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WELL-POSEDNESS, CONDITIONING AND REGULARIZATION OF MINIMIZATION, INCLUSION AND FIXED-POINT PROBLEMS

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Well-posedness, conditioning and regularization of fixed-point problems are studied in connexion with well-posedness, conditioning and Tikhonov regularization of minimization and inclusion problems. Equivalence theorems are proved. Coupling iteration and well-posedness as well as iteration and regularization are also considered.

Keywords: conditioning, inclusion, maximal monotone, minimization, fixed-point, well-posed, regularization.

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1 Introduction

A la Tikhonov well-posedness is introduced for mapping fixed-point problems in connexion with well-posedness of minimization and inclusion problems. This well-posedness leads to strong convergence of the iteration method for nonexpansive mappings.

Conditioning of functions is a useful notion connected with well-posedness in optimization ([28, 7, 5, 14]). An analogue is considered for multivalued operators and mappings in connexion with inclusion and fixed-point well-posedness.

The Tikhonov regularization method for ill-posed problems is well known for minimization and inclusion ([27, 9, 26]). We extend this method to fixed-point. The iteration method suitably combined with regularization allows to select the same solution (fixed-point) than the sole regularization method, akin to recent results ([6, 11, 22], see also [18] for a more general result).

The paper is organized as follows. In section 2 we introduce well-posedness notions for minimization, inclusion and fixed-point problems and we study their connexions. Conditioning for operators and mappings is considered in section 3 in connexion with conditioning of functions. In section 4, known equivalence results between well-posedness and conditioning for minimization are extended to inclusion and fixed-point problems. Section 5 is devoted to the convergence of the iteration method for a firmly nonexpansive mapping on a Banach space under fixed-point well-posedness. Tikhonov regularization is introduced in section 6 for fixed-point problems in connexion with minimization and inclusion; the well known selection property remains true in this general situation. In section 7 we show that exact regularization (the regularized solution is a solution to the original problem provided the perturbation in the regularization process be small enough) holds true under a special conditioning. Finally, in section 8 we present, in the context of fixed-point, a general framework of recent results on coupling iteration and Tikhonov regularization.

2 Well-posedness

Let X be a real normed vector space equipped with the norm $\|.\|$. We note $\langle .,\rangle$ the duality pairing between X and its dual $X^\star$ and $\|.\|_\star$ the dual norm on $X^\star$. X will be often a Hilbert space identified with its dual by the Riesz theorem.

All along this work we consider three classes of problems on X.

Minimization.

Data: $f : X \to \overline{\mathbb{R}}$, solution set: $\operatorname{Argmin} f := \{x \in X; f(x) = \inf f\}$.

Inclusion.

Data: Y real normed vector space with norm $\|.\|_Y$, $T : X \to 2^Y$, solution set: $T^{-1}(0) := \{x \in X; 0 \in T(x)\}$.

Fixed-point.

Data: $P : X \to X$, solution set: $\operatorname{Fix} P := \{x \in X; x = P(x)\}$.

It is worth noting that Fixed-point is reducible to Inclusion taking $Y := X$ and $T := I - P$ where I denotes the identity mapping on X.

A general way to define well-posedness relies on the notion of **asymptotically solving sequence**. Namely, let us consider some class (P) of problems with data set D and, for $d \in D$, solution set S defined by some relation on the cartesian product $X \times D$. Roughly speaking, an asymptotically solving sequence for d is a sequence $\{x_n\}$ in X such that (x_n, d) satisfies the relation asymptotically. We will be more precise for the three classes above. Nevertheless the notion of asymptotically solving sequence being well defined, we say that $d \in D$ is (P) well-posed iff

(i) S is nonempty,

(ii) any asymptotically solving sequence $\{x_n\}$ converges to S in the sense that $d(x_n, S) \to 0$.

