\nabla'_x y = \nabla_x y + \sigma(x, y);$$
$$\nabla'_x \xi = -A_\xi x + D_x \xi,$$

which define the second fundamental tensor σ, the normal connection D and the shape operator A_ξ with respect to ξ. The mean curvature vector field H of the surface M^2 is defined as $H = \dfrac{1}{2}\,\mathrm{tr}\,\sigma$.

Let $M^2 : z = z(u, v),\ (u, v) \in \mathcal{D}$ ($\mathcal{D} \subset \mathbb{R}^2$) be a local parametrization of M^2. The tangent space T_pM^2 to M^2 at an arbitrary point $p = z(u, v)$ is $\mathrm{span}\{z_u, z_v\}$.

We choose an orthonormal normal frame field $\{e_1, e_2\}$ of M^2 so that the quadruple $\{z_u, z_v, e_1, e_2\}$ is positive oriented in $\mathbb{R}^4$. Then the following derivative formulas hold:

$$\begin{aligned}
\nabla'_{z_u} z_u &= z_{uu} = \Gamma_{11}^1\, z_u + \Gamma_{11}^2\, z_v + c_{11}^1\, e_1 + c_{11}^2\, e_2,\\
\nabla'_{z_u} z_v &= z_{uv} = \Gamma_{12}^1\, z_u + \Gamma_{12}^2\, z_v + c_{12}^1\, e_1 + c_{12}^2\, e_2,\\
\nabla'_{z_v} z_v &= z_{vv} = \Gamma_{22}^1\, z_u + \Gamma_{22}^2\, z_v + c_{22}^1\, e_1 + c_{22}^2\, e_2,
\end{aligned}$$

where Γ_{ij}^k are the Christoffel's symbols and the functions $c_{ij}^k,\ i,j,k = 1,2$ are given by

$$\begin{array}{lll}
c_{11}^1 = \langle z_{uu}, n_1\rangle; & c_{12}^1 = \langle z_{uv}, n_1\rangle; & c_{22}^1 = \langle z_{vv}, n_1\rangle;\\
c_{11}^2 = \langle z_{uu}, n_2\rangle; & c_{12}^2 = \langle z_{uv}, n_2\rangle; & c_{22}^2 = \langle z_{vv}, n_2\rangle.
\end{array}$$

Obviously, the surface M^2 lies in a 2-plane if and only if M^2 is totally geodesic, i.e. $c_{ij}^k = 0,\ i,j,k = 1,2$. So, we assume that at least one of the coefficients c_{ij}^k is not zero.

We use the standard denotations $E = g(z_u, z_u),\ F = g(z_u, z_v),\ G = g(z_v, z_v)$ for the coefficients of the first fundamental form

$$I(\lambda, \mu) = E\lambda^2 + 2F\lambda\mu + G\mu^2, \quad \lambda, \mu \in \mathbb{R}$$

and we set $W = \sqrt{EG - F^2}$.

The second fundamental form II of the surface M^2 at a point $p \in M^2$ is introduced by the following functions

$$L = \frac{2}{W}\begin{vmatrix} c_{11}^1 & c_{12}^1 \\ c_{11}^2 & c_{12}^2 \end{vmatrix}; \quad M = \frac{1}{W}\begin{vmatrix} c_{11}^1 & c_{22}^1 \\ c_{11}^2 & c_{22}^2 \end{vmatrix}; \quad N = \frac{2}{W}\begin{vmatrix} c_{12}^1 & c_{22}^1 \\ c_{12}^2 & c_{22}^2 \end{vmatrix}.$$

Let $X = \lambda z_u + \mu z_v,\ (\lambda, \mu) \neq (0,0)$ be a tangent vector at a point $p \in M^2$. Then

$$II(\lambda, \mu) = L\lambda^2 + 2M\lambda\mu + N\mu^2, \quad \lambda, \mu \in \mathbb{R}.$$

The second fundamental form II is invariant up to the orientation of the tangent space or the normal space of the surface.

Such a bilinear form has been considered for an arbitrary 2-dimensional surface in a 4-dimensional affine space $\mathbb{A}^4$ (see for example [3, 22, 34]). Here we consider this form for surfaces in $\mathbb{R}^4$ taking into consideration also the first fundamental form of the surface.

The condition $L = M = N = 0$ characterizes points at which the space $\{\sigma(x, y) : x, y \in T_pM^2\}$ is one-dimensional. We call such points *flat points* of the surface. These points are analogous to flat points in the theory of surfaces in $\mathbb{R}^3$. In [22] and [23] such points are called inflection points. E. Lane [22] has shown that every point of a surface is an inflection point if and only if the surface is developable or lies in a 3-dimensional space.

Further we consider surfaces free of flat points, i.e. $(L, M, N) \neq (0, 0, 0)$.

Using the functions L, M, N and E, F, G we introduce in [10] the linear map γ in the tangent space at any point of M^2

$$\gamma : T_pM^2 \to T_pM^2$$

similarly to the theory of surfaces in $\mathbb{R}^3$.

The linear map γ is invariant with respect to changes of parameters on M^2 as well as to motions in $\mathbb{R}^4$. Thus the functions

$$k = \frac{LN - M^2}{EG - F^2}, \qquad \varkappa = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

are invariants of the surface.

The sign of k is a geometric invariant and the sign of $\varkappa$ is invariant under motions in $\mathbb{R}^4$. However, the sign of $\varkappa$ changes under symmetries with respect to a hyperplane in $\mathbb{R}^4$. It turns out that the invariant $\varkappa$ is the curvature of the normal connection of the surface.

The map γ plays a similar role in the theory of surfaces in $\mathbb{R}^4$ as the Weingarten map in the theory of surfaces in $\mathbb{R}^3$. Analogously to $\mathbb{R}^3$ the invariants k and $\varkappa$ divide the points of M^2 into the following types: *elliptic* ($k > 0$), *parabolic* ($k = 0$), and *hyperbolic* ($k < 0$).

The second fundamental form II determines conjugate, asymptotic, and principal tangents at a point p of M^2 in the standard way.

Two tangents $g_1 : X = \alpha_1 z_u + \beta_1 z_v$ and $g_2 : X = \alpha_2 z_u + \beta_2 z_v$ are said to be *conjugate tangents* if

$$L\alpha_1\alpha_2 + M(\alpha_1\beta_2 + \alpha_2\beta_1) + N\beta_1\beta_2 = 0.$$

A tangent $g : X = \alpha z_u + \beta z_v$ is said to be *asymptotic* if it is self-conjugate, i.e.

$$L\alpha^2 + 2M\alpha\beta + N\beta^2 = 0.$$

The first fundamental form I and the second fundamental form II generate principal tangents and principal lines, as in $\mathbb{R}^3$.

