PLISKA
STUDIA MATHEMATICA
BULGARICA
ПЛИСКА
БЪЛГАРСКИ
МАТЕМАТИЧЕСКИ
СТУДИИ

Pliska Stud. Math. Bulgar. **21** (2012), 147–176

PLISKA
STUDIA MATHEMATICA
BULGARICA

POLYHARMONIC HARDY SPACES ON THE KLEIN-DIRAC QUADRIC WITH APPLICATION TO POLYHARMONIC INTERPOLATION AND CUBATURE FORMULAS*

Ognyan Kounchev, Hermann Render

Abstract. In the present paper we introduce a new concept of Hardy type space naturally defined on the Klein-Dirac quadric. We study different properties of the functions belonging to these spaces, in particular boundary value problems. We apply these new spaces to polyharmonic interpolation and to interpolatory cubature formulas.

1. Introduction. In one-dimensional mathematical analysis, Interpolation Theory and Quadrature formulas are intimately related, cf. [35], [11]. This relation causes a similarity between the approaches for estimation of the remainders of Interpolation and Quadrature. One approach for estimation of the error in Interpolation theory is related to Lagrange formula and uses higher derivatives of the interpolated function (cf. [35], chapter 3, Theorem 4, and [11], Theorem 3.1.1). The second approach uses analyticity of the interpolated function and Hermite formula (cf. [35], chapter 3, Theorem 5, and [11], Theorema 3.6.1). In a similar way, already A. Markov has estimated the error of a quadrature formula for differentiable functions in $C^N(I)$ defined on the interval I by means of its $N-$th derivative (cf. [35], chapter 7.1, and Davis [11], p. 344). The second

2010 *Mathematics Subject Classification:* Primary 65D30, 32A35, Secondary 41A55.
Key words: Hardy spaces, numerical integration, cubature formulas, error estimate.
*Both authors thank the Alexander von Humboldt Foundation and Grant DO-02-275 with Bulgarian National Foundation.

approach estimates the error of a quadrature formula for certain classes of functions f which are *analytic* on some open set D in $\mathbb{C}$ containing the interval I, (cf. [12], chapter 4.6, see also [35], chapter 12.2). However, in both Interpolation and Quadrature, the first approach is usually not very practical beyond derivatives of order five, see [56].

Interpolation and Approximation of integrals in the multivariate case is a much more difficult task. In Numerical Analysis, instead of quadrature formula the notion of cubature formula is often used, see [51], [55], [52] and the recent survey [10]. In contrast to the univariate case there is no satisfactory error analysis available in the multivariate case, cf. [55], [5], and part 4 in the last Russian edition of the classical monograph [35]. Let us mention that the area of quadrature domains which has received a lot of interest recently presents an interesting multidimensional alternative and we refer to [14].

The present research continues the study of estimates of *polyharmonic interpolation* and *polyharmonic interpolatory cubature formulas* initiated in [30]; in the present paper we consider interpolation and cubature formulas in the ball in $\mathbb{R}^d$, while in [30] the case of an annular region was considered[1].

1.1. Gauss-Almansi formula. In [18] *polyharmonic interpolation* has been considered for functions defined in the ball in $\mathbb{R}^d$. In the same spirit, in [27] and [25] we have introduced new multivariate cubature formulae $C_N(f)$ in the ball depending on a parameter $N \in \mathbb{N}$ which approximates the integral

$$\int_{B_R} f(x)\, d\mu(x) \tag{1}$$

for continuous functions $f : B_R \to \mathbb{C}$ defined on the *ball*

$$B_R = \left\{ x \in \mathbb{R}^d : |x| < R \right\}, \tag{2}$$

where $|x|$ denotes the euclidean norm of $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

The exact definition of the *polyharmonic interpolation* formula and of the *polyharmonic cubature* formula $C_N(f)$ will be explained in Section 6. A major purpose of the present paper is to provide an error analysis for a class of functions on the ball B_R which exhibit a certain type of analytical behavior.

Let us introduce the necessary notions and notations. Let $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ be the unit sphere endowed with the rotation invariant measure $d\theta$. We shall write $x \in \mathbb{R}^d$ in spherical coordinates $x = r\theta$ with $\theta \in \mathbb{S}^{d-1}$. Let $\mathcal{H}_k(\mathbb{R}^d)$

[1] Extended electronic version of the present paper with full proofs of the results appear in `http://arxiv.org/abs/1205.6414`.

be the set of all harmonic homogeneous complex-valued polynomials of degree k. Then $f \in \mathcal{H}_k\left(\mathbb{R}^d\right)$ is called a *solid harmonic* and the restriction of f to $\mathbb{S}^{d-1}$ a *spherical harmonic* of degree k and we set

$$a_k := \dim \mathcal{H}_k\left(\mathbb{R}^d\right), \tag{3}$$

see [54], [49], [1], [24] for details. Throughout the paper we shall assume that the set of functions

$$Y_{k,\ell} : \mathbb{R}^d \to \mathbb{C}, \text{ for } \ell = 1, \ldots, a_k, \tag{4}$$

is an *orthonormal basis* of $\mathcal{H}_k\left(\mathbb{R}^d\right)$ with respect to the scalar product

$$\langle f, g\rangle_{\mathbb{S}^{d-1}} := \int_{\mathbb{S}^{d-1}} f(\theta)\,\overline{g(\theta)}d\theta.$$

Recall that due to the homogeneity of $Y_{k,\ell}(x)$ we have the identity $Y_{k,\ell}(x) = r^k Y_{k\ell}(\theta)$ for $x = r\theta$.

Our polyharmonic Interpolation and polyharmonic Cubature $C_N(f)$ approximating the integral (1) are based on the *Laplace–Fourier series* of the continuous function $f : B_R \to \mathbb{C}$, defined by the formal expansion

$$f(r\theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} f_{k,\ell}(r)\, Y_{k,\ell}(\theta) \tag{5}$$

where the *Laplace–Fourier coefficient* $f_{k,\ell}(r)$ is defined by

$$f_{k,\ell}(r) = \int_{\mathbb{S}^{d-1}} f(r\theta)\, Y_{k,\ell}(\theta)\, d\theta \tag{6}$$

for any positive real number r with $r < R$ and a_k is defined in (3). There is a strong interplay between algebraic and analytic properties of the function f and those of the Laplace-Fourier coefficients $f_{k,\ell}$. For example, if $f(x)$ is a polynomial in the variable $x = (x_1, \ldots, x_d)$ then the Laplace-Fourier coefficient $f_{k,\ell}$ is of the form $f_{k,\ell}(r) = r^k p_{k,\ell}\left(r^2\right)$ where $p_{k,\ell}$ is a univariate polynomial, see e.g. in [54] or [51]. Hence, the *Laplace-Fourier* series (5) of a polynomial $f(x)$ is equal to

$$f(x) = \sum_{k=0}^{\deg f} \sum_{\ell=1}^{a_k} p_{k,\ell}(|x|^2) Y_{k,\ell}(x) = \sum_{k=0}^{\deg f} \sum_{\ell=1}^{a_k} |x|^k\, p_{k,\ell}(|x|^2) Y_{k,\ell}(\theta) \tag{7}$$

where $\deg f$ is the total degree of f and $p_{k,\ell}$ is a univariate polynomial of degree $\leq \deg f - k$. This representation is often called the *Gauss representation.* A similar formula is valid for a much larger class of functions. Let us recall that a function $f : G \to \mathbb{C}$ defined on an open set G in $\mathbb{R}^d$ is called *polyharmonic of order* N if f is $2N$ times continuously differentiable and

$$\Delta^N u\left(x\right) = 0 \tag{8}$$

for all $x \in G$ where $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$ is the Laplace operator and Δ^N the N-th iterate of Δ. The theorem of *Almansi* states that for a polyharmonic function f of order N defined on the ball $B_R = \left\{x \in \mathbb{R}^d : |x| < R\right\}$ there exist univariate polynomials $p_{k,\ell}\left(r\right)$ of degree $\leq N-1$ such that

$$f\left(x\right) = \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k} p_{k,\ell}(|x|^2) Y_{k,\ell}\left(x\right) = \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k} |x|^k\, p_{k,\ell}(|x|^2) Y_{k,\ell}\left(\theta\right) \tag{9}$$

where convergence of the sum is uniform on compact subsets of B_R, see e.g. [51], [3], [2].

To end this technical introduction, let us remind some estimates for spherical harmonics which we will need below:

1. For every multiindex α and for every integer $k \geq 1$ holds

   $$\left|D_\theta^\alpha Y_{k,\ell}\left(\theta\right)\right| \leq C k^{|\alpha| + \frac{d-2}{2}} \qquad \text{for } \theta \in \mathbb{S}^{d-1}, \tag{10}$$

   see [49], p. 120. Here D_θ^α is the multi-index notation for the derivative with respect to $\theta \in \mathbb{S}^{d-1}$.

2. A function $f\left(\theta\right)$ defined on $\mathbb{S}^{d-1}$ is real analytic if its Laplace-Fourier expansion $f\left(\theta\right) = \sum_{k=0}^{\infty} Y_k\left(\theta\right)$ (where we have put $Y_k\left(\theta\right) = \sum_{\ell=1}^{a_k} f_{k,\ell} Y_{k,\ell}\left(\theta\right)$) satisfies

   $$\left\|Y_k\left(\theta\right)\right\|_{L_2\left(\mathbb{S}^{d-1}\right)} < Ce^{-\eta k} \qquad \text{for } k \geq 0,$$

   for some constants C, $\eta > 0$; see [51].