If any subsequence of an asymptotically solving sequence is also asymptotically solving (as it is the case in the three situations below) this notion of well-posedness is more general than the notion of well-posedness in the generalized sense introduced in [8, 19] for minimization: S is nonempty and any asymptotically solving sequence has a subsequence converging to some point in S. The two notions are equivalent if S is compact.

For the three classes above we will consider the following notions of asymptotically solving sequence and therefore the corresponding well-posedness notions.

Minimization.

f-minimizing: $f(x_n) \to \inf f$.

Inclusion.

(Y, T)-stationary: $d_Y(0, T(x_n)) \to 0$ or equivalently:

$\forall n \in \mathbb{N}, \exists y_n \in T(x_n), \|y_n\|_Y \to 0$.

Fixed-point.

P-asymptotically regular: $x_n - P(x_n) \to 0$.

In case of minimization the corresponding notion of well-posedness is nothing but the (generalized to nonuniqueness) Tikhonov one, and in case of inclusion we recover the notion of well asymptotical behaviour introduced in [3] for the subdifferential of a convex function and in [2] for a general maximal monotone operator. It is proved in [17] that for variational inequalities (subclass of Inclusion), a sequence is asymptotically solving in the sense of [20] for a given variational inequality iff it is asymptotically solving for the equivalent inclusion problem.

Of course, with $Y := X$, fixed-point well-posedness for P is nothing but inclusion well-posedness for $I - P$.

If, in addition to well-posedness, the problem with data d has a unique solution $\overline{x}$, d will be said (P) Tikhonov well-posed. It is worth noting that this implies: "there exists $\overline{x}$ in X such that any asymptotically solving sequence converges to $\overline{x}$", the converse being true if there exists an asymptotically solving sequence, if the limit of any convergent asymptotically solving sequence is in S, and if any solution defines a (constant) asymptotically solving sequence, which is the case in the three considered classes if, respectively,

f is lower-semi-continuous (minimization),

T has a closed graph and 0 belongs to the closure of the image of T (inclusion),

P is continuous and 0 belongs to the closure of the image of $I - P$.

In the following we give examples of Tikhonov well-posed problems if the solution set S is not empty.

Minimization.

f is α-strongly convex. Indeed this implies $S = \{\overline{x}\}$ and

$$\forall x \in X, \;\; f(x) \geq \min f + \alpha \|x - \overline{x}\|^2,$$

S being nonempty if, in addition, X is a reflexive Banach space and f is closed proper.

Inclusion.

$Y := X^\star$ and T is γ-strongly monotone. Indeed this implies $S = \{\overline{x}\}$ and

$$\forall (x, y) \in T, \ \ \|y\|_\star \geq \gamma \|x - \overline{x}\|,$$

S being nonempty if, in addition, X is a Hilbert space and T is maximal monotone.

Fixed-point.

P is σ-strongly nonexpansive. Indeed this implies $S = \{\overline{x}\}$ and

$$\forall x \in X, \ \ \|x - P(x)\| \geq (1 - \sigma)\|x - \overline{x}\|,$$

S being nonempty if, in addition, X is a Banach space.

It is well known that, if f is α-strongly convex, then its subdifferential ∂f is 2α-strongly monotone and (X being a Hilbert space) if T is maximal monotone and γ-strongly monotone, then its resolvent $J_\lambda^T := (I + \lambda T)^{-1}$ ($\lambda > 0$) is $1/(1 + \lambda\gamma)$-strongly nonexpansive. So (X being a Hilbert space) if f closed proper convex is strongly convex, then f is minimization Tikhonov well-posed, its subdiferential is inclusion Tikhonov well-posed and its proximal mapping $\operatorname{prox}_{\lambda f}$ ($= J_\lambda^{\partial f}$) is fixed-point Tikhonov well-posed. Moreover the three problems: minimization for f, inclusion for ∂f, fixed-point for $\operatorname{prox}_{\lambda f}$ are known to be equivalent in the sense that they have the same solution set ([21, 25]):

$$S = \operatorname{Argmin} f = (\partial f)^{-1}(0) = \operatorname{Fix} \operatorname{prox}_{\lambda f}.$$

More generally, the connexion between the three notions of well-posedness for equivalent problems is given in the two following propositions.