A tangent $g : X = \alpha z_u + \beta z_v$ is said to be *principal* if it is perpendicular to its conjugate. The equation for the principal tangents at a point $p \in M^2$ is

$$\begin{vmatrix} E & F \\ L & M \end{vmatrix} \lambda^2 + \begin{vmatrix} E & G \\ L & N \end{vmatrix} \lambda\mu + \begin{vmatrix} F & G \\ M & N \end{vmatrix} \mu^2 = 0.$$

A line $c : u = u(q),\ v = v(q);\ q \in J \subset \mathbb{R}$ on M^2 is said to be an *asymptotic line*, respectively a *principal line*, if its tangent at any point is asymptotic, respectively principal. The surface M^2 is parameterized by principal lines if and only if $F = 0,\ M = 0$.

The notion of conjugacy can also be introduced in a geometric way as follows. Let g be a tangent at the point $p \in M^2$ determined by the vector $X = \lambda z_u + \mu z_v$. We consider the linear map $\sigma_g : T_pM^2 \to (T_pM^2)^\perp$, defined by

$$\sigma_g(Y) = \sigma\left(\frac{\lambda z_u + \mu z_v}{\sqrt{I(\lambda, \mu)}}, Y\right), \quad Y \in T_pM^2.$$

Let $g_1 : X_1 = \lambda_1 z_u + \mu_1 z_v$ and $g_2 : X_2 = \lambda_2 z_u + \mu_2 z_v$ be two tangents at $p \in M^2$. The oriented areas of the parallelograms spanned by the pairs of normal vectors $\sigma_{g_1}(z_u)$, $\sigma_{g_2}(z_v)$ and $\sigma_{g_2}(z_u)$, $\sigma_{g_1}(z_v)$ are denoted by $S(\sigma_{g_1}(z_u), \sigma_{g_2}(z_v))$, and $S(\sigma_{g_2}(z_u), \sigma_{g_1}(z_v))$, respectively. We assign the quantity ζ_{g_1,g_2} to the pair of tangents g_1, g_2, defined by

$$\zeta_{g_1,g_2} = \frac{S(\sigma_{g_1}(z_u), \sigma_{g_2}(z_v))}{W} + \frac{S(\sigma_{g_2}(z_u), \sigma_{g_1}(z_v))}{W}.$$

We prove that ζ_{g_1,g_2} is an invariant (under any change of the parameters) of the pair of tangents g_1, g_2.

Using this invariant we give the following definition.

Definition 1 ([11]). Two tangents $g_1 : X_1 = \lambda_1 z_u + \mu_1 z_v$ and $g_2 : X_2 = \lambda_2 z_u + \mu_2 z_v$ are said to be *conjugate tangents* if $\zeta_{g_1,g_2} = 0$.

Calculating the oriented areas in ζ_{g_1,g_2}, we find that

$$\zeta_{g_1,g_2} = \frac{L\lambda_1\lambda_2 + M(\lambda_1\mu_2 + \mu_1\lambda_2) + N\mu_1\mu_2}{\sqrt{I(\lambda_1, \mu_1)}\sqrt{I(\lambda_2, \mu_2)}} = \frac{II(\lambda_1, \mu_1; \lambda_2, \mu_2)}{\sqrt{I(\lambda_1, \mu_1)}\sqrt{I(\lambda_2, \mu_2)}}.$$

Thus, $\zeta_{g_1,g_2} = 0$ if and only if $L\lambda_1\lambda_2 + M(\lambda_1\mu_2 + \lambda_2\mu_1) + N\mu_1\mu_2 = 0$. Hence, the tangents g_1 and g_2 are conjugate according to Definition 1 if and only if they are conjugate with respect to the second fundamental form II.

In terms of the invariant ζ_{g_1,g_2} we define two invariants ν_g and α_g of any tangent g of the surface as follows:

$$\nu_g = \zeta_{g,g}; \qquad \alpha_g = \zeta_{g,g^\perp}.$$

The invariant ν_g is expressed by the first and the second fundamental forms of the surface in the same way as the normal curvature of a tangent in the theory of surfaces in $\mathbb{R}^3$, i.e. $\nu_g = \dfrac{II(\lambda,\mu)}{I(\lambda,\mu)}$.

The invariant α_g can be written in the following way:

$$\alpha_g = \frac{\lambda^2(EM - FL) + \lambda\mu(EN - GL) + \mu^2(FN - GM)}{W I(\lambda,\mu)}.$$

Hence, α_g is expressed by the coefficients of the first and the second fundamental forms in the same way as the geodesic torsion in the theory of surfaces in $\mathbb{R}^3$.

That is why we call ν_g the *normal curvature* of g, and α_g - the *geodesic torsion* of g.

As in the classical case we have that a tangent g is asymptotic if and only if $\nu_g = 0$; a tangent g is principal if and only if $\alpha_g = 0$.

If p is an elliptic point of M^2 ($k > 0$) then there are no asymptotic tangents through p; if p is a hyperbolic point ($k < 0$) then there are two asymptotic tangents passing through p, and if p is a parabolic point ($k = 0$) then there is one asymptotic tangent through p. Thus, the sign of the invariant k determines the number of asymptotic tangents at the point.

As in the classical case the following inequality holds at each point of the surface:

$$\varkappa^2 - k \geq 0.$$

If $\varkappa^2 - k = 0$, every tangent is principal, and if $\varkappa^2 - k > 0$, there exist exactly two principal tangents.

In [14] we apply the same approach to the theory of spacelike surfaces in Minkowski space. Let $\mathbb{R}^4_1$ be the four-dimensional Minkowski space endowed with the metric $\langle , \rangle$ of signature $(3,1)$. A surface $M^2 : z = z(u,v),\ (u,v) \in \mathcal{D}$ ($\mathcal{D} \subset \mathbb{R}^2$) in $\mathbb{R}^4_1$ is said to be *spacelike* if $\langle , \rangle$ induces a Riemannian metric g

on M^2. Thus at each point p of a spacelike surface M^2 we have the following decomposition:

$$\mathbb{R}^4_1 = T_pM^2 \oplus N_pM^2$$

with the property that the restriction of the metric $\langle , \rangle$ onto the tangent space T_pM^2 is of signature $(2,0)$, and the restriction of the metric $\langle , \rangle$ onto the normal space N_pM^2 is of signature $(1,1)$.