1.2. Complexification of the ball in $\mathbb{R}^d$, related to the ball of the Klein-Dirac quadric. We want to study analytical extensions of functions f defined on the ball using the Laplace-Fourier series (9). Our strategy is to require minimal assumptions on the functions f; thus instead of the standard approach where one works with functions f which are *a priori* analytically extendible to a fixed domain U in the complex space $\mathbb{C}^d$ (as in [2]) we shall require only that we can extend the function $x = r\theta \longmapsto f\left(r\theta\right)$ to an analytic function $z\theta \longmapsto f\left(z\theta\right)$, so we only complexify the radial variable r to a complex variable z. Henceforth we will use the following **terminological convention**: If the function $f(r\theta)$ possesses an analytic extension with respect to r we call the extended function $f(z\theta)$ "r-**complexification**" or "r-**analytic continuation**". Thus, for the $r-$complexification of the function f one should expect from equation (6) that the Laplace-Fourier coefficient $f_{k,\ell}(r)$ extends to an analytic function of one variable. Hence, we consider the analytically continued functions on domains in the set $\mathbb{C}\times\mathbb{S}^{d-1}$.

We obtain the following important proposition.

Proposition 1. *The set of functions*

$$B_N := \left\{ \begin{array}{c} b_{k,\ell;j}(r,\theta) = r^{k+2j} Y_{k,\ell}(\theta) : \\ k\geq 0,\ \ell = 1,2,\ldots,a_k,\ j=0,1,\ldots,N-1 \end{array} \right\} \tag{11}$$

is ***a basis*** *for the multivariate polynomials which are polyharmonic of order N.*

The proof follows by representation (7).

Remark 2. We will see later that the basis B_N is a natural generalization of the basis $\left\{r^j\right\}_{j=0}^{N-1}$ for the polynomials in the one-dimensional case.

The main approach in the present paper is to consider the $r-$complexification of the functions f defined on the ball $B_R\subset\mathbb{R}^d$ in the form $f(z\theta)$. In particular, for polyharmonic functions f we can provide a formal expression for the $r-$complexification, using representation (9), by the following formula:

$$f(z\theta) = \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k} p_{k,\ell}(z^2) Y_{k,\ell}(z\theta) = \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k} z^k p_{k,\ell}(z^2) Y_{k,\ell}(\theta) \tag{12}$$

The question of convergence will be addressed in the course of the paper. From this formula we make a **crucial observation**: the r-complexification $f(z\theta)$ depends only on the values $z\theta$ and not on the coordinates of the pair (z,θ). Indeed, this is due to the equality $z^2=(z\theta,z\theta)$ where for the vectors $w,u\in\mathbb{C}^d$ we have the *non-Hermitian* product by putting $(u,w) := \sum_{j=1}^{d} u_j w_j$. Thus the function $f(z\theta)$ is defined on the space $\mathbb{C}\times\mathbb{S}^{d-1}/\mathbb{Z}_2 = \left\{z\theta : z\in\mathbb{C},\ \theta\in\mathbb{S}^{d-1}\right\}$, where the factor in $\mathbb{Z}_2$ means identification of the points (z,θ) and $(-z,-\theta)$ in $\mathbb{C}\times\mathbb{S}^{d-1}$. But the set $\mathbb{C}\times\mathbb{S}^{d-1}/\mathbb{Z}_2\subset\mathbb{C}^d$ is one of the possible representations of the famous *Klein-Dirac quadric.*

Definition 3. *We define the **Klein-Dirac** quadric by putting*

$$\mathrm{KDQ} := \mathbb{C} \times \mathbb{S}^{d-1}/\mathbb{Z}_2. \tag{13}$$

We shall see that the $r-$analytic continuation of the solutions of the polyharmonic equation are in fact "the analytic functions" naturally defined on the Klein-Dirac quadric KDQ. The main interest of the present paper is devoted to the Function theory on the **complexified ball** $\mathcal{B}_R$ in KDQ defined by

$$\mathcal{B}_R := \left\{ z\theta : \ |z| < R, \ \theta \in \mathbb{S}^{d-1} \right\} = \mathbb{D}_R \times \mathbb{S}^{d-1}/\mathbb{Z}_2, \tag{14}$$

where $\mathbb{D}_R$ is the open disc of radius R in $\mathbb{C}$, i.e. $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$; for $R = 1$ as usually one puts $\mathbb{D}_1 = \mathbb{D}$ and $\mathcal{B} = \mathcal{B}_1$.

Some enlightening comments about the **Klein-Dirac quadric** KDQ defined in (13) are in order.

Remark 4. This quadric has been originally introduced in a special case by Felix Klein in his Erlangen program in 1870, where he put forth his correspondence between the lines in complex projective 3-space and a general quadric in projective 5-space. The physical relevance of the quadric and the relation to the conformal motions of compactified Minkowski space-time had been exploited by Paul Dirac in 1936 [13]. The Klein-Dirac quadric plays an important role also in Twistor theory, where it is related to the complexified compactified Minkowski space [42]. Apparently, the term "Klein-Dirac quadric" for arbitrary dimension d, has been coined by the theoretical physicist I. Todorov, cf. e.g. [39], [40]. In these references important aspects of the Function theory on the ball in KDQ have been considered in the context of Conformal Quantum Field Theory (CFT). In the context of CFT Laurent expansions appear in a natural way as the field functions in the higher dimensional conformal vertex algebras (using a complex variable parametrization of compactified Minkowski space); see in particular formula (4.43) in [41], as well as the references [39], [40].

Our main novelty will be a multivariate generalization of the classical Hardy space $H^2(\mathbb{D})$ called **(polyharmonic) Hardy space on the ball** $\mathcal{B}_R$ to be introduced in Definition 11. We will denote this space by $H^2(\mathcal{B}_R)$. For simplicity sake we will restrict ourselves to considering the space $H^2(\mathcal{B})$.

Running ahead of the events, let us say that the name *polyharmonic* comes from the fact that $H^2(\mathcal{B}_R)$ may be obtained as a limit of the r-complexifications of the polyharmonic functions in the ball (12): we take the closure of all finite sums of the type

$$u(z,\theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} u_{k,\ell}\left(z^2\right) z^k Y_{k,\ell}(\theta), \tag{15}$$

where $u_{k,\ell}(\cdot)$ are algebraic polynomials of degree $\leq N-1$; such functions u satisfy $\Delta^N u(x) = 0$. In view of the Polyharmonic Paradigm announced in [24], the space $H^2(\mathcal{B}_R)$ generalizes the classical Hardy spaces which are obtained as limits of algebraic polynomials, where the degree of a polynomial is replaced by the **degree of polyharmonicity**.

The Hardy space $H^2(\mathcal{B}_R)$ will be a Hilbert space and we will provide a *Cauchy type kernel*, which is the analog and a generalization to the Hua-Aronszajn kernel in the ball (cf. [2], p. 125, Corollary 1.1). Let us note that the last is a multidimensional generalization of the classical Cauchy kernel $\frac{1}{z-a}$ from Complex Analysis (more about Cauchy kernels see in [22], [32]).

Remark 5. The reader familiar with the Cartan classification of classical domains, may remark that the boundary $\partial\mathcal{B}_R$ is equal to the **Shilov boundary**

$$\left\{ \left(e^{i\varphi} R\theta\right) : \varphi \in [0,\pi],\ \theta \in \mathbb{S}^{d-1} \right\}$$

of the so-called **Cartan classical domai**n $\mathcal{R}_{IV}$ (called also "Lie-ball") equal to

$$\widehat{B}_R := \left\{ \xi + i\eta \in \mathbb{C}^d : \xi, \eta \in \mathbb{R}^d,\ q(\xi + i\eta) < R \right\} \qquad \text{where}$$

$$q(\xi + i\eta) = \sqrt{|\xi|^2 + |\eta|^2 + 2\sqrt{|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2}}.$$

This has been considered from the point of view of several complex variables in the monograph of Hua [21], and for the study of the polyharmonic functions of infinite order in the monographs [2] (see in particular p. 59 and 126) and [3]. The Hardy spaces defined in Definition 11 can be identified with the Hardy space of holomorphic functions on the Lie ball in $\mathbb{C}^d$, cf. [50]; this correspondence will be given a thorough consideration in [28].

Remark 6. Let us define the annulus in the Klein-Dirac quadric KDQ as the set

$$\widetilde{A}_{a,b} := \left\{ z\theta \in \mathrm{KDQ} : a < |z| < b,\ \theta \in \mathbb{S}^{d-1} \right\}.$$

The Function theory on $\widetilde{A}_{a,b}$ would help to relate the present results to previous obtained by us. We have seen in [30] that an interesting, consistent and fruitful

Function theory is available only on the set

$$\mathcal{A}_{a,b} = \left\{ (z, \theta) : a < |z| < b, \ \theta \in \mathbb{S}^{d-1} \right\}$$

which is a subset of $\mathbb{C} \times \mathbb{S}^{d-1}$. This is due to the fact that the $r-$analytic continuations of the solutions of the polyharmonic equations in the annulus $A_{a,b} \subset \mathbb{R}^d$ live on the set $\mathcal{A}_{a,b}$ but not on the set $\widetilde{A}_{a,b}$! The point is that in the case of the annulus $\mathcal{A}_{a,b}$ we cannot identify the point $z\theta$ with (z, θ), in other words, (z, θ) is not identified with $(-z, -\theta)$.

The paper is organized as follows: in Section 2 we recall background material about the Hardy space $H^2(\mathbb{B}_R)$. In Section 3 we introduce the polyharmonic Hardy space $H^2(\mathcal{B})$ on the ball $\mathcal{B}$ of the Klein-Dirac quadric. We prove that it is a Hilbert space, a maximum principle, and infinite-differentiability of the functions in $H^2(\mathcal{B}_b)$. In Section 3.1 we construct a Cauchy type kernel for $H^2(\mathcal{B}_b)$. In Section 4 we prove other main properties which generalize similar properties of the one-dimensional Hardy spaces. In Section 5 we characterize the polyharmonic functions which are extendible to the Hardy space $H^2(\mathcal{B}_b)$. In Section 6 we prove some of the main results of the paper, about the error estimate of the polyharmonic interpolation, and about the polyharmonic interpolatory cubature formulas, generalizing the polyharmonic Gauß-Jacobi cubature formulas introduced in [25], [27].