Proposition 2.1 ([14, 4]).

Let X be a real Banach space and f be a closed proper convex function on X. Then f is minimization well-posed iff ∂f is inclusion well-posed. Of course, $S = \operatorname{Argmin} f = (\partial f)^{-1}(0)$.

Proposition 2.2 *Let X be a real Hilbert space and T be a maximal monotone operator on X. Then, for all positive λ, T is inclusion well-posed iff J_λ^T is fixed-point well-posed. Of course, $S = T^{-1}(0) = \operatorname{Fix} J_\lambda^T$.*

Proof. (i) Let $\{x_n\}$ be an asymptotically regular sequence for J_λ^T. Therefore, $e_n := x_n - J_\lambda^T(x_n) \to 0$. But $e_n/\lambda \in T(x_n - e_n)$. Thanks to inclusion well-posedness we get $d(x_n - e_n, S) \to 0$ and therefore $d(x_n, S) \to 0$.

(ii) Let $\{x_n\}$ be a stationary sequence for T. So there exits $\{y_n\} \subset X$ such that $\|y_n\| \to 0$ and $y_n \in T(x_n)$, which is equivalent to $x_n = J_\lambda^T(x_n + \lambda y_n)$. Moreover, $\|x_n + \lambda y_n - J_\lambda^T(x_n + \lambda y_n)\| = \lambda \|y_n\| \to 0$. Thanks to fixed-point well posedness we get $d(x_n + \lambda y_n, S) \to 0$ and therefore $d(x_n, S) \to 0$. □

3 Conditioning

Recall ([14, 5, 28]) that a function $f : X \to \overline{\mathbb{R}}$ with $S := \mathrm{Argmin}\, f \neq \emptyset$ is said ψ-conditioned iff there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ with $\psi(0) = 0$ such that

$$\forall x \in X, \;\; f(x) \geq \min f + \psi(d(x, S)).$$

Let $T : X \to 2^Y$ with $S := T^{-1}(0) \neq \emptyset$. We say that T is ψ-conditioned iff there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ with $\psi(0) = 0$ such that

$$\forall x \in X, \;\; d_Y(0, T(x)) \geq \psi(d(x, S))$$

or, equivalently,

$$\forall (x, y) \in T, \;\; \|y\|_Y \geq \psi(d(x, S)).$$

Let $P : X \to X$ with $S := \mathrm{Fix}\ P \neq \emptyset$. We say that P is ψ-conditioned iff there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ with $\psi(0) = 0$ such that

$$\forall x \in X, \;\; \|x - P(x)\| \geq \psi(d(P(x), S)).$$

The two last definitions are motivated by the following two propositions.

Proposition 3.1 *Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ with $\psi(0) = 0$. Let X be a Banach space and f be a closed proper convex function on X. Then f is ψ-conditioned iff ∂f is $\underline{\psi}$-conditioned, where $\underline{\psi}$ denotes the function equal to $\psi(t)/t$ for $t > 0$ and equal to zero for $t = 0$.*

Proof. First we note that $S := \mathrm{Argmin}\, f = \partial f^{-1}(0)$. Let f be ψ-conditioned and $(x, y) \in \partial f$. We have

$$\forall \overline{x} \in S, \;\; \min f \geq f(x) + \langle y, \overline{x} - x\rangle \geq \min f + \psi(d(x, S)) + \langle y, \overline{x} - x\rangle.$$

So, we get

$$\forall (x, y) \in \partial f, \; x \notin S, \;\; \|y\|_\star \geq \psi(d(x, S))/d(x, S),$$

that is, ∂f is $\underline{\psi}$-conditioned.