Let M^2 be a spacelike surface in $\mathbb{R}^4_1$, i.e. $\langle z_u, z_u \rangle > 0$, $\langle z_v, z_v \rangle > 0$. We choose a normal frame field $\{n_1, n_2\}$ such that $\langle n_1, n_1 \rangle = 1$, $\langle n_2, n_2 \rangle = -1$, and the quadruple $\{z_u, z_v, n_1, n_2\}$ is positively oriented in $\mathbb{R}^4_1$. Then we have the following derivative formulas:

$$\begin{array}{l}
\nabla'_{z_u} z_u = z_{uu} = \Gamma_{11}^1 \, z_u + \Gamma_{11}^2 \, z_v + c_{11}^1 \, n_1 - c_{11}^2 \, n_2;\\
\nabla'_{z_u} z_v = z_{uv} = \Gamma_{12}^1 \, z_u + \Gamma_{12}^2 \, z_v + c_{12}^1 \, n_1 - c_{12}^2 \, n_2;\\
\nabla'_{z_v} z_v = z_{vv} = \Gamma_{22}^1 \, z_u + \Gamma_{22}^2 \, z_v + c_{22}^1 \, n_1 - c_{22}^2 \, n_2.
\end{array}$$

In the same way as in $\mathbb{R}^4$ we define conjugate tangents at any point of the surface M^2. Again we assign the quantity ζ_{g_1,g_2} to the pair of tangents g_1, g_2, defined by the formula

$$\zeta_{g_1,g_2} = \frac{S(\sigma_{g_1}(z_u), \sigma_{g_2}(z_v))}{W} + \frac{S(\sigma_{g_2}(z_u), \sigma_{g_1}(z_v))}{W},$$

where $S(\sigma_{g_1}(z_u), \sigma_{g_2}(z_v))$, and $S(\sigma_{g_2}(z_u), \sigma_{g_1}(z_v))$ denote the oriented areas of the parallelograms determined by the pairs of normal vectors $\sigma_{g_1}(z_u)$, $\sigma_{g_2}(z_v)$ and $\sigma_{g_2}(z_u)$, $\sigma_{g_1}(z_v)$ in the Lorentz plane span$\{n_1, n_2\}$.

We prove that ζ_{g_1,g_2} is invariant under any change of the parameters on M^2. By means of this invariant we define conjugate tangents and the notions of *normal curvature* and *geodesic torsion* of a tangent g in the same way as in the Euclidean case:

$$\nu_g = \zeta_{g,g}; \qquad \alpha_g = \zeta_{g,g^\perp}.$$

The second fundamental form II of the surface M^2 at a point $p \in M^2$ is introduced on the base of conjugacy of two tangents at the point. The second fundamental form determines an invariant linear map of Weingarten-type $\gamma : T_pM^2 \rightarrow T_pM^2$ at any point of M^2 and generates the invariants k and $\varkappa$:

$$k = \frac{LN - M^2}{EG - F^2}, \qquad \varkappa = \frac{EN + GL - 2FM}{2(EG - F^2)}.$$

As in the theory of surfaces in $\mathbb{R}^4$ the following inequality holds at each point of the surface:

$$\varkappa^2 - k \geq 0.$$

If $\varkappa^2 - k = 0$, every tangent is principal, and if $\varkappa^2 - k > 0$, there exist exactly two principal tangents.

3. Classes of surfaces characterized in terms of the invariants $\boldsymbol{k}$ and $\boldsymbol{\varkappa}$. The minimal surfaces and the surfaces with flat normal connection in $\mathbb{R}^4$ or $\mathbb{R}^4_1$ are characterized in terms of the invariants k and $\varkappa$ as follows.

Proposition 3.1. *Let M^2 be a surface in $\mathbb{R}^4$ or a spacelike surface in $\mathbb{R}^4_1$ free of flat points. Then M^2 is minimal if and only if*

$$\varkappa^2 - k = 0.$$

The last equality is equivalent to

$$L = \rho E, \quad M = \rho F, \quad N = \rho G.$$

It is interesting to note that the "*umbilical*" points, i.e. points at which the above equalities hold good, are exactly the points at which the mean curvature vector H is zero. So, the surfaces consisting of "umbilical" points in $\mathbb{R}^4$ or $\mathbb{R}^4_1$ are exactly the minimal surfaces.

The surfaces with flat normal connection are characterized by

Proposition 3.2. *Let M^2 be a surface in $\mathbb{R}^4$ or a spacelike surface in $\mathbb{R}^4_1$ free of flat points. Then M^2 is a surface with flat normal connection if and only if*

$$\varkappa = 0.$$

We note that the condition $\varkappa = 0$ implies that $k < 0$ and each surface with flat normal connection has two families of orthogonal asymptotic lines.

4. Classes of surfaces characterized in terms of the tangent indicatrix and the normal curvature ellipse. The minimal surfaces and the surfaces with flat normal connection can also be characterized in terms of a geometric figure in the tangent space at any point of the surface.

Let M^2 be a surface in $\mathbb{R}^4$ or a spacelike surface in $\mathbb{R}^4_1$. The normal curvatures of the principal tangents are said to be *principal normal curvatures* of M^2. Similarly to the theory of surfaces in $\mathbb{R}^3$, we introduce a geometric figure - the *indicatrix* χ in the tangent space T_pM^2 at an arbitrary point p of M^2, defined by

$$\chi : \nu' X^2 + \nu'' Y^2 = \varepsilon, \qquad \varepsilon = \pm 1,$$

where ν' and ν'' are the principal normal curvatures.

Then the elliptic, hyperbolic and parabolic points of a surface, defined by the sign of the invariant k, are characterized in terms of the indicatrix χ as in $\mathbb{R}^3$: if p is an elliptic point ($k > 0$), then the indicatrix χ is an ellipse; if p is a hyperbolic point ($k < 0$), then the indicatrix χ consists of two hyperbolas (for the sake of simplicity we say that χ is a hyperbola); if p is a parabolic point ($k = 0$), then the indicatrix χ consists of two straight lines parallel to the principal tangent with non-zero normal curvature.

The conjugacy in terms of the second fundamental form coincides with the conjugacy with respect to the indicatrix χ, i.e. the following statement holds true.

Proposition 4.1. *Two tangents g_1 and g_2 are conjugate tangents of M^2 if and only if g_1 and g_2 are conjugate with respect to the indicatrix χ.*

The minimal surfaces and the surfaces with flat normal connection are characterized in terms of the tangent indicatrix of the surface as follows.

Proposition 4.2. *Let M^2 be a surface in $\mathbb{R}^4$ or a spacelike surface in $\mathbb{R}^4_1$ free of flat points. Then M^2 is minimal if and only if at each point of M^2 the tangent indicatrix χ is a circle.*

Proposition 4.3. *Let M^2 be a surface in $\mathbb{R}^4$ or a spacelike surface in $\mathbb{R}^4_1$ free of flat points. Then M^2 is a surface with flat normal connection if and only if at each point of M^2 the tangent indicatrix χ is a rectangular hyperbola.*

Now we shall characterized the minimal surfaces and the surfaces with flat normal connection in terms of the ellipse of normal curvature.