2. Classical Hardy spaces – a reminder. Hardy spaces are a bridge between Harmonic and Complex Analysis. This is based on the fact that the Taylor coefficients of a function $f(x)$ on the real line $\mathbb{R}$ are at the same time the coefficients of an orthogonal expansion of the analytic continuation $f(z)$ with respect to the basis $\left\{z^j\right\}_{j\geq 0}$ which is orthogonal on the circle. Thus in a certain sense the setting of the Hardy spaces represents a study of the properties of the real functions having Taylor coefficients with $\sum |a_j|^2 < \infty$ by the methods of Complex Analysis. Since we are generalizing before all the Hardy space H^2 we will recall the main results about it. Let us put

$$M(f; r) := \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\varphi}\right)\right|^2 d\varphi \right\}^{1/2}.$$

Then the for every function f which is analytic in the disc $\mathbb{D}$ we define the Hardy space norm

$$\|f\|_{H^2(\mathbb{D})} := \sup_{r<1} M(f; r).$$

Remark 7. For every analytic function f the function $M(f;r)$ is an increasing function of r, cf. Theorem 17.6 in [44].

A basic fact is that H^2 is a Hilbert space and may be identified with a subspace of $L^2(\mathbb{S}^1)$. For every $g \in L^2(\mathbb{S}^1)$ we have the norm defined by

$$\|g\|^2_{L_2(\mathbb{S})} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|g\left(e^{i\varphi}\right)\right|^2 d\varphi,$$

and the Fourier coefficients

$$\widehat{g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g\left(e^{i\varphi}\right) e^{-in\varphi} d\varphi \qquad \text{for } n \in \mathbb{Z}.$$

Some of the main properties of the classical Hardy spaces H^2 are summarized in the following theorem (cf. Theorem 17.10 in [44]).

Theorem 8. *1. An analytic function f on $\mathbb{D}$ of the type*

$$f(z) = \sum_{j=0}^{\infty} f_j z^j \qquad \textit{for } z \in \mathbb{D} \tag{16}$$

belongs to $H^2(\mathbb{D})$ if and only if

$$\sum_{j=0}^{\infty} |f_j|^2 < \infty; \tag{17}$$

in that case

$$\|f\|^2_{H^2(\mathbb{D})} = \sum_{j=0}^{\infty} |f_j|^2 .$$

2. If $f \in H^2(\mathbb{D})$ then f has radial limits $f^*\left(e^{i\varphi}\right)$ at almost all points on the circle $\mathbb{S}$ and $f^* \in L^2(\mathbb{S})$. The Riesz condition holds, i.e.

$$\begin{aligned} f_j^* &= 0 \qquad \textit{for all } j < 0; \\ f_j^* &= f_j \qquad \textit{for all } j \geq 0. \end{aligned} \tag{18}$$

The L^2-approximation holds

$$\lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\varphi}\right) - f^*\left(e^{i\varphi}\right)\right|^2 d\varphi = 0.$$

The integral of Poisson and of Cauchy of f^* recover f, i.e.

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\varphi - t) f^*\left(e^{it}\right) dt$$

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f^*(\zeta)}{\zeta - z} d\zeta$$

where Γ is the positively oriented circle $\mathbb{S}$.

3. The mapping $f \longmapsto f^*$ is an isometry of H^2 on the subspace of $L^2(\mathbb{S})$ which consists of those g for which $\widehat{g}(j) = 0$ for all $j < 0$.

Let us recall the famous theorem of brothers F. and M. Riesz which concludes the absolute continuity of a Borel measure on $\mathbb{S}^1$ only from the annihilation of half of its Fourier coefficients, [44, Theorem 17.13].

Theorem 9. *Let μ be a complex valued Borel measure on the circle $\mathbb{S}$. If*

$$\int_0^{2\pi} e^{imt} d\mu(t) = 0 \qquad \text{for } m \geq 1, \tag{19}$$

then the measure μ is absolutely continuous with respect to the Lebesgue measure, i.e. there exists a function $f^* \in L^1(\mathbb{S})$ such that $d\mu(t) = \frac{1}{2\pi} f^*(t)\, dt$ for $t \in [0, 2\pi]$.

3. The polyharmonic Hardy space $H^2(\mathcal{B})$ on the ball of the Klein-Dirac quadric. At first we observe that the basis functions $b_{k,\ell;j}(r,\theta)$ defined in (11) have a natural $r-$analytic extension

$$b_{k,\ell,j}(z,\theta) := z^{2j+k} Y_{k,\ell}(\theta). \tag{20}$$

Let us present some heuristics for explaining our main goal: We want to define the Hardy space $H^2(\mathcal{B}_R)$ as a space of functions which are uniform limits of sequences of complexified polynomials $P(z\theta)$ on compacts of the ball $\mathcal{B}_R$ in KDQ. For that reason we need an appropriate inner product. Let us make the **important observation** that there is a natural *inner product* where the basic functions (20) are **orthogonal**. Hence, for functions defined on the set $\mathbb{S}^1 \times \mathbb{S}^{d-1}$ we introduce the inner product

$$\langle f, g \rangle_* := \frac{1}{2\pi} \int_{\mathbb{S}^{d-1}} \int_0^{2\pi} f\left(e^{i\varphi}, \theta\right) \overline{g\left(e^{i\varphi}, \theta\right)} d\varphi d\theta. \tag{21}$$

The **crux** of our approach is the orthogonality of the basis functions $b_{k,\ell,j}$ in (11) and (20) on the boundary of the Klein-Dirac quadric $\mathbb{S}^1 \times \mathbb{S}^{d-1}/\mathbb{Z}_2$, i.e.

$$\left\langle b_{k,\ell,j}, b_{k',\ell,j} \right\rangle = \delta_{k,k'}\delta_{\ell,\ell'}\delta_{j,j'}, \tag{22}$$

where the Kronecker symbol δ means $\delta_{\alpha,\beta} = 1$ for $\alpha = \beta$ and 0 for $\alpha \neq \beta$. This property is a remarkable generalization of the orthogonality of the basis $\left\{z^j\right\}_{j\geq 0}$ on the circle $\mathbb{S}^1$ and traces the analogy to the one-dimensional case.

Further, we provide some arguments about the proper definition of the norm of the prospective Hardy space. The objects of our polyharmonic Hardy space $H^2(\mathcal{B})$ will be functions $f(z,\theta)$ which are representable as infinite sums in the L^2 sense, and are absolutely and uniformly convergent on compacts with $|z| < 1$:

$$f(z,\theta) = \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k}\left(\sum_{j=0}^{\infty} f_{k,\ell,j} z^{2j}\right) z^k Y_{k,\ell}(\theta). \tag{23}$$

$$= \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k} f_{k,\ell}(z)\, Y_{k,\ell}(\theta) \tag{24}$$

If $f_{k,\ell,j} = 0$ for all $j \geq N$ and $k \leq K$ for some $K \geq 0$, then the function f is a polynomial satisfying $\Delta^N f(x) = 0$ for all $x \in \mathbb{R}^d$. Hence, formula (23) represents the $r-$complexification for the polynomials.

In view of representation (23) we introduce the following naturally defined subspaces of the classical Hardy spaces on the unit disc $\mathbb{D} \subset \mathbb{C}$:

Definition 10. *The "component spaces" $H^{2,k}(\mathbb{D}) \subset H^2(\mathbb{D})$ consist of functions $f \in H^2(\mathbb{D})$ having the representation $f(z) = f_1\left(z^2\right) z^k$. We put*

$$H^{2,k}(\mathbb{D}) := \left\{ f(z) : f(z) = f_1\left(z^2\right) z^k,\ f_1 \in H^2(\mathbb{D}) \right\}. \tag{25}$$

Thus the space $H^{2,k}(\mathbb{D})$ consists of the analytic functions in $H^2(\mathbb{D})$ having Taylor series

$$f(z) = \sum_{j=0}^{\infty} a_j z^{k+2j},$$

respectively the norm on $H^{2,k}(\mathbb{D})$ is the inherited from $H^2(\mathbb{D})$.

Definition 11. *We define the polyharmonic Hardy space $H^2(\mathcal{B})$ on the unit ball $\mathcal{B} = \mathcal{B}_1$ of the Klein-Dirac quadric defined in* (14), *as the space of functions f given by the Laplace-Fourier series* (24) *with coefficients $f_{k,\ell} \in H^{2,k}(\mathbb{D})$ satisfying*

$$\|f\| := \sqrt{\sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k} \|f_{k,\ell}\|^2_{H^2(\mathbb{D})}} < \infty.$$

Remark 12. The reader may note that Definition 11 mimics the definition of the classical Hardy spaces which are obtained as the closure of the polynomials.

Now let $f \in H^2(\mathcal{B})$. By Definition 11, since all $f_{k,\ell} \in H^2(\mathbb{D})$ it follows that for $r \to 1^-$ and $z = re^{i\varphi}$ all $f_{k,\ell}\left(re^{i\varphi}\right)$ have limiting values $f^*_{k,\ell}\left(e^{i\varphi}\right)$ in $L_2\left(\mathbb{S}^1\right)$, hence

$$\begin{aligned}(26)\qquad \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k} \|f_{k,\ell}\|^2_{H^2(\mathbb{D})} &= \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k} \int_0^{2\pi} \left|f^*_{k,\ell}\left(e^{i\varphi}\right)\right|^2 d\varphi \\ &= \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k} \int_0^{2\pi} \left|f^*_{k,\ell}\left(z^2\right) z^k\right|^2 d\varphi < \infty.\end{aligned}$$

This implies

$$\lim_{r\to 1^-} \int_0^{2\pi}\int_{\mathbb{S}^{d-1}} \left|f\left(re^{i\varphi},\theta\right)\right|^2 d\varphi d\theta = \sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k} \int_0^{2\pi} \left|f^*_{k,\ell}\left(e^{i\varphi}\right)\right|^2 d\varphi < \infty,$$

hence, it follows that $f \in L_2\left(\mathbb{D} \times \mathbb{S}^{d-1}\right)$, and due to (23), also $f \in L_2(\mathcal{B})$.