Reciprocally, noting that f is ψ-conditioned iff, for all positive real $\theta < 1$, f is $\theta\,\psi$-conditioned, let ∂f be $\underline{\psi}$-conditioned and assume that f is not ψ-conditioned. So, there exist a positive real $\theta < 1$ and $x_\psi \in X$ such that

$$f(x_\psi) < \min f + \theta\; \psi(d(x_\psi, S)).$$

This implies $x_\psi \notin S$ and $0 \in \partial_\epsilon f(x_\psi)$, where $0 < f(x_\psi) - \min f \leq \epsilon < \theta\; \psi(d(x_\psi, S))$. Thanks to Brøndstedt-Rockafellar's theorem, there exists $(\tilde{x}, \tilde{y}) \in \partial f$ such that

$$\|\tilde{x} - x_\psi\| \leq \theta\; d(x_\psi, S), \;\; \|\tilde{y}\|_\star \leq \epsilon/(\theta\; d(x_\psi, S)).$$

Hence, $(\tilde{x}, \tilde{y})$ satifies

$$(\tilde{x}, \tilde{y}) \in \partial f, \;\; \tilde{x} \notin S, \;\; \|\tilde{y}\|_\star < \psi(d(x_\psi, S))/d(x_\psi, S),$$

a contradiction with $\underline{\psi}$-conditioning of ∂f. $\square$

Proposition 3.2 *Let X be a real Hilbert space and T be a maximal monotone operator on X. Then, for all $\lambda > 0$, T is ψ-conditioned iff J_λ^T is $\lambda\, \psi$-conditioned.*

Proof. First we note that, for all positive λ, $S := T^{-1}(0) = \text{Fix } J_\lambda^T$.

Let T be ψ-conditioned. As $x - J_\lambda^T(x) \in \lambda\, T(J_\lambda^T(x))$, we have $\|x - J_\lambda^T(x)\| \geq \lambda\, \psi(d(J_\lambda^T(x), S))$.

Reciprocally, let $(x, y) \in T$. We have $x = J_\lambda^T(x + \lambda y)$. As J_λ^T is $\lambda\, \psi$-conditioned we have

$$\lambda\, \|y\| = \|x + \lambda y - J_\lambda^T(x + \lambda y)\| \geq \lambda\, \psi(d(J_\lambda^T(x + \lambda y), S) = \lambda\, \psi(d(x, S)). \qquad \square$$

The two last propositions lead immediately to the following result.

Corollary 3.1 *Let X be a real Hilbert space and f a closed proper convex function on X. Then, for all $\lambda > 0$, f is ψ-conditioned iff ∂f is $\underline{\psi}$-conditioned, iff* $\text{prox}_{\lambda f}$ *is λ $\underline{\psi}$-conditioned.*

4 Well-posedness and conditioning

Let X be a Banach space and f a closed proper convex function on X. It is known ([14]) that f is minimisation well-posed iff f is strongly firmly conditioned, that is, f is ψ-conditioned where ψ is strongly firm, i.e. $\underline{\psi}$ is firm:

$$\forall \{t_n\} \subset \mathbb{R}_+ \backslash \{0\}, \;\; \psi(t_n)/t_n \to 0 \Rightarrow t_n \to 0.$$

So, putting together Propositions 2.1 and 3.1 leads to: ∂f is inclusion well-posed iff ∂f is firmly conditioned. Actually this can also be obtained as an immediate consequence of the following proposition.

Proposition 4.1 *Let X and Y be two real normed spaces and $T : X \to 2^Y$ such that $S := T^{-1}(0)$ is nonempty and closed. Then T is inclusion well-posed iff T is firmly conditioned.*

Proof. That firm conditioning implies inclusion well-posedness is easy to prove. Reciprocally, let us consider the radial regularized of T, i.e. the function $\psi_T : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$\psi_T(t) := \inf\{\|y\|_Y; (x, y) \in T, \; d(x, S) \geq t\}.$$