The notion of the ellipse of normal curvature of a surface in $\mathbb{R}^4$ was introduced by Moore and Wilson [29, 30]. The ellipse of normal curvature associated to the second fundamental form of a spacelike surface in $\mathbb{R}^4_1$ was first considered in [20]. The *ellipse of normal curvature* at a point p of a surface M^2 is the ellipse in the normal space at the point p given by $\{\sigma(x,x) : x \in T_pM^2, \langle x,x\rangle = 1\}$. Let $\{x,y\}$ be an orthonormal base of the tangent space T_pM^2 at p. Then, for any $v = \cos\psi\, x + \sin\psi\, y$, we have

$$\sigma(v,v) = H + \cos 2\psi\, \frac{\sigma(x,x) - \sigma(y,y)}{2} + \sin 2\psi\, \sigma(x,y),$$

where H is the mean curvature vector of M^2 at p. So, when v goes once around the unit tangent circle, the vector $\sigma(v,v)$ goes twice around the ellipse centered at H.

A surface M^2 in $\mathbb{R}^4$ is called *super-conformal* [2] if at any point of M^2 the ellipse of curvature is a circle. An explicit construction of any simply connected super-conformal surface in $\mathbb{R}^4$ that is free of minimal and flat points is given in [8].

Obviously, M^2 is a minimal surface if and only if for each point $p \in M^2$ the ellipse of curvature is centered at p. We give a characterization of the surfaces with flat normal connection in terms of the ellipse of normal curvature by

Proposition 4.4. *Let M^2 be a surface in $\mathbb{R}^4$ or a spacelike surface in $\mathbb{R}^4_1$ free of flat points. Then M^2 is a surface with flat normal connection if and only if for each point $p \in M^2$ the ellipse of normal curvature is a line segment, which is not collinear with the mean curvature vector field.*

Proposition 4.3 and Proposition 4.4 give us the following

Corollary 4.5. *Let M^2 be a surface in $\mathbb{R}^4$ or a spacelike surface in $\mathbb{R}^4_1$ free of flat points. Then the tangent indicatrix χ is a rectangular hyperbola if and only if the ellipse of normal curvature is a line segment, which is not collinear with the mean curvature vector field.*

5. Fundamental theorems. In the local theory of surfaces in Euclidean space a statement of significant importance is a theorem of Bonnet-type giving the natural conditions under which the surface is determined up to a motion.

A theorem of this type was proved for surfaces with flat normal connection in Euclidean space by B.-Y. Chen in [4].

The general fundamental existence and uniqueness theorems for submanifolds of pseudo-Riemannian manifolds are formulated in terms of tensor fields and connections on vector bundles (e.g. [5], Theorem 2.4 and Theorem 2.5). In [12], we introduce a geometric moving frame field of Frenet-type on a surface in $\mathbb{R}^4$ and using the corresponding derivative formulas, we find eight invariant functions which determine the surface up to a motion in $\mathbb{R}^4$. We formulate and prove the fundamental theorem for surfaces in $\mathbb{R}^4$ in terms of these invariant functions. This theorem is a special case of the general fundamental theorem but in the present form it is more appropriate and easier to apply.

Let M^2 be a surface in $\mathbb{R}^4$ free of minimal points, i.e. $\varkappa^2 - k > 0$ at each point. We assume that M^2 is parameterized by principal lines and denote the unit vector fields $x = \dfrac{z_u}{\sqrt{E}}$, $y = \dfrac{z_v}{\sqrt{G}}$. The equality $M = 0$ implies that the normal vector fields $\sigma(x,x)$ and $\sigma(y,y)$ are collinear. We denote by b a unit normal vector field collinear with $\sigma(x,x)$ and $\sigma(y,y)$, and by l the unit normal vector field such that $\{x, y, b, l\}$ is a positive oriented orthonormal frame field of M^2 (the vectors b, l are determined up to a sign). Thus we obtain a geometrically determined orthonormal frame field $\{x, y, b, l\}$ at each point $p \in M^2$. With respect to this frame field we have the following derivative formulas of Frenet-type:

$$\begin{array}{ll}
\nabla'_x x = \quad\ \ \gamma_1\, y + \nu_1\, b; & \nabla'_x b = -\nu_1\, x - \lambda\, y \qquad + \beta_1\, l;\\
\nabla'_x y = -\gamma_1\, x \qquad + \lambda\, b + \mu\, l; & \nabla'_y b = -\lambda\, x - \nu_2\, y \qquad + \beta_2\, l;\\
\nabla'_y x = \quad -\gamma_2\, y + \lambda\, b + \mu\, l; & \nabla'_x l = \qquad -\mu\, y - \beta_1\, b;\\
\nabla'_y y = \ \gamma_2\, x \qquad + \nu_2\, b; & \nabla'_y l = -\mu\, x \qquad - \beta_2\, b,
\end{array}$$

where $\gamma_1, \gamma_2, \nu_1, \nu_2, \lambda, \mu, \beta_1, \beta_2$ are invariant functions determined by the geometric frame field, $\mu \neq 0$. The invariants k, $\varkappa$, and the Gauss curvature K of M^2 are expressed as follows:

$$k = -4\nu_1\, \nu_2\, \mu^2, \qquad \varkappa = (\nu_1 - \nu_2)\mu, \qquad K = \nu_1\, \nu_2 - (\lambda^2 + \mu^2).$$

We prove the following fundamental theorem for surfaces in $\mathbb{R}^4$ free of minimal points.

Theorem 5.1 ([12]). *Let γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2 be smooth functions, de-*

fined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, *satisfying the conditions*

$$\frac{\mu_u}{2\mu\,\gamma_2 + \nu_1\,\beta_2 - \lambda\,\beta_1} > 0; \qquad \frac{\mu_v}{2\mu\,\gamma_1 - \lambda\,\beta_2 + \nu_2\,\beta_1} > 0;$$
$$-\gamma_1\sqrt{E}\sqrt{G} = (\sqrt{E})_v; \qquad -\gamma_2\sqrt{E}\sqrt{G} = (\sqrt{G})_u;$$
$$\nu_1\,\nu_2 - (\lambda^2 + \mu^2) = \frac{1}{\sqrt{E}}\,(\gamma_2)_u + \frac{1}{\sqrt{G}}\,(\gamma_1)_v - \big((\gamma_1)^2 + (\gamma_2)^2\big);$$
$$2\lambda\,\gamma_2 + \mu\,\beta_1 - (\nu_1 - \nu_2)\,\gamma_1 = \frac{1}{\sqrt{E}}\,\lambda_u - \frac{1}{\sqrt{G}}\,(\nu_1)_v;$$
$$2\lambda\,\gamma_1 + \mu\,\beta_2 + (\nu_1 - \nu_2)\,\gamma_2 = -\frac{1}{\sqrt{E}}\,(\nu_2)_u + \frac{1}{\sqrt{G}}\,\lambda_v;$$
$$\gamma_1\,\beta_1 - \gamma_2\,\beta_2 + (\nu_1 - \nu_2)\,\mu = -\frac{1}{\sqrt{E}}\,(\beta_2)_u + \frac{1}{\sqrt{G}}\,(\beta_1)_v,$$