Note that as in the classical Hardy spaces the inner product (21) is well-defined for the polynomials but might not be well defined for arbitrary functions having bad boundary behavior. For that reason we have to change the definition of this inner product.

Definition 13. *We put*

$$(27)\qquad \langle f, g\rangle_{H^2(\mathcal{B})} := \frac{1}{2\pi} \lim_{\substack{r\to 1\\ r<1}} \int_{\mathbb{S}^{d-1}} \int_0^{2\pi} f\left(re^{i\varphi},\theta\right) \overline{g\left(re^{i\varphi},\theta\right)} d\varphi d\theta.$$

The following theorem justifies our arguments above and is an analog to results for the classical Hardy spaces, cf. Theorem 8 (or Theorem 17.10 in [44]).

Theorem 14. *1. The space $H^2(\mathcal{B})$ is complete.*

2. It coincides with the space of functions f having representation (23)

$$f(z,\theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^{\infty} f_{k,\ell;j} z^{k+2j} Y_{k,\ell}(\theta) \tag{28}$$

with coefficients satisfying

$$\sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^{\infty} |f_{k,\ell;j}|^2 < \infty. \tag{29}$$

3. The norm of f is given by

$$\|f\|_{H^2(\mathcal{B})} = \left\{ \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^{\infty} |f_{k,\ell;j}|^2 \right\}^{1/2}.$$

The following equality holds, for $z = re^{i\varphi}$,

$$\|f\|^2_{H^2(\mathcal{B})} = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \left| z^k \sum_{j=0}^{\infty} f_{k,\ell,j} z^{2j} \right|^2 d\varphi = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^{\infty} |f_{k,\ell,j}|^2, \tag{30}$$

4. Every element $f \in H^2(\mathcal{B})$ is the limit of a sequence of polynomials $P_N \in \mathcal{P}$ which satisfy

$$\Delta^N P_N(x) = 0 \qquad \text{for } x \in B \subset \mathbb{R}^d.$$

We skip the proof.

Remark 15. The essence of Theorem 14 is that, as in the classical Hardy spaces, only the information about f in the real domain, provided by the Laplace-Fourier coefficients

$$f(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} f_{k,\ell}(r^2) r^k Y_{k,\ell}(\theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^{\infty} f_{k,\ell;j} r^{k+2j} Y_{k,\ell}(\theta),$$

determine when does f belong to $H^2(\mathcal{B})$. All we need to know is that they satisfy the convergence condition (29), $\sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^{\infty} |f_{k,\ell,j}|^2 < \infty$.

The following result follows from the representation in (25).

Proposition 16. *For every function* $f \in H^{2,k}(\mathbb{D})$ *the following formula of Cauchy type holds:*

$$f(\zeta) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{z}{z^2 - \zeta^2} \frac{\zeta^k}{z^k} f(z)\, dz;$$

here $\zeta \in \mathbb{D}$.

3.1. Cauchy type kernel for $H^2(\mathcal{B})$ and Hua-Aronszajn type formula. The orthogonality of the basis $\{b_{k,\ell,j}\}$ allows us to construct a **Cauchy type kernel** and a corresponding **formula** which reproduces the multivariate polynomials by using their values on the set $\mathbb{S}^1 \times \mathbb{S}^{d-1}/\mathbb{Z}_2$. By exploiting the above orthogonality, we obtain such formula easily, following the general principles of constructing kernels, by putting:

$$\begin{aligned} K(z', \theta'; z, \theta) &:= \sum_{j\geq 0} \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} b_{k,\ell,j}(\zeta;\theta')\, b_{k,\ell,j}(z;\theta) \\ &= \sum_{j\geq 0} \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \zeta^{2j+k} Y_{k,\ell}(\theta')\, z^{2j+k} Y_{k,\ell}(\theta). \end{aligned} \tag{31}$$

Note that this kernel is **absolutely convergent** for every ζ and θ with $|\zeta| < 1$, $\theta' \in \mathbb{S}^{d-1}$, and $z = e^{i\varphi}$, due to the estimates for the spherical harmonics (10). By the orthogonality property (22), for every polynomial P we obtain the Cauchy type formula formula

$$\begin{aligned} P(\zeta\theta') &= \langle K(\zeta, \theta'; \cdot), P(\cdot) \rangle_* \\ &= \frac{1}{2\pi} \int_{\mathbb{S}^{d-1}} \int_0^{2\pi} K(\zeta, \theta'; z, \theta)\, \overline{P(z,\theta)} d\varphi d\theta. \end{aligned} \tag{32}$$

Let us remark that no such formula is available in the real domain. Formula (32) is a strong motivation to consider further the consequences of the inner product (21).

Obviously,

$$\begin{aligned} K(\zeta, \theta'; z, \theta) &= \sum_{j\geq 0} (\zeta z)^{2j} \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} (\zeta z)^k Y_{k,\ell}(\theta')\, Y_{k,\ell}(\theta) \\ &= \frac{1}{1 - \zeta^2 z^2} \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} (\zeta z)^k Y_{k,\ell}(\theta')\, Y_{k,\ell}(\theta) \end{aligned}$$

which shows the absolute convergence for $|\zeta z| < 1$. Let us recall that the usual **Poisson kernel** (see [54], chapter 2, Theorem 1.9) is given by

$$K_P\left(r,\theta,\theta'\right) := \frac{1-r^2}{|\theta - r\theta'|^d} = \sum_{k=0}^{\infty}\sum_{\ell=1}^{d_k} r^k Y_{k,\ell}\left(\theta\right) Y_{k,\ell}\left(\theta'\right) \qquad \text{for } r<1$$

in every dimension $d \geq 2$. This expression is obviously close to the above expression for $K\left(\zeta,\theta';z,\theta\right)$. We will apply the idea for the $r-$complexification to the kernel $K_P\left(r,\theta,\theta'\right)$ and we will relate it to the Cauchy type kernel $K\left(\zeta,\theta';z,\theta\right)$.

Proposition 17. *For every complex number w with $|w| < 1$ the following equality holds:*

$$\sum_{k=0}^{\infty}\sum_{\ell=1}^{d_k} w^k Y_{k,\ell}\left(\theta\right) Y_{k,\ell}\left(\theta'\right) = \sum_{k=0}^{\infty} w^k Z_\theta^{(k)}\left(\theta'\right) = \frac{1-w^2}{\left(1-2w\left\langle\theta,\theta'\right\rangle + w^2\right)^{\frac{d}{2}}}, \tag{33}$$

where $Z_\theta^{(k)}\left(\theta'\right)$ are the zonal harmonics, see [54]. Let us put $\cos\phi = \left\langle\theta,\theta'\right\rangle$. For $d=2$ we have

$$K_P\left(w,\theta,\theta'\right) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} w^k \frac{\cos k\phi}{\pi} = \frac{1}{2\pi}\frac{1-w^2}{1-2w\cos\phi + w^2} \tag{34}$$

and for $d > 2$,

$$K_P\left(w,\theta,\theta'\right) = \sum_{k=0}^{\infty} w^k c_{k,d} P_k^\lambda\left(\cos\phi\right) \tag{35}$$

where

$$c_{k,d} = \frac{1}{\omega_{d-1}}\frac{2k+d-2}{d-2}, \qquad \lambda = \frac{d-2}{2}$$

and P_k^λ are the Legendre polynomials.

*Hence, the $r-$complexification of the **Poisson kernel** $K_P\left(r,\theta,\theta'\right)$ is given by*

$$K_P\left(w,\theta,\theta'\right) = \frac{1-w^2}{\left(1-2w\left\langle\theta,\theta'\right\rangle + w^2\right)^{\frac{d}{2}}} = \frac{1-w^2}{\left(\left(1-e^{i\phi}w\right)\left(1-e^{-i\phi}w\right)\right)^{d/2}}. \tag{36}$$

*The **Cauchy type kernel** $K\left(\zeta,\theta';z,\theta\right)$ defined in (31) is related to the r-complexification of the Poisson kernel by the equality*

$$K\left(\zeta,\theta';z,\theta\right) = \frac{1}{1-\zeta^2 z^2} K_P\left(\zeta z,\theta,\theta'\right) = \frac{1}{\left(1-2\zeta z\left\langle\theta,\theta'\right\rangle + \zeta^2 z^2\right)^{\frac{d}{2}}}. \tag{37}$$

We skip the proof.