It is clear that $\psi_T(0) = 0$ and that T is ψ_T-conditioned. Let $\{t_n\}$ be a sequence of nonnegative reals such that $\psi(t_n) \to 0$. ¿From the definition of the infimum, for all $n \in \mathbb{N}$, there exists $(x_n, y_n) \in T$, such that $d(x_n, S) \geq t_n$ and $\|y_n\|_Y \leq \psi(t_n) + 1/n$. Therefore, $\{x_n\}$ is an asymptotically solving sequence and, thanks to well-posedness, $d(x_n, S) \to 0$ and hence $t_n \to 0$. So ψ_T is firm. $\square$

Now, let X be a real Hilbert space and T a maximal monotone operator on X such that $S := T^{-1}(0) \neq \emptyset$. Putting together Propositions 2.2 and 3.2 leads to: for all $\lambda > 0$, J_λ^T is fixed-point well-posed iff J_λ^T is firmly conditioned. Actually, this can also be obtained as an immediate consequence of the following proposition.

Proposition 4.2 *Let X be a normed space and $P : X \to X$ such that $S := \operatorname{Fix} P$ is nonempty and closed. Then, P is fixed-point well-posed iff P is firmly conditioned.*

Proof. The proof is analogue to the one of Proposition 4.1 considering the radial regularized of P defined by

$$\psi_P(t) := \inf\{\|x - P(x)\|;\ x \in X,\ d(P(x), S) \geq t\},$$

and noticing that, for an asymptotically regular sequence $\{x_n\}$, $d(x_n, S) \to 0$ and $d(P(x_n), S) \to 0$ are equivalent. $\square$

5 Iteration and well-posedness

Let X be a real Banach space and P a self mapping on X. We consider the approximate iterative scheme

$$\|x_n - P(x_n)\| \leq \epsilon_n,\ n = 1, 2, \ldots$$

Proposition 5.1 *Let us assume that P is θ-firmly nonexpansive, i.e.*

$$\exists \theta > 0,\ \forall x, y \in X,\ \|P(x) - P(y)\|^2 \leq \|x - y\|^2 - \theta\|(I - P)(x) - (I - P)(y)\|^2,$$

that P is fixed-point well-posed (which implies $S := \operatorname{Fix} P \neq \emptyset$), and that $\sum_{n=1}^{+\infty} \epsilon_n < +\infty$. Then x_n converges in norm to some x_∞ in S.

Proof. Thanks to nonexpansiveness we have

$$\forall n \in \mathbb{N}, \forall \overline{x} \in X,\ \|x_n - \overline{x}\| \leq \|x_{n-1} - \overline{x}\| + \epsilon_n.$$

Therefore,

$$\forall m > n,\ \|x_m - x_n\| \leq 2d(x_n, S) + \sum_{k=n+1}^{m} \epsilon_k.$$

Let $e_n := x_n - P(x_{n-1})$. Thanks to θ-firm nonexpansiveness we have

$$\forall n \in \mathbb{N}, \forall \overline{x} \in S,\ \|x_n - e_n - \overline{x}\|^2 \leq \|x_{n-1} - \overline{x}\|^2 - \theta\|(I - P)(x_{n-1})\|^2.$$

Therefore, $\{x_n\}$ is asymptotically regular for P. Thanks to well-posedness, $d(x_n, S) \to 0$. Finally, $\{x_n\}$ is a Cauchy sequence so converges to some x_∞ and, as S is closed and $d(., S)$ is continuous, $x_\infty \in S$. □

As a direct consequence of Propositions 2.1, 2.2 and 5.1 we get

Corollary 5.1 ([15]).

Let T be a maximal monotone operator on the real Hilbert space X, inclusion well-posed (for instance $T := \partial f$ with f closed proper convex, minimization well-posed). Then, for all positive λ, any sequence $\{x_n\}$ generated by the approximate proximal iterative scheme

$$\|x_n - J_\lambda^T(x_{n-1})\| \leq \epsilon_n,$$

with $\sum_{n=1}^{+\infty} \epsilon_n < +\infty$, converges in norm to some zero of T.