where $\sqrt{E} = \dfrac{\mu_u}{2\mu\,\gamma_2 + \nu_1\,\beta_2 - \lambda\,\beta_1}$, $\sqrt{G} = \dfrac{\mu_v}{2\mu\,\gamma_1 - \lambda\,\beta_2 + \nu_2\,\beta_1}$. *Let* x_0, y_0, b_0, l_0 *be an orthonormal frame at a point* $p_0 \in \mathbb{R}^4$. *Then there exist a subdomain* $\mathcal{D}_0 \subset \mathcal{D}$ *and a unique surface* $M^2 : z = z(u,v)$, $(u,v) \in \mathcal{D}_0$, *passing through* p_0, *such that* γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2 *are the geometric functions of* M^2 *and* x_0, y_0, b_0, l_0 *is the geometric frame of* M^2 *at the point* p_0.

The Bonnet-type fundamental theorem for surfaces free of minimal points shows that the eight invariants under some natural conditions determine the surface up to a motion in $\mathbb{R}^4$.

In [14] we consider spacelike surfaces in $\mathbb{R}^4_1$ whose mean curvature vector at each point is a non-zero spacelike vector or timelike vector. In a similar way as in $\mathbb{R}^4$ we introduce a geometric moving frame field of Frenet-type on such a surface and using the corresponding derivative formulas, we formulate and prove the fundamental theorem for this class of surfaces in terms of their invariant functions. The theorem states that any spacelike surface with spacelike or timelike mean curvature vector field is determined up to a motion in $\mathbb{R}^4_1$ by its eight invariant functions satisfying some natural conditions.

6. Marginally trapped surfaces in Minkowski 4-space. The concept of trapped surfaces was introduced by Roger Penrose in [31] and it plays an important role in general relativity. These surfaces were defined in order to study global properties of spacetime. A surface in a 4-dimensional spacetime is called marginally trapped if it is closed, embedded, spacelike and its mean curvature vector is lightlike at each point of the surface. In Physics similar or weaker definitions attract attention. Recently, marginally trapped surfaces have been studied from a mathematical viewpoint. In the mathematical literature, it is customary to call a codimension-two surface in a semi-Riemannian manifold *marginally trapped* it its mean curvature vector H is lightlike at each point, and removing the other hypotheses, i.e. the surface does not need to be closed or embedded.

Classification results in 4-dimensional Lorentz manifolds were obtained imposing some extra conditions on the mean curvature vector, the Gauss curvature or the second fundamental form. Marginally trapped surfaces with positive relative nullity in Lorenz space forms were classified in [6]. Marginally trapped surfaces with parallel mean curvature vector in Lorenz space forms were classified in [7]. In [17] marginally trapped surfaces which are invariant under a boost transformation in 4-dimensional Minkowski space were studied, and marginally trapped surfaces in Minkowski 4-space which are invariant under spacelike rotations were classified in [18]. The classification of marginally trapped surfaces in Minkowski 4-space which are invariant under a group of screw rotations (a group of Lorenz rotations with an invariant lightlike direction) is obtained in [19].

In [15] we consider marginally trapped surfaces in the four-dimensional Minkowski space $\mathbb{R}^4_1$. Our approach to the study of these surfaces is based on the principal lines generated by the second fundamental form similarly to the case of spacelike surfaces whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector.

Let $M^2 : z = z(u,v),\ (u,v) \in \mathcal{D}$ be a local parametrization on a marginally trapped surface. The geometric moving frame field for a marginally trapped surface is introduced as follows. Since the mean curvature vector is lightlike at each point of the surface, i.e. $\langle H, H \rangle = 0$, there exists a pseudo-orthonormal normal frame field $\{n_1, n_2\}$, such that $n_1 = H$ and

$$\langle n_1, n_1 \rangle = 0; \quad \langle n_2, n_2 \rangle = 0; \quad \langle n_1, n_2 \rangle = -1.$$

We assume that M^2 is free of flat points, i.e. $(L, M, N) \neq (0,0,0)$. Then at each point of the surface there exist principal lines and without loss of generality we assume that M^2 is parameterized by principal lines. Let us denote $x = \dfrac{z_u}{\sqrt{E}}$, $y = \dfrac{z_v}{\sqrt{G}}$. Thus we obtain a special frame field $\{x, y, n_1, n_2\}$ at each point $p \in M^2$, such that x, y are unit spacelike vector fields collinear with the principal directions; n_1, n_2 are lightlike vector fields, $\langle n_1,\, n_2 \rangle = -1$, and n_1 is the mean curvature vector field. We call such a frame field a *geometric frame field* of M^2. With respect to this frame field we have the following Frenet-type derivative formulas of M^2:

$$\begin{array}{ll}
\nabla'_x x = \gamma_1\, y + (1+\nu)\, n_1; & \nabla'_x n_1 = \mu\, y + \beta_1\, n_1;\\
\nabla'_x y = -\gamma_1\, x + \lambda\, n_1 + \mu\, n_2; & \nabla'_y n_1 = \mu\, x + \beta_2\, n_1;\\
\nabla'_y x = -\gamma_2\, y + \lambda\, n_1 + \mu\, n_2; & \nabla'_x n_2 = (1+\nu)\, x + \lambda\, y - \beta_1\, n_2;\\
\nabla'_y y = \gamma_2\, x + (1-\nu)\, n_1; & \nabla'_y n_2 = \lambda\, x + (1-\nu)\, y - \beta_2\, n_2,
\end{array}$$

where γ_1, γ_2, ν, λ, μ, β_1, β_2 are invariant functions determined by the geometric frame field. In terms of these invariants we prove the following Bonnet-type theorem for marginally trapped surfaces in $\mathbb{R}^4_1$ free of flat points.