The definition of the kernel $K(\zeta,\theta';z,\theta)$ shows that for all polynomials P with real coefficients holds

$$P(\zeta\theta') = \frac{1}{2\pi\omega_d}\int_{\mathbb{S}^{d-1}}\int_0^{2\pi} K(\zeta,\theta';z,\theta)\,\overline{P(z,\theta)}d\varphi d\theta \qquad \text{for } z=e^{i\varphi}.$$

Since $\overline{z}=z^{-1}$, a change in the integration $\varphi\to-\varphi$ shows that

$$\begin{aligned} P(\zeta\theta') &= \frac{1}{2\pi\omega_d}\int_{\mathbb{S}^{d-1}}\int_0^{2\pi} K(\zeta,\theta';z,\theta)\,\overline{P(z,\theta)}d\varphi d\theta \\ &= \frac{1}{2\pi i\omega_d}\int_{\mathbb{S}^{d-1}}\int_{\Gamma_1}\frac{1}{z}K\left(\zeta,\theta';\frac{1}{z},\theta\right)P(z,\theta)\,dzd\theta \end{aligned}$$

where Γ_1 is the positively oriented circle $\mathbb{S}^1$. On the other hand we see that

$$\frac{1}{z}K\left(\zeta,\theta';\frac{1}{z},\theta\right) = \frac{z^{d-1}}{\left(z^2-2\zeta z\langle\theta,\theta'\rangle+\zeta^2\right)^{\frac{d}{2}}}.$$

The last by definition is up to a factor the Hua-Aronszajn kernel, see [2], p. 126.

The above identities motivate the definition of the well-known **Hua-Aronszajn kernel** $H(\zeta,\theta';z,\theta)$ given by

$$H(\zeta,\theta';z,\theta) = \frac{1}{\omega_d}\frac{z^{d-1}}{\left(\zeta^2-2\zeta z\langle\theta,\theta'\rangle+z^2\right)^{\frac{d}{2}}}, \tag{38}$$

where $\omega_d=\pi^{d/2}/\Gamma(d/2)$ is the surface of the sphere (see [2], p. 122 Theorem 1.1, and p. 126, Remark 1.4, where up to a factor it is called *Cauchy kernel* for the *Cartan classical* domain $\mathcal{R}_{IV}$).

From above we see that the following equality holds:

$$\begin{aligned} H(\zeta,\theta';z,\theta) &= \frac{1}{\omega_d}\frac{1}{z}K\left(\zeta,\theta';\frac{1}{z},\theta\right) \\ &= \frac{1}{z}\frac{1}{1-\zeta^2/z^2}\sum_{k=0}^{\infty}\sum_{\ell=1}^{a_k}(\zeta/z)^k\,Y_{k,\ell}(\theta')\,Y_{k,\ell}(\theta) \\ &= \frac{1}{\omega_d}\frac{1}{z}\frac{1}{\left(1-2\zeta/z\langle\theta,\theta'\rangle+\zeta^2/z^2\right)^{\frac{d}{2}}}. \end{aligned} \tag{39}$$

4. Main properties of the Hardy spaces $\boldsymbol{H^2(\mathcal{B})}$. In the next theorem we provide a generalization of the classical boundary value properties of the Hardy spaces H^2, see e.g. Theorem 8 (or Theorem 17.10, 17.12 and 17.13 in [44]).

Theorem 18. *Let $f \in H^2(\mathcal{B})$.*

1. ***Fatou type theorem:*** *For $r \to 1^-$, and for almost all $\varphi \in [0, 2\pi]$ and $\theta \in \mathbb{S}^{d-1}$, the function $f\left(re^{i\varphi}\theta\right)$ has a radial limit which we denote by $f^*\left(e^{i\varphi}, \theta\right)$, and which satisfies $f^*\left(e^{i\varphi}, \theta\right) \in L^2\left(\mathbb{S} \times \mathbb{S}^{d-1}\right)$. If the expansion of the function f is given by (28),*

$$f(z, \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^{\infty} f_{k,\ell;j} z^{k+2j} Y_{k,\ell}(\theta) \qquad \text{for all } |z| < 1, \theta \in \mathbb{S}^{d-1},$$

then $f^*\left(e^{i\varphi}, \theta\right)$ is given by the Laplace-Fourier series

$$f^*\left(e^{i\varphi}, \theta\right) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^{\infty} f_{k,\ell;j} z^{k+2j} Y_{k,\ell}(\theta) \qquad \text{for } z = e^{i\varphi}. \tag{40}$$

*2. The following **limiting relation** holds,*

$$\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{S}^{d-1}} \left| f\left(re^{i\varphi}\theta\right) - f^*\left(e^{i\varphi}\theta\right)\right|^2 d\varphi d\theta = 0.$$

3. Let the Laplace-Fourier series of the function $f^*\left(e^{i\varphi}, \theta\right) \in L^2\left(\mathbb{S} \times \mathbb{S}^{d-1}\right)$ be given by

$$f^*\left(e^{i\varphi}, \theta\right) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} f^*_{k,\ell}\left(e^{i\varphi}\right) Y_{k,\ell}(\theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^{\infty} f^*_{k,\ell;j} z^{k+2j} Y_{k,\ell}(\theta) \qquad \text{for } z = e^{i\varphi}.$$

*Then the Fourier coefficients of the functions $f^*_{k,\ell}\left(e^{i\varphi}\right)$ satisfy the following zero conditions, which we call **Riesz type conditions**:*

$$f^*_{k,\ell;j} = 0 \qquad \text{for all } j \neq k, k+2, k+4, \ldots\ . \tag{41}$$

These conditions are equivalent to (the usual form of Riesz conditions)

$$\int_0^{2\pi} \int_{\mathbb{S}^{d-1}} f^*\left(e^{i\varphi}, \theta\right) e^{-ij\varphi} Y_{k,\ell}(\theta)\, d\varphi d\theta = 0 \quad \text{for all } \ j \neq k, k+2, k+4, \ldots. \tag{42}$$

*4. The following **Cauchy-Hua-Aronszajn** formula holds*

$$f\left(\zeta,\theta'\right)=\frac{1}{2\pi i}\int_{\Gamma_1}\int_{\mathbb{S}^{d-1}}H\left(\zeta,\theta';z,\theta\right)f^{*}\left(z,\theta\right)dzd\theta \tag{43}$$
$$=\frac{1}{2\pi\omega_d}\int_0^{2\pi}P_{r^2}\left(2\varphi-2\varphi'\right)\int_{\mathbb{S}^{d-1}}K_P\left(\frac{\zeta}{z},\theta,\theta'\right)f^{*}\left(z,\theta\right)d\theta d\varphi,$$

*where we use the notations $z=e^{i\varphi}$ and $\zeta=re^{i\varphi'}$, the kernel K is the **Cauchy type kernel** (31), and $P_r(\varphi)$ is the usual two-dimensional Poisson kernel. We call the kernel*

$$P_{r^2}\left(2\varphi-2\varphi'\right)K_P\left(\frac{\zeta}{z},\theta,\theta'\right) \tag{44}$$

*the **modified Poisson type kernel**. Here the contour Γ_1 is the positively oriented circle $\mathbb{S}^1$, or a bigger contour which encircles it.*

*5. **Dirichlet problem** with L^2 data: If a function $f^{*}\left(e^{i\varphi},\theta\right)\in L^2\left(\mathbb{S}\times\mathbb{S}^{d-1}\right)$ satisfies the Riesz type conditions (41) then there exists an unique function $f\in H^2(\mathcal{B})$ which has as a "non-tangential limit" the function $f^{*}\left(e^{i\varphi},\theta\right)$ in the sense*

$$\lim_{r\longrightarrow 1}\int_{\mathbb{S}^{d-1}}\int_0^{2\pi}\left|f\left(re^{i\varphi},\theta\right)-f^{*}\left(e^{i\varphi},\theta\right)\right|^2d\varphi d\theta=0.$$

*6. Every function $f^{*}\in L^2\left(\mathbb{S}^1\times\mathbb{S}^{d-1}\right)$ which satisfies the Riesz tyep conditions (41) belongs to the space $L^2\left(\mathbb{S}^1\times\mathbb{S}^{d-1}/\mathbb{Z}_2\right)$. The map $f\longrightarrow f^{*}$ is an **isometry** between $H^2(\mathcal{B})$ and the subspace of $L^2\left(\mathbb{S}^1\times\mathbb{S}^{d-1}/\mathbb{Z}_2\right)$.*

7. The norm is given by

$$\|f\|^2_{H^2(\mathcal{B})}=\frac{1}{2\pi}\int_{\mathbb{S}^{d-1}}\int_0^{2\pi}\left|f^{*}\left(e^{i\varphi}\theta'\right)\right|^2d\varphi d\theta'.$$

We skip the proof.

Theorem 18 shows that we may solve Boundary Value Problems in the space $H^2(\mathcal{B})$ which is an essential advantage over the situation with the holomorphic functions in $\mathbb{C}^d$ and alternative definitions of Hardy space in several dimensions, see [53], [45], [54], [46].

By Theorem 18 the following space:

$$\left\{f^{*}\in L^2\left(\mathbb{S}\times\mathbb{S}^{d-1}\right):\ f^{*}\text{ satisfies the Riesz type condition (41)}\right\}$$

is isomorphic to the space $H^2(\mathcal{B})$. The Cauchy-Hua-Aronszajn formula (43) generalizes the Cauchy formula in $\mathbb{C}$ and the Poisson formula in $\mathbb{R}^d$ at the same time.

Remark 19. Let us formulate a **conjecture** about an analog to brothers' Riesz theorem: Let the complex valued Borel measure $\mu(\varphi,\theta)$ be given on $\mathbb{S}^1 \times \mathbb{S}^{d-1}$ with $\varphi \in [0,2\pi]$ and $\theta \in \mathbb{S}^{d-1}$. Assume that for all indices (k,ℓ) holds

$$\int_0^{2\pi}\int_{\mathbb{S}^{d-1}} \overline{z}^j Y_{k,\ell}(\theta)\, d\mu(\varphi,\theta) = 0 \qquad \text{for } j \neq k, k+2, k+4, \ldots \tag{45}$$

Is the measure μ "absolutely continuous", i.e. does there exist a function f^* which is in $L_1\left(\mathbb{S}^1 \times \mathbb{S}^{d-1}\right)$ such that $d\mu(\varphi,\theta) = f^*(\varphi,\theta)\, d\varphi d\theta$? It seems that the answer in this form is negative, but a positive answer needs some additional properties of the measure μ. A thorough discussion to this question will be considered in [28]. Let us remark that a genuine analog of the brothers Riesz theorem is difficult to achieve for all approaches to Hardy spaces, cf. [7], [19], [45], [53], [54], [46].