6 Regularization

As it is well known for minimization ([27]), the Tikhonov regularization method consists in replacing an ill-posed problem by a family (in practice a sequence) of Tikhonov well-posed ones of same type. Let X be a real Hilbert space. For a special subclass of each of the three classes above we define below the regularized problem and the regularized solution, that is, the unique solution of the regularized problem.

Convex minimization.

Let f be a closed proper convex function on X, x in X and $t > 0$. The regularized problem of the minimization of f is the minimization of

$$f_{x,t} := f + \frac{t}{2}\|. - x\|^2.$$

As $f_{x,t}$ is closed proper, strongly convex, it has a unique minimizer, namely the f-proximal point to x with parameter $\dfrac{1}{t}$: $\operatorname{prox}_{\frac{1}{t}f} x$.

Maximal monotone inclusion.

Let T be a maximal monotone operator on X, x in X and $s > 0$. The regularized problem of the inclusion for T is the inclusion for

$$T_{x,s} := T + s(I - x).$$

As $T_{x,s}$ is maximal monotone and strongly monotone, it has a unique zero, namely the T-proximal point to x with parameter $\frac{1}{s}$: $J^T_{\frac{1}{s}} x$.

When T is the subdifferential of a closed proper saddle function ([24]) L on the product $X := X_1 \times X_2$ then the inclusion problem for $T_{x,s}$ with $x := (x_1, x_2)$ is equivalent to the saddle-point problem for

$$L_{x,s} := L + \frac{s}{2}(\|. - x_1\|_1^2 - \|. - x_2\|_2^2).$$

So, convergence for saddle-point regularization can be deduced from convergence for inclusion regularization (see Proposition 6.1 (iii) below).

Nonexpansive mapping fixed-point.

Let P be a nonexpansive self mapping on X, x in X and $0 < r \leq 1$. The regularized problem of fixed-point for P is the fixed-point problem for

$$P_{x,r} := P((1-r).+rx).$$

As $P_{x,r}$ is strongly nonexpansive it has a unique fixed-point we call P-proximal point to x with parameter r noted $R_r^P x$.

This new proximal mapping R_r^P has the following easy to prove properties:

(i) $\operatorname{Fix} R_r^P = \operatorname{Fix} P$,

(ii) R_r^P is nonexpansive, 1-firmly nonexpansive if P is 1-firmly nonexpansive,

(iii) For T maximal monotone, $R_r^{J_\lambda^T} = J_{\lambda/r}^T$.

Now let $\{r_n\}$ be a sequence of positive reals that tends to 0 and x be fixed.

Proposition 6.1 (i) $f(\operatorname{prox}_{\frac{1}{r_n}f}\, x) \to \inf f$,

(ii) *If* $S := \operatorname{Argmin} f \neq \emptyset$, *then* $\operatorname{prox}_{\frac{1}{r_n}f}\, x$ *converges in norm to* $proj_S x$,

(iii) *If* $S := T^{-1}(0) \neq \emptyset$, *then* $J^T_{\frac{1}{r_n}} x$ *converges in norm to* $proj_S x$,

(iv) *If* $S := \operatorname{Fix} P \neq \emptyset$, *then* $R^P_{r_n} x$ *converges in norm to* $proj_S x$.

Proof. (i), (ii) and (iii) are well known ([27, 12, 1, 26]). In fact (ii) is a consequence of (iii) which in turns is a consequence of (iv) the proof of which is analogue to the one of (iii) ([26]) using Lemma 6.1 below and the fact that $I - P$ is maximal monotone. □

Lemma 6.1 *Let* P *be a nonexpansive self mapping on the real Hilbert space* X *such that* $S := \operatorname{Fix} P \neq \emptyset$. *Then*

(i) $\forall x \in X,\ \forall 0 < r (\leq 1),\ \ \|x - R_r^P x\| \leq \dfrac{2}{2-r}\, d(x, S)$,

(ii) *If* P *is* 1*-firmly nonexpansive then* $\forall x \in X,\ \forall r > 0,\ \ \|x - R_r^P x\| \leq d(x, S)$.