Theorem 6.1 ([15]). *Let γ_1, γ_2, ν, λ, μ, β_1, β_2 be smooth functions, defined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions*

$$\begin{array}{ll}
\dfrac{\mu_u}{\mu(2\,\gamma_2 + \beta_1)} > 0; & \dfrac{\mu_v}{\mu(2\,\gamma_1 + \beta_2)} > 0;\\[2ex]
-\gamma_1 \sqrt{E}\sqrt{G} = (\sqrt{E})_v; & -\gamma_2 \sqrt{E}\sqrt{G} = (\sqrt{G})_u;
\end{array}$$

$$2\lambda\,\mu = \frac{1}{\sqrt{E}}\,(\gamma_2)_u + \frac{1}{\sqrt{G}}\,(\gamma_1)_v - \big((\gamma_1)^2 + (\gamma_2)^2\big);$$

$$2\lambda\,\gamma_2 - 2\nu\,\gamma_1 - \lambda\,\beta_1 + (1+\nu)\,\beta_2 = \frac{1}{\sqrt{E}}\,\lambda_u - \frac{1}{\sqrt{G}}\,\nu_v;$$

$$2\lambda\,\gamma_1 + 2\nu\,\gamma_2 + (1-\nu)\,\beta_1 - \lambda\,\beta_2 = \frac{1}{\sqrt{E}}\,\nu_u + \frac{1}{\sqrt{G}}\,\lambda_v;$$

$$\gamma_1\,\beta_1 - \gamma_2\,\beta_2 + 2\nu\,\mu = -\frac{1}{\sqrt{E}}\,(\beta_2)_u + \frac{1}{\sqrt{G}}\,(\beta_1)_v,$$

where $\sqrt{E} = \dfrac{\mu_u}{\mu(2\gamma_2 + \beta_1)}$, $\sqrt{G} = \dfrac{\mu_v}{\mu(2\gamma_1 + \beta_2)}$. Let $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ be vectors at a point $p_0 \in \mathbb{R}^4_1$, such that x_0, y_0 are unit spacelike vectors, $\langle x_0, y_0\rangle = 0$, $(n_1)_0, (n_2)_0$ are lightlike vectors, and $\langle (n_1)_0, (n_2)_0\rangle = -1$. Then there exist a subdomain $\mathcal{D}_0 \subset \mathcal{D}$ and a unique marginally trapped surface $M^2 : z = z(u,v)$, $(u,v) \in \mathcal{D}_0$ free of flat points, such that M^2 passes through p_0, the functions $\gamma_1, \gamma_2, \nu, \lambda, \mu, \beta_1, \beta_2$ are the geometric functions of M^2 and $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is the geometric frame of M^2 at the point p_0.

Theorem 6.1 shows that each marginally trapped surface in $\mathbb{R}^4_1$ is determine up to a motion in $\mathbb{R}^4_1$ by seven invariants satisfying some natural conditions.

7. Examples. We apply our theory to some special classes of surfaces in $\mathbb{R}^4$ or $\mathbb{R}^4_1$.

7.1. Rotational surfaces with two-dimensional axis in $\mathbb{R}^4$. The rotational surfaces with two-dimensional axis in the Euclidean space $\mathbb{R}^4$ are defined as follows. Let $Oe_1e_2e_3e_4$ be a fixed orthonormal base of $\mathbb{R}^4$ and $\mathbb{R}^3$ be the subspace spanned by e_1, e_2, e_3. We consider a smooth curve $c : \widetilde{z} = \widetilde{z}(u),\ u \in J$ in $\mathbb{R}^3$, parameterized by $\widetilde{z}(u) = (x_1(u), x_2(u), r(u))$. Without loss of generality we assume that c is parameterized by the arc-length, i.e. $(x_1')^2 + (x_2')^2 + (r')^2 = 1$. We assume also that $r(u) > 0,\ u \in J$.

The rotational surface M^2, obtained by the rotation of the curve c about the two-dimensional axis Oe_1e_2 (the rotation of c that leaves the plane Oe_1e_2 fixed), is given by

$$z(u, v) = (x_1(u), x_2(u), r(u)\cos v, r(u)\sin v)\,; \quad u \in J,\ v \in [0; 2\pi).$$

In [10] we find the invariants of this rotational surface and describe all rotational surfaces, for which the invariant k is constant.

7.2. Rotational surfaces with two-dimensional axis in $\mathbb{R}^4_1$. There are three types of spacelike rotational surfaces with two-dimensional axis in the Minkowski space $\mathbb{R}^4_1$: elliptic, hyperbolic, and parabolic. They can be defined as follows. Let $Oe_1e_2e_3e_4$ be a fixed orthonormal coordinate system in $\mathbb{R}^4_1$, such that $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \langle e_3, e_3\rangle = 1$, $\langle e_4, e_4\rangle = -1$.

First we consider a smooth spacelike curve $c : \widetilde{z} = \widetilde{z}(u),\ u \in J$, lying in the three-dimensional subspace $\mathbb{R}^3_1 = \operatorname{span}\{e_1, e_2, e_4\}$ of $\mathbb{R}^4_1$ and parameterized by

$$\widetilde{z}(u) = (x_1(u), x_2(u), 0, r(u))\,.$$

We assume that $(x_1')^2 + (x_2')^2 - (r')^2 = 1$, $r(u) > 0,\ u \in J$. Let us consider the surface M^2 in $\mathbb{R}^4_1$ given by

$$z(u, v) = (x_1(u), x_2(u), r(u)\sinh v, r(u)\cosh v)\,; \quad u \in J,\ v \in \mathbb{R}.$$

It is a spacelike surface, which is an orbit of a spacelike curve under the action of the orthogonal transformations of $\mathbb{R}^4_1$ which leave a spacelike plane point-wise fixed (in our case the plane Oe_1e_2 is fixed). It is called a *spacelike rotational surface of hyperbolic type.*

A classification of all spacelike rotational surfaces of hyperbolic type with non-zero constant mean curvature in the three-dimensional de Sitter space $\mathbb{S}^3_1$ is given in [24] and the spacelike Weingarten rotational surfaces in $\mathbb{S}^3_1$ are classified in [25]. Marginally trapped rotational surface of hyperbolic type are found in [17].

In a similar way we consider a spacelike surface in $\mathbb{R}^4_1$ which is an orbit of a spacelike curve c under the action of the orthogonal transformations of $\mathbb{R}^4_1$ which leave a timelike plane point-wise fixed. Now we consider a spacelike curve $c : \widetilde{z} = \widetilde{z}(u),\ u \in J$, parameterized by

$$\widetilde{z}(u) = (r(u), 0, x_1(u), x_2(u))\,; \quad u \in J.$$

The curve c lies in the three-dimensional subspace $\mathbb{R}^3_1 = \operatorname{span}\{e_1, e_3, e_4\}$ of $\mathbb{R}^4_1$. We assume that $(r')^2 + (x_1')^2 - (x_2')^2 = 1$, $r(u) > 0,\ u \in J$. Let us consider the surface M^2 given by

$$z(u, v) = (r(u)\cos v, r(u)\sin v, x_1(u), x_2(u))\,; \quad u \in J,\ v \in [0; 2\pi).$$

It is a spacelike surface in $\mathbb{R}^4_1$, obtained by the rotation of the curve c about the two-dimensional Lorentz plane Oe_3e_4. It is called a *spacelike rotational surface of elliptic type.*

A local classification of spacelike surfaces in $\mathbb{R}^4_1$, which are invariant under spacelike rotations, and with mean curvature vector either vanishing or lightlike, is obtained in [18].