Remark 20. Another **conjecture** about the boundary properties of the remarkable Poisson type kernel in (43) is the following: Assume that the function $g\left(e^{i\varphi},\theta\right)$ belongs to $C\left(\mathbb{S}^1 \times \mathbb{S}^{d-1}/\mathbb{Z}_2\right)$ and satisfies the brothers Riesz conditions (42). Then by means of formula (40) (or equivalently, by (43)) we may define a function $F_g(z,\theta)$ on the interior (for $|z| = r < 1$) of the ball $\mathcal{B}$ of the Klein-Dirac quadric. Let us put $F_g\left(e^{i\varphi},\theta\right) = g\left(e^{i\varphi},\theta\right)$ for $r = 1$. We conjecture that the function F_g is continuos on the closure $\overline{\mathcal{B}}$. Let us note that in the classical case the Poisson kernel is used to prove similar statement, see chapter 2, Theorem 1.9 in [54]. Here we expect that the modified Poisson kernel (44) will of central importance for the solution.

4.1. Maximum principle. In the classical Hardy space, the maximum principle is intimately related to the Cauchy formula in $\mathbb{C}$ or to the Poisson formula in $\mathbb{R}^d$ (see the proof of the completeness of H^p in Remark 17.8 in [44]). A weak form of maximum principle alows to prove that the elements of H^2 are uniform limits of polynomials on compact subsets of $\mathbb{D}$. Here we prove analog to this for the polyharmonic Hardy space $H^2(\mathcal{B})$. In the next theorem we see that the explicit form for the Cauchy-Hua-Aronszajn kernel is essential for proving a maximum principle.

Theorem 21. *Let $f \in H^2(\mathcal{B})$. For every q with $0 < q < 1$ we have the following (weak) maximum principle holds:*

$$|f(\zeta\theta)| \leq (1-q)^{-d}\, \|f\|_{H^2(\mathcal{B})} \qquad \textit{for all } |\zeta| \leq q,\ \theta \in \mathbb{S}^{d-1}.$$

More generally, for every mixed derivative $D^{\alpha}_{\zeta,\theta}$ with respect to the variables ζ and θ, we have the maximum principle

$$\left|D^{\alpha} f(\zeta\theta)\right| \leq C_1 \times (1-q)^{-d-|\alpha|} \left[\left(\frac{d}{2}\right)\left(\frac{d}{2}+1\right)\cdots\left(\frac{d}{2}+|\alpha|\right)\right] |\alpha| \, \|f\|_{H^2(\mathcal{B})}$$

for all $|\zeta| \leq q$, and $\theta \in \mathbb{S}^{d-1}$;

here the constant $C_1 > 0$ is independent of α. Respectively, for real $\zeta = r$ with $x = r\theta$ this gives an estimate for $D^{\alpha}_{x} f(x)$.

We skip the proof.

We have the following immediate corollary about the regularity of the functions in the space $H^2(\mathcal{B})$.

Corollary 22. *The functions in $H^2(\mathcal{B})$ belong to $C^{\infty}(\mathcal{B})$.*

The proof follows from the maximum principle in Theorem 21 since every $f \in H^2(\mathcal{B})$ and the derivatives of f are uniform limits of a sequence of polynomials $P_N(z\theta)$ and the respective derivatives of $P_N(z\theta)$ on every compact sets $K \times \mathbb{S}^{d-1}$ where the compact $K \subset \mathbb{D}$.

4.2. Real analytic functions and the space $H^2(\mathcal{B})$. As is the case with the classical analytic functions the elements of $H^2(\mathcal{B})$ are real analytic which we prove in the next theorem.

Theorem 23. *Let $F \in H^2(\mathcal{B})$. Then the function $f(x) = F(r\theta)$ is real-analytic in the ball $B \subset \mathbb{R}^d$.*

5. BVPs for the polyharmonic operator Δ^N and the spaces $H^2(\mathcal{B})$. The polyharmonic function in the one-dimension case satisfy $d^N P_{N-1}(t)/dt^N = 0$ and hence they are polynomials. Their complexifications $P_N(z)$ belong to the Hardy spaces in the disc. On the other hand, not all polyharmonic function (of fixed finite order N) belong to the polyharmonic Hardy spaces $H^2(\mathcal{B})$. In the present section we characterize those polyharmonic functions which belong to $H^2(\mathcal{B})$.

We will characterize the polyharmonic functions by means of their boundary properties. The main point is that the polyharmonic functions in a domain

$D \subset \mathbb{R}^d$ (and more general domains) may be "parametrized" by considering the Dirichlet problem

$$\Delta^N u = 0 \qquad \text{in } D$$
$$\Delta^j u = g_j \qquad \text{on } \partial D \quad \text{for } j = 0, 1, 2, \ldots, N-1.$$

By means of the classical Green formulas we may find $u(x)$ for $x \in D$. On the other hand, if the function u has a r-complexification to ball $\mathcal{B}$ of the Klein-Dirac quadric, then by means of Cauchy type formula (43) we may recover $u(z, \theta)$ using its values $u\left(e^{i\varphi}, \theta\right)$ on the boundary $\partial\mathcal{B} = \mathbb{S} \times \mathbb{S}^{d-1}/\mathbb{Z}_2$. Thus we see the boundary values $u\left(e^{i\varphi}, \theta\right)$ provide an alternative parametrization for the polyharmonic functions. It is essential, as in the one-dimensional case, to provide a relation between the boundary data $\{g_j\}_{j=0}^{N-1}$ and $u\left(e^{i\varphi}, \theta\right)$.

The following theorem establishes the link between the polyharmonic BVPs and the boundary values of the elements in $H^2(\mathcal{B})$.

Theorem 24. *Let u be a polyharmonic function of order $N \geq 1$ in the ball $B \subset \mathbb{R}^d$, i.e. $\Delta^N u(x) = 0$ in B. Then the $r-$complexification of u satisfies $u \in H^2(\mathcal{B})$ if and only if*

$$\Delta^j u_{|\partial B} \in H^{-j}\left(\mathbb{S}^{d-1}\right), \qquad \text{for } j = 0, 1, \ldots, N-1,$$

where $H^s\left(\mathbb{S}^{d-1}\right)$ denotes the Sobolev space of exponent s.

We skip the proof.

We see that Theorem 24 provides us with a large class of functions defined on the ball $B \subset \mathbb{R}^d$ which are extendable to the ball $\mathcal{B}$ of the Klein-Dirac quadric. It is also possible to consider functions which are in a certain sense polyharmonic of infinite order, as those studied by Aronszajn, Lelong, Avanissian, and others, cf. the references in [2], [3], [29].

6. Error estimate of Polyharmonic Interpolation and Cubature formulas. The topic of estimation of quadrature formulas for analytic functions is a widely studied one. Beyond the classical monographs [35], [12], we provide further and more recent publications, as [5], [16], [17], [23], [31], [37]. No references may be found though for the multivariate case, for cubature formulas, even in the fundamental monographs as [51], [55], [52]; see also the recent survey [10].

Our main framework of Interpolation and Cubature was defined in [27], [18], [25]. It has further brought to life the multivariate complexification and the polyharmonic Hardy spaces.

We will consider first *polyharmonic interpolation,* and as second, *polyharmonic Cubature formulas* for approximating integrals of the type

$$\int_B f(x)\, d\mu(x)$$

over the unit ball $B \subset \mathbb{R}^d$, where $\mu(x)$ is a signed measure of special type. The main purpose of the present section is to find error estimates for the Interpolation and Cubature for functions $f \in H^2(\mathcal{B})$. Interpolation and Quadrature are intimately related in the one-dimensional case, and we will demonstrate similar relation in our multivariate setting.

6.1. One-dimensional case. First, following the classical scheme outlined in [12], p. 303 – 306, (see also chapter 12 in [35]), we will remind how one finds the error for the classical Quadrature formulas.

Let the points $t_0, t_1, \ldots, t_N$ belong to the interval $[a, b]$. We define

$$\omega_N(z) = (z - t_0)(z - t_1) \cdots (z - t_N).$$

Let the function f be analytic in a simply connected (open) domain $D \subset \mathbb{C}$ containing the interval $[a, b]$ with boundary $\partial D = \Gamma$. Then the interpolation polynomial $P_N(t)$ satisfying $P_N(t_j) = f(t_j)$ for $j = 0, 1, \ldots, N$ is given by

$$P_N(z) = \sum_{j=0}^{N} f(t_j) \frac{\omega(z)}{\omega'(t_j)(z - t_j)} = f(z) - \frac{1}{2\pi i} \int_\Gamma \frac{\omega(z) f(t)}{\omega(t)(t - z)} dt,$$

where Γ is considered as a contour oriented counterclockwise. Hence, the remainder is

$$f(z) - P_N(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\omega(z) f(t)}{\omega(t)(t - z)} dt.$$

Now, if μ is a non-negative Stieltjes measure, say $d\mu(t) = w(t)\, dt$, the quadrature formula

$$\int_a^b f(t)\, d\mu(t) \approx \sum_{j=0}^{N} \lambda_j f(t_j)$$

is called **interpolatory quadrature** formulas of degree N if it satisfies the following equality

$$\sum_{j=0}^{N} \lambda_j Q(t_j) = \int_a^b P_N(t)\, d\mu(t)$$

for every polynomial Q_N of degree $\leq N$. This implies that

$$\lambda_j = \int_a^b \frac{\omega(t)}{(t - t_j)\, \omega'(t_j)} d\mu(t);$$

cf. [12], p. 303, or Krylov, [35], chapter 12. Hence, for the error of such formula we obtain

$$E(f) := \int_a^b (f(z) - P_N(z))\, d\mu(z) = \frac{1}{2\pi i} \int_a^b \left(\int_\Gamma \frac{\omega(z) f(t)}{\omega(t)(t - z)} dt \right) d\mu(z).$$

This may be directly estimated by

$$|E(f)| \leq \frac{L_\Gamma}{2\pi} \max_{t \in \Gamma} |f(t)| \times \frac{1}{d^{N+1}} \frac{(b-a) D^{N+1}}{\delta_\Gamma} \int_a^b |d\mu(z)|$$

where $d := \min_j (\operatorname{dist}(t_j, \Gamma))$, $D := \max_j (\operatorname{dist}(t_j, \{a, b\}))$, $\delta_\Gamma := \operatorname{dist}([a, b], \Gamma)$, and L_Γ is the length of the contour Γ.