Proof. (i) Let $\overline{x} \in S$ and $x_r := R_r^P x$. Thanks to the nonexpansiveness of P we get

$$\|x_r - \overline{x}\|^2 \leq \|x_r - \overline{x} + r(x - x_r)\|^2$$

$$= \|x_r - \overline{x}\|^2 + 2r\langle x_r - \overline{x}, x - x_r\rangle + r^2\|x - x_r\|^2,$$

from which we deduce easily the result.

(ii) In the righthandside of the first inequality of (i), thanks to firmness, we can substract $\|r(x - x_r)\|^2$. □

Remark 6.1 *Of course, we can define fixed-point Tikhonov regularization of P as inclusion Tikhonov regularization of $I - P$, leading to the regularized fixed-point problem*

$$y_s = \frac{1}{1+s} P(y_s) + \frac{s}{1+s} x.$$

A simple calculation shows that, with the correspondance of parameters $1 - r = \dfrac{1}{1+s}$, then $x_r = (1+s)y_s - sx$. So, as $r \to 0$ iff $s \to 0$, $x_r \to \overline{x}$ iff $y_s \to \overline{x}$.

7 Exact regularization

In the framework of the previous section we prove that, under a specific kind of conditioning, exact regularization holds true, that is, the regularized solution is a solution to the original problem for all r small enough. In fact we obtain more, namely that the selected solution (the projection of x onto the solution set S) is achieved if x is close enough to S with given r or, equivalently, if r is small enough with given x.

Let f be a closed proper convex function on X such that $S := \operatorname{Argmin} f \neq \emptyset$. Let $\gamma > 0$. Recall that f is said γ-linear conditioned if

$$\forall x \in X,\ f(x) \geq \min f + \gamma\, d(x, S).$$

We note that linear conditioning is a particular strongly firm conditioning. In fact, in this case, f is ψ-conditioned with $\psi(t) := \gamma\, t$ and hence $\underline{\psi}(t) = \gamma$ if $t > 0$ and $\underline{\psi}(0) = 0$. More precisely, if $\underline{\psi}(t_n) \to 0$ then $t_n = 0$ for all n large enough.

It has been proved ([15]) that, under γ-linear conditioning, if $d(x, S) < \gamma/r$ then $\operatorname{prox}_{\frac{1}{r}f}\, x = \operatorname{proj}_S x$, and consequently that the proximal point algorithm has finite termination, more precisely, denoting $\{x_n\}$ the generated sequence, $\exists N, \forall n > N, x_n = \operatorname{proj}_S x_N$.

This exact regularization result for convex minimization can be extended to maximal monotone inclusion and nonexpansive mapping fixed-point as follows.

First we introduce constant conditioning for this two classes.

Let $\gamma > 0$. An operator $T : X \to 2^Y$ such that $S := T^{-1}(0) \neq \emptyset$ is said γ-constant conditioned if T is ψ-conditioned with $\psi(t) = \gamma$ if $t > 0$ and $\psi(0) = 0$. We note that this is equivalent to

$$\forall (x, y) \in T,\ \text{if } \|y\| < \gamma,\ \text{then } x \in S.$$

Let $\delta > 0$. We say that a self mapping P of the real normed vector space X such that $S := \operatorname{Fix} P \neq \emptyset$ is δ-constant conditioned if P is ψ-conditioned with $\psi(t) = \delta$ if $t > 0$ and $\psi(0) = 0$. We note that this is equivalent to

$$\forall x \in X,\ \text{if } \|x - P(x)\| < \delta,\ \text{then } P(x) \in S.$$

As corollaries of Propositions 3.1 and 3.2 we get immediately the following two propositions.