In [13] we study spacelike rotational surfaces of elliptic or hyperbolic type in $\mathbb{R}^4_1$ and describe all such surfaces, for which the invariant k is constant. We also describe the class of Chen spacelike rotational surfaces of elliptic or hyperbolic type.

A spacelike surface in $\mathbb{R}^4_1$, which is the orbit of a spacelike curve under the action of the orthogonal transformations of $\mathbb{R}^4_1$ leaving a degenerate plane point-wise fixed, is called a *spacelike rotational surface of parabolic type* or *screw invariant surface*. Screw invariant surfaces can be parameterized as follows. We consider a spacelike curve $c : \widetilde{z} = \widetilde{z}(u),\ u \in J$, lying in the three-dimensional subspace $\mathbb{R}^3_1 = \operatorname{span}\{e_1, e_3, e_4\}$ of $\mathbb{R}^4_1$ and parameterized by

$$\widetilde{z}(u) = (x_1(u), 0, x_3(u), x_4(u))\,.$$

We assume that $(x_1')^2 + (x_3')^2 - (x_4')^2 = 1$, $x_1(u) \neq 0$, $u \in J$. The spacelike rotational surface of parabolic type is defined by

$$z(u,v) = \Big(x_1(u), v(x_3(u) + x_4(u)), \\ x_3(u) - \frac{v^2}{2}(x_3(u) + x_4(u)), x_4(u) + \frac{v^2}{2}(x_3(u) + x_4(u))\Big).$$

The spacelike Weingarten rotation surfaces of parabolic type in $\mathbb{S}^3_1$ are found in [25]. The screw invariant surfaces with vanishing or lightlike mean curvature vector field are classified in [19].

7.3. General rotational surfaces in $\mathbb{R}^4$. Considering general rotations in $\mathbb{R}^4$, C. Moore introduced general rotational surfaces as follows [28]. Let $c : x(u) = \big(x^1(u), x^2(u), x^3(u), x^4(u)\big)$; $u \in J \subset \mathbb{R}$ be a smooth curve in $\mathbb{R}^4$, and α, β be constants. A general rotation of the meridian curve c in $\mathbb{R}^4$ is given by

$$X(u,v) = \big(X^1(u,v), X^2(u,v), X^3(u,v), X^4(u,v)\big),$$

where

$$\begin{aligned} X^1(u,v) &= x^1(u)\cos\alpha v - x^2(u)\sin\alpha v;\\ X^2(u,v) &= x^1(u)\sin\alpha v + x^2(u)\cos\alpha v;\\ X^3(u,v) &= x^3(u)\cos\beta v - x^4(u)\sin\beta v;\\ X^4(u,v) &= x^3(u)\sin\beta v + x^4(u)\cos\beta v. \end{aligned}$$

In the case $\beta = 0$ the plane Oe_3e_4 is fixed and one gets the classical rotation about a fixed two-dimensional axis.

In [27] we consider a special case of such surfaces, given by

$$\mathcal{M} : z(u,v) = (f(u)\cos\alpha v, f(u)\sin\alpha v, g(u)\cos\beta v, g(u)\sin\beta v),$$

where $u \in J \subset \mathbb{R}$, $v \in [0; 2\pi)$, $f(u)$ and $g(u)$ are smooth functions, satisfying $\alpha^2 f^2(u) + \beta^2 g^2(u) > 0$, $f'^{\,2}(u) + g'^{\,2}(u) > 0$, and α, β are positive constants. In the case $\alpha \neq \beta$ each parametric curve $u = const$ is a curve in $\mathbb{R}^4$ with constant Frenet curvatures, and in the case $\alpha = \beta$ each parametric curve $u = const$ is a circle. The parametric curves $v = const$ of $\mathcal{M}$ are plane curves with Frenet curvature $\varkappa = \dfrac{|g'f'' - f'g''|}{(\sqrt{f'^{\,2} + g'^{\,2}})^3}$. So, for each $v = \text{const}$ the parametric curves are congruent in $\mathbb{R}^4$. These curves are the meridians of $\mathcal{M}$.

The surfaces given above are general rotational surfaces in the sense of C. Moore with plane meridian curves. We find the invariant functions γ_1, γ_2, ν_1, ν_2, λ, μ, β_1, β_2 of these surfaces and find all minimal super-conformal general rotational surfaces in $\mathbb{R}^4$ [27].

7.4. General rotational surfaces in $\mathbb{R}^4_1$. In [14] we consider a class of spacelike surfaces in $\mathbb{R}^4_1$ which are analogous to the general rotational surfaces with plane meridians in $\mathbb{R}^4$.

Let us consider the surface $\mathcal{M}_1$ parameterized by

$$\mathcal{M}_1 : z(u,v) = (f(u)\cos\alpha v, f(u)\sin\alpha v, g(u)\cosh\beta v, g(u)\sinh\beta v)\,,$$

where $u \in J \subset \mathbb{R}$, $v \in [0; 2\pi)$, $f(u)$ and $g(u)$ are smooth functions, satisfying $\alpha^2 f^2(u) - \beta^2 g^2(u) > 0$, $f'^{\,2}(u) + g'^{\,2}(u) > 0$, and α, β are positive constants.

$\mathcal{M}_1$ is a spacelike surface with spacelike mean curvature vector field, parameterized by principal parameters (u, v).

Analogously, we consider the surface $\mathcal{M}_2$ given by

$$\mathcal{M}_2 : z(u,v) = (f(u)\cos\alpha v, f(u)\sin\alpha v, g(u)\sinh\beta v, g(u)\cosh\beta v)\,,$$

where $u \in J \subset \mathbb{R}$, $v \in [0; 2\pi)$, $f(u)$ and $g(u)$ are smooth functions, satisfying $f'^{\,2}(u) - g'^{\,2}(u) > 0$, $\alpha^2 f^2(u) + \beta^2 g^2(u) > 0$, and α, β are positive constants.

$\mathcal{M}_2$ is a spacelike surface with timelike mean curvature vector field, parameterized by principal parameters (u, v).

We find the invariants of the general rotational spacelike surfaces $\mathcal{M}_1$ and $\mathcal{M}_2$ [14].