6.2. Multivariate Interpolation. Now we consider the multivariate case. Let us assume that the number b satisfies

$$0 < b < 1.$$

Choose the domain $D = \mathcal{B}$ and a function $f \in H^2(\mathcal{B})$, assuming that f has the expansion

$$f(z, \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} f_{k,\ell}(z^2)\, z^k Y_{k,\ell}(\theta) \qquad \text{for } |z| < 1, \text{ and } \theta \in \mathbb{S}^{d-1}.$$

Let $N \geq 0$ be a fixed integer. We will consider *polyharmonic interpolation* which has been studied in [18]. Let the points $\{r_{k,\ell;j}\}_{j=0}^N$ belong to the interval $[0, b]$. We consider the following series as interpolant to the function f:

$$P_N(z, \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} p_{k,\ell}(z^2)\, z^k Y_{k,\ell}(\theta), \tag{46}$$

where $p_{k,\ell}$ are polynomials of degree $\leq N$ satisfying the interpolation conditions

$$p_{k,\ell}\left(r_{k,\ell;j}^2\right) = f_{k,\ell}\left(r_{k,\ell;j}^2\right) \qquad \text{for } j = 0, 1, \ldots, N.$$

We prove below that the series (46) is convergent.

For the remainder of this interpolation we have

$$f(z, \theta) - P_N(z, \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \left[f_{k,\ell}(z^2) - p_{k,\ell}(z^2) \right] z^k Y_{k,\ell}(\theta).$$

Now define as above the functions

$$\omega_{k,\ell}(z) = \left(z - r_{k,\ell;0}^2\right)\left(z - r_{k,\ell;1}^2\right)\cdots\left(z - r_{k,\ell;N}^2\right),$$

and consider the oriented contour $\Gamma(t) = e^{it}$ for $t \in [0, 2\pi]$.

For all z with $|z| \leq b$ and $\theta \in \mathbb{S}^{d-1}$, and obtain the estimate

$$|f(z,\theta) - P_N(z,\theta)| \leq \left| \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \left[f_{k,\ell}\left(z^2\right) - p_{k,\ell}\left(z^2\right)\right] z^k Y_{k,\ell}(\theta) \right| \leq$$

$$\leq C \frac{2^{N+2}}{(1-b)^{N+2}} \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \left\| f_{k,\ell} \right\|_{H^2(\mathbb{D})} b^k k^{\frac{d-2}{2}}. \tag{47}$$

Since $f \in H^2(\mathcal{B})$ the last inequality shows, after application of Cauchy-Bunyakovski-Schwarz inequality, that the series $\sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \left[f_{k,\ell}\left(z^2\right) - p_{k,\ell}\left(z^2\right)\right] z^k Y_{k,\ell}(\theta)$ is absolutely and uniformly convergent. Hence, the series (46) representing the function $P_N(z,\theta)$ is also such.

Above we have outlined the most important arguments for proving the following:

Theorem 25. *Let $f \in H^2(\mathcal{B})$. Let the points $\{r_{k,\ell;j}\}_{j=0}^N$ belong to the interval $[0,b]$, where $b < 1$. Then the function $P_N(z,\theta)$ defined by the series* (46) *is polyharmonic of order $N+1$ in the ball $B_b \subset \mathbb{R}^d$ and belongs to the polyharmonic Hardy space $H^2(\mathcal{B}_b)$, while the following inequality holds:*

$$\|P_N\|_{H^2(\mathcal{B}_b)} \leq C_{N,b} \|f\|_{H^2(\mathcal{B})}.$$

6.3. Multivariate Polyharmonic Cubature. The class of **pseudo-positive** measures used for our cubature formula $C_N(f)$ is now defined in the following way: a signed measure μ with support in $B_R \subset \mathbb{R}^d$ is *pseudo-positive with respect to the orthonormal basis* $Y_{k,\ell}, \ell = 1, \ldots, a_k$, $k \in \mathbb{N}_0$ if the inequality

$$\int_{\mathbb{R}^d} h(|x|) Y_{k,\ell}(x) \, d\mu(x) \geq 0 \tag{48}$$

holds for every non-negative continuous function $h : [a,b] \to [0,\infty)$ and for all $k \in \mathbb{N}_0$, $\ell = 1, 2, \ldots, a_k$. Let us note that every signed measure $d\mu$ with bounded variation may be represented (non-uniquely) as a difference of two pseudo-positive measures. We refer to [27] for instructive examples of pseudo-positive measures.

Let the *pseudo-positive* (signed) measure $d\mu$ be given in the ball $B_b \subset \mathbb{R}^d$. For all indices (k,ℓ) the component measures are defined by

$$d\mu_{k,\ell}(r) := \int_{\mathbb{S}^{d-1}} Y_{k,\ell}(\theta)\, d\mu(r\theta) \geq 0 \qquad \text{for all } r \in [0,b]; \tag{49}$$

here the integral is symbolical with respect to the variables θ. Rigorously, the component measure $d\mu_{k,\ell}(r)$ is defined for the functions $g(r)$ on the interval $[0,b]$ by means of the equality $\int_0^b g(r)\, d\mu_{k,\ell}(r) := \int_{B_b} g(r)\, Y_{k,\ell}(\theta)\, d\mu(x)$; cf. [25], [27].

In [25], [27], we have considered a special type of Cubature formula, the so-called *polyharmonic Gauss-Jacobi Cubature formula.* Here however we will consider more generally, *interpolatory polyharmonic Cubature formulas* and will prove their convergence and error estimate for them. The case of the annulus has been considered by us in [30].

Let us fix (k,ℓ). We assume that there exist points $t_{k,\ell;j}$, $j = 0,1,\ldots,N$, belonging to the interval $[0,b]$, and numbers $\{\lambda_{k,\ell;j}\}_{j=1}^N$, such that the following **interpolatory quadrature** formula holds:

$$\int_0^b Q(r)\, d\mu_{k,\ell}(r) = \sum_{j=0}^{N} \lambda_{k,\ell;j} Q(t_{k,\ell;j}) \qquad \text{for every } Q \in V_{k,N}; \tag{50}$$

here the set $V_{k,N}$ is given by

$$V_{k,N} = \left\{ r^{k+2j} \right\}_{j=0}^N .$$

We define the **polyharmonic interpolatory Cubature formula** by

$$C_N(f) := \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^{N} \lambda_{k,\ell;j} f_{k,\ell}(t_{k,\ell;j}). \tag{51}$$

For the interpolation polyharmonic function P_N defined in (46) we obtain equality

$$\int_B P_N(x)\, d\mu(x) = C_N(P_N).$$

Hence, the remainder of the polyharmonic Cubature formula is given by

$$\begin{aligned} E(f) &= \int_B (f(r,\theta) - P_N(r,\theta))\, d\mu(z,\theta) \\ &= \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \int_0^1 \left[f_{k,\ell}(r^2) - p_{k,\ell}(r^2) \right] r^k d\mu_{k,\ell}(r), \end{aligned}$$

which implies the estimate

$$|E(f)| \leq \frac{D_N}{(1-b)^{N+2}} \frac{L_\Gamma}{2\pi} \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \|f_{k,\ell}\|_{H^2(\mathcal{B})} \times \int_0^1 r^k d\mu_{k,\ell}(r).$$

This proves the following result:

Theorem 26. *Let $f \in H^2(\mathcal{B})$. Let the points $\{r_{k,\ell;j}\}_{j=0}^N$ belong to the interval $[0,b]$ with $b < 1$. Then the polyharmonic cubature formula defined by* (51) *with remainder $E(f) = \int_B f(x)\, d\mu(x) - C_N(f)$ satisfies the following estimate*

$$|E(f)| \leq \frac{D_N}{(1-b)^{N+2}} \frac{L_\Gamma}{2\pi} \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \|f_{k,\ell}\|_{H^2(\mathcal{B})} \times \int_0^1 r^k d\mu_{k,\ell}(r).$$

7. Conclusions.

1. Our research may be considered as a contribution to the topic of analytic continuation of solutions to elliptic equations (in particular, harmonic functions), see the discussion and references to the works of V. Avanissian, P. Lelong, C. Kiselman, J. Siciak, M. Jarnicki, T. du Cros, on p. 54–55 in [2], [3], [38], [15], [26], [29]. Our construction of r-analytic continuation is applicable to domains as annuli, strips and other domains with symmetry in $\mathbb{R}^d$. The case of the annulus has been considered in [30], while the case of strip and other domains will be considered in [28].

2. The concept of *polyharmonic Hardy spaces* appears to be a new multivariate concept which differs from the existing approaches in several complex variables, cf. [53], [54], [45], [46], [9], [48], [50].