7.5. Meridian surfaces in $\mathbb{R}^4$. In [12] we construct a family of surfaces lying on a rotational hypersurface in the Euclidean space $\mathbb{R}^4$ as follows. Let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal frame in $\mathbb{R}^4$, and $S^2(1)$ be the 2-dimensional sphere in $\mathbb{R}^3 = \operatorname{span}\{e_1, e_2, e_3\}$, centered at the origin O. Let $f = f(u)$, $g = g(u)$ be smooth functions, defined in an interval $I \subset \mathbb{R}$, such that $\dot{f}^2(u) + \dot{g}^2(u) = 1,\ u \in I$, where $\dot{f}(u) = \dfrac{df(u)}{du}$ and $\dot{g}(u) = \dfrac{dg(u)}{du}$. The standard rotational hypersurface M in $\mathbb{R}^4$, obtained by the rotation of the meridian curve $m : u \to (f(u), g(u))$ about the Oe_4-axis, is parameterized as follows:

$$M : Z(u, w^1, w^2) = f(u)\, l(w^1, w^2) + g(u)\, e_4,$$

where $l(w^1, w^2)$ is the unit position vector of $S^2(1)$ in $\mathbb{R}^3$.

We consider a smooth curve $c : l = l(v) = l(w^1(v), w^2(v)),\ v \in J,\ J \subset \mathbb{R}$ on $S^2(1)$, parameterized by the arc-length. Using this curve we construct a surface $\mathcal{M}_m$ in $\mathbb{R}^4$ in the following way:

$$\mathcal{M}_m : z(u,v) = f(u)\, l(v) + g(u)\, e_4, \quad u \in I,\, v \in J.$$

The surface $\mathcal{M}_m$ lies on the rotational hypersurface M. Since $\mathcal{M}_m$ is a one-parameter system of meridians of the rotational hypersurface, we call $\mathcal{M}_m$ a *meridian surface*.

We describe the meridian surfaces with constant Gauss curvature, with constant mean curvature, and with constant invariant k [12].

7.6. Meridian surfaces in $\mathbb{R}^4_1$. In [15] we use the same idea to construct a special family of two-dimensional spacelike surfaces lying on rotational hypersurfaces in $\mathbb{R}^4_1$. We consider a rotational hypersurface with timelike axis and a rotational hypersurface with spacelike axis.

Let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal frame in $\mathbb{R}^4_1$, i.e. $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \langle e_3, e_3\rangle = 1$, $\langle e_4, e_4\rangle = -1$. First we consider a rotational hypersurface with timelike axis. Let $S^2(1)$ be the 2-dimensional sphere in the Euclidean space $\mathbb{R}^3 = \operatorname{span}\{e_1, e_2, e_3\}$, centered at the origin O. Let $f = f(u)$, $g = g(u)$ be smooth functions, defined in an interval $I \subset \mathbb{R}$, such that $\dot{f}^2(u) - \dot{g}^2(u) > 0,\ u \in I$. We assume that $f(u) > 0,\ u \in I$. The standard rotational hypersurface $\mathcal{M}'$ in $\mathbb{R}^4_1$, obtained by the rotation of the meridian curve $m : u \to (f(u), g(u))$ about the Oe_4-axis, is parameterized as follows:

$$\mathcal{M}' : Z(u, w^1, w^2) = f(u)\, l(w^1, w^2) + g(u)\, e_4,$$

where $l(w^1, w^2)$ is the unit position vector of $S^2(1)$ in $\mathbb{R}^3$. The hypersurface $\mathcal{M}'$ is a rotational hypersurface in $\mathbb{R}^4_1$ with timelike axis.

We consider a smooth curve $c : l = l(v) = l(w^1(v), w^2(v)),\ v \in J,\ J \subset \mathbb{R}$ on $S^2(1)$, parameterized by the arc-length, and construct a surface $\mathcal{M}'_m$ in $\mathbb{R}^4_1$ in the following way:

$$\mathcal{M}'_m : z(u,v) = f(u)\, l(v) + g(u)\, e_4, \quad u \in I,\, v \in J.$$

The surface $\mathcal{M}'_m$ lies on the rotational hypersurface $\mathcal{M}'$ in $\mathbb{R}^4_1$. Since $\mathcal{M}'_m$ is a one-parameter system of meridians of $\mathcal{M}'$, we call $\mathcal{M}'_m$ a *meridian surface on* $\mathcal{M}'$.

In a similar way we consider meridian surfaces lying on a rotational hypersurface in $\mathbb{R}^4_1$ with spacelike axis. Let $S^2_1(1)$ be the timelike sphere in the Minkowski space $\mathbb{R}^3_1 = \operatorname{span}\{e_2, e_3, e_4\}$, i.e. $S^2_1(1) = \{V \in \mathbb{R}^3_1 : \langle V, V\rangle = 1\}$. $S^2_1(1)$ is a timelike surface in $\mathbb{R}^3_1$ known as the *de Sitter space*. Let $f = f(u)$, $g = g(u)$ be smooth functions, defined in an interval $I \subset \mathbb{R}$, such that $\dot{f}^2(u) + \dot{g}^2(u) > 0$, $f(u) > 0$, $u \in I$. We denote by $l(w^1, w^2)$ the unit position vector of $S^2_1(1)$ in $\mathbb{R}^3_1$ and consider the rotational hypersurface $\mathcal{M}''$ in $\mathbb{R}^4_1$, obtained by the rotation of the meridian curve $m : u \to (f(u), g(u))$ about the Oe_1-axis. It is parameterized as follows:

$$\mathcal{M}'' : Z(u, w^1, w^2) = f(u)\, l(w^1, w^2) + g(u)\, e_1.$$

The hypersurface $\mathcal{M}''$ is a rotational hypersurface in $\mathbb{R}^4_1$ with spacelike axis.

Now we consider a smooth spacelike curve $c : l = l(v) = l(w^1(v), w^2(v))$, $v \in J$, $J \subset \mathbb{R}$ on $S^2_1(1)$, parameterized by the arc-length, and construct a surface $\mathcal{M}''_m$ in $\mathbb{R}^4_1$ as follows:

$$\mathcal{M}''_m : z(u, v) = f(u)\, l(v) + g(u)\, e_1, \quad u \in I,\ v \in J.$$

The surface $\mathcal{M}''_m$ lies on the rotational hypersurface $\mathcal{M}''$. We call $\mathcal{M}''_m$ a *meridian surface on* $\mathcal{M}''$, since $\mathcal{M}''_m$ is a one-parameter system of meridians of $\mathcal{M}''$.

We find all meridian surfaces in $\mathbb{R}^4_1$ which are marginally trapped [15].

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