3. We have seen that the space of $r-$analytic functions on the Klein-Dirac quadric provides an useful setting for estimation of the remainders in Interpolation and Cubature. Although the space of such functions is $1-1$ mapped to a Hardy space of holomorphic functions of several complex variables on the Lie ball, our approach has a non-trivial counterpart on the annulus which is not obtained from $\mathbb{C}^d$ constructions, cf. [30]. Our approach which is based on $r-$analytic continuation of solutions to elliptic equations (in particular, polyharmonic functions) provides non-trivial constructions of Hardy spaces on complexified annulus, strip and other symmetric domains in $\mathbb{R}^d$, which are not obtained by the standard approach to holomorphic functions in $\mathbb{C}^d$.

REFERENCES

[1] G. E. Andrews, R. Askey, R. Roy. Special functions. Encyclopedia of Mathematics and its Applications, vol. **71**. Cambridge, Cambridge University Press, 1999.

[2] N. Aronszajn, T. M. Creese, L. J. Lipkin. Polyharmonic Functions. Oxford, Clarendon Press, 1983.

[3] V. Avanissian. Cellule d'harmonicité et prolongement analytique complexe. Paris, Hermann, 1985.

[4] S. Axler, P. Bourdon, W. Ramey. Harmonic Function Theory, second edition. New York, Springer, 2001.

[5] N. S. Bakhvalov. On the optimal speed of integrating analytic functions. *U.S.S.R. Comput. Math. Math. Phys.* **7** (1967), 63–75.

[6] S. Bergman. The Kernel Function and Conformal Mapping. Providence, RI, Amer. Math. Soc., 1970.

[7] S. Bochner. Boundary values of analytic functions in several variables and of almost periodic functions. *Ann. Math.* **45** (1944), 708—722.

[8] M. M. Chawla, M. K. Jain. Error Estimates for Gauss Quadrature Formulas for Analytic Functions. *Mathematics of Computation* **22** (Jan. 1968), No. 101, 82–90.

[9] R. Coifman, G. Weiss. Extensions of Hardy spaces and their use in analysis. *Bull. Amer. Math. Soc.* **83**, *4* (1977), 569–645.

[10] R. Cools. An Encyclopaedia of Cubature Formulas. *J. Complexity* **19** (2003), 445–453.

[11] P. Davis. Interpolation and Approximation. New York, Dover Publications Inc., 1975.

[12] P. Davis, P. Rabinowitz. Methods of Numerical Integration, second edition. Computer Science and Applied Mathematics. Orlando, FL, Academic Press, Inc.,1984.

[13] P. Dirac. Wave equations in conformal space. *Ann. of Math.* **37**, *2* (1936), 429–442.

[14] P. Ebenfelt et al. (Eds) Quadrature Domains and Their Applications. The Harold S. Shapiro anniversary volume. Expanded version of talks and papers presented at a conference on the occasion of the 75th birthday of Harold S. Shapiro, Santa Barbara, CA, USA, March 2003. Operator Theory: Advances and Applications, vol. **156**. Basel, Birkhäuser, 2005.

[15] K. Fujita, M. Morimoto. On the double expansion of holomorphic functions. *J. Math. Anal. Appl.* **272** (2002), 335–348.

[16] W. Gautschi, R. S. Varga. Error Bounds for Gaussian Quadrature of Analytic Functions. *SIAM J. Numer. Anal.* **20** (1983), 1170–1186.

[17] M. Goetz. Optimal quadrature for analytic functions. *J. Comput. Appl. Math.* **137** (2001), 123–133.

[18] W. Haussmann, O. Kounchev. On polyharmonic interpolation. *J. of Math. Analysis and Applications* **331** (2007), 840–849.

[19] H. Helson, D. Lowdenslager. Prediction theory and Fourier series in several variables. *Acta Mathematica* **99** (1958), 165–202.

[20] K. Hoffman. Banach spaces of analytic functions. Prentice-Hall Series in Modern Analysis. Englewood Cliffs, N.J., Prentice-Hall, Inc., 1962.

[21] Hua Loo-Keng. Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains. Providence, RI, Amer. Math. Soc., 1963.

[22] N. Kerzman, E. M. Stein. The Cauchy Kernel, the Szegö Kernel, and the Riemann Mapping Function. *Math. Ann.* **236** (1978), 85–93.

[23] M. Kzaz. Convergence acceleration of some Gaussian quadrature formulas for analytic functions. *J. Appl. Numer. Math.* **10** (1992), 481–496.

[24] O. Kounchev. Multivariate Polysplines. Applications to Numerical and Wavelet Analysis. San Diego, Academic Press, 2001.

[25] O. Kounchev, H. Render. Reconsideration of the multivariate moment problem and a new method for approximating multivariate integrals. electronic version at `arXiv:math/0509380v1 [math.FA]`

[26] O. Kounchev, H. Render. Holomorphic Continuation via Laplace-Fourier series. *Contemporary Mathematics* **455** (2008), 197–205.

[27] O. Kounchev, H. Render. The moment problem for pseudo-positive definite functionals. *Arkiv fœr Matematik* **48** (2010), 97–120.

[28] O. Kounchev, H. Render. Multivariate Moment Problem, Hardy Spaces, and Orthogonality, in preparation.

[29] O. Kounchev, H. Render. Polyharmonic functions of infinite order on annular regions. *Tôhoku Math. J.* (to appear).

[30] O. Kounchev, H. Render. Polyharmonic Hardy spaces on the complexified annulus and error estimates fo cubature formulas. *Result. Math.* **62** (2012), 377–403.

[31] M. A. Kowalski, A. G. Werschulz, H. Wozniakowski. Is Gauss quadrature optimal for analytic functions? *Numerische Mathematik* **47**, *1* (March, 1985), 89–98.

[32] S. Krantz. Explorations in Harmonic Analysis. Berlin, Springer, 2009.

[33] S. Krantz. Geometric Function Theory. Basel, Birkhäuser, 2005.

[34] M. Krein, A. Nudelman. The Markov moment problem and extremal problems. Providence, R.I., Amer. Math. Soc., 1977.

[35] V. Krylov. Approximate calculation of integrals. Translated by Arthur H. Stroud. New York-London, The Macmillan Co., 1962.

[36] S. McCullough, Li-Chien Shen. On the Szegö kernel of an annulus. *Proc. Amer. Math. Soc.* **121** (1994), 1111–1121.

[37] G. Milovanovic, M. M. Spalevic. Error bounds for Gauss-Turán quadrature formulae of analytic functions. *Math. Comp.* **72** (2003), 1855–1872.

[38] M. Morimoto. Analytic Functionals on the Sphere. Translation of Mathematical Monographs, vol. **178**. Providence, Rhode Island, Amer. Math. Soc., 1998.

[39] N. Nikolov, I. Todorov. Conformal Quantum Field Theory in Two and Four Dimensions. In: Proceedings of the Summer School in Modern Mathematical Physics (Eds B. Dragovich, B. Sazdović), Belgrade 2002, 1–49, online available at arxiv.

[40] N. M. Nikolov, I. T. Todorov. Conformal invariance and rationality in an even dimensional quantum field theory. *Int. J. Mod. Phys.* **A19** (2004) 3605–3636; `math-ph/0405005`.

[41] N. Nikolov, I. Todorov. Lectures on Elliptic Functions and Modular Forms in Conformal Field Theory, `math-ph/0412039`.

[42] R. Penrose. On the Origins of Twistor Theory. In: Gravitation and Geometry, a volume in honour of I. Robinson, Biblipolis, Naples 1987; online in `http://users.ox.ac.uk/~tweb/00001/`

[43] R. M. Range. Holomorphic Functions and Integral Representations in Several Complex Variables. Berlin, Springer, 1986.

[44] W. Rudin. Real and Complex Analysis. New York, McGraw-Hill, 1976.

[45] W. Rudin. Function theory in polydiscs. New York, Benjamin, Inc., 1969.

[46] W. Rudin. Function Theory in the Unit Ball of C^n. New York, Springer-Verlag, 1980.

[47] D. Sarason. The H^p spaces of an annulus. Mem. Amer. Math. Soc., vol. **56**, 1965.

[48] D. Sarason. Holomorphic Spaces: A Brief and Selective Survey. In: Holomorphic Spaces, MSRI Publications, vol. **33**. Cambridge, Cambridge University Press, 1998.

[49] R. Seeley. Spherical harmonics. *Amer. Math. Monthly*, **73** (1966), 115–121.

[50] B. A. Shaimkulov. On holomorphic extendability of functions from part of the Lie sphere to the Lie ball. *Siberian Math. J.* **44** (2003), 1105–1110.

[51] S. L. Sobolev. Cubature formulas and modern analysis. An introduction. Translated from the 1988 Russian edition. Gordon and Breach Science Publishers, Montreux, 1992.

[52] S. Sobolev, V. Vaskevich. The theory of cubature formulas. Berlin, Springer, 1997.

[53] E. M. Stein. Singular Integrals and Differentiability Properties of Functions. Princeton, N. J., Princeton University Press, 1970.

[54] E. M. Stein, G. Weiss. Introduction to Fourier Analysis on Euclidean spaces. Princeton, N. J., Princeton University Press, 1971.

[55] A. H. Stroud. Approximate calculation of multiple integrals. Prentice-Hall Series in Automatic Computation. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1971.

[56] A. H. Stroud. Numerical quadrature and solution of ordinary differential equations. New York-Heidelberg, Springer-Verlag, 1974.

[57] A. N. Tikhonov, A. A. Samarskii. Equations of Mathematical Physics. Reprint of the 1963 translation. New York, Dover Publications Inc., 1990.

[58] I. Vekua. New Methods for Solving Elliptic Equations. New York, Wiley, 1967.

Ognyan Kounchev
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: `kounchev@math.bas.bg`
and
IZKS, University of Bonn
Bonn, Germany

Hermann Render
School of Mathematical Sciences
Belfield
Dublin 4, Ireland
e-mail: `hermann.render@ucd.ie`