Proposition 7.1 ([23]). *Let X be a Banach space and f be a closed proper convex function on X. Then f is γ-linear conditioned iff ∂f is γ-constant conditioned.*

Proposition 7.2 *Let X be a real Hilbert space and T be a maximal monotone operator on X. Then, for all $\lambda > 0$, T is γ-constant conditioned iff J_λ^T is $\lambda\,\gamma$-constant conditioned.*

We can now present the general exact regularization results.

Proposition 7.3 *Let P be a nonexpansive self mapping of the real Hilbert space X with δ-constant conditioning. Let $S := \mathrm{Fix}\, P$.*

(i) *If $d(x,S) < \dfrac{2-r}{2r}\delta$ or, equivalently, $0 < r < \min\{1, \frac{2\delta}{2d(x,S)+\delta}\}$, then $R_r^P x \in S$.*

(ii) *If P is 1-firmly nonexpansive and $d(x,S) < \delta/r$, then $R_r^P x =$proj$_S x$.*

Proof. By definition of $x_r := R_r^P x$ we have $\|(1-r)x_r + rx - P((1-r)x_r) + rx)\| = r\|x - x_r\|$.

(i) From lemma 6.1 (i) we have $r\|x - x_r\| < \delta$. Therefore, thanks to constant conditioning, we get $x_r = P((1-r)x_r + rx) \in S$.

(ii) From lemma 6.1 (ii) we have $r\|x - x_r| < \delta$ and therefore $x_r \in S$. Thanks again to lemma 6.1 (ii) we get $\|x - x_r\| = d(x,S)$. □

Corollary 7.1 *Let T be a maximal monotone operator on the real Hilbert space X, with γ-constant conditioning. Let $S := T^{-1}(0)$. If $d(x,S) < \gamma/r$, then $J_{\frac{1}{r}}^T x =$ proj$_S x$.*

Proof. Apply (ii) of Proposition 7.3 with $P := J_1^T$ (so $R_r^{J_1^T} = J_{\frac{1}{r}}^T$) and, thanks to Proposition 7.2, $\delta := \gamma$. □

Linear conditioning can be defined for a saddle function which also implies constant conditioning of its subdifferential and hence, exact regularization for saddle-point problems ([10]).

8 Iteration and regularization

Let X be a real Hilbert space and P a nonexpansive self mapping on X with set of fixed-points S. As shown in the previous section, the Tikhonov regularization method allows to approximate a particular fixed-point, namely the projection onto S of a given x. Now the iteration method applied to the regularized mapping $P_{x,r}$ for fixed r allows to approximate the regularized solution $x_r := R_r^P x$, since classicaly the generated sequence converges in norm to x_r. So this give a two stages approximation. We prove in the following that if in the iteration method we use variable r_n tending to 0 not too fast, then the sequence generated by this diagonal iterative scheme converges also to the projection of x onto S.

More precisely we consider a sequence $\{x_n\}$ generated by the following approximate iterative scheme

$$\|x_n - P((1-r_n)x_{n-1} + r_n x)\| \leq \epsilon_n, \; n = 1, 2, \ldots$$

where $0 < r_n \leq 1$ and $\epsilon_n \geq 0$.

Proposition 8.1 *Let us assume:*
P is nonexpansive, $r_n \to 0$, $\sum_{n=1}^{+\infty} r_n = +\infty$, $\epsilon_n / r_n \to 0$, $|\frac{1}{r_n} - \frac{1}{r_{n-1}}| \to 0$, $S \neq \emptyset$. Then x_n converges in norm to $proj_S x$.

Proof. Though the result can be deduced from [18] (Proposition 5.1), we prefer to give here a self-contained proof. Noting $x(n) := R^P_{r_n} x$, we get easily the estimate

$$\|x_n - x(n)\| \leq (\|x_{n-1} - x(n-1)\| + \|x(n-1) - x(n)\| + (1+r_n)\epsilon_n)/(1+r_n).$$

We invoke the following direct consequence of [6] (Corollary 5.4):
let sequences of nonnegative reals a_n, r_n, γ_n be such that $\sum_{n=1}^{+\infty} r_n = +\infty$, $\gamma_n \to 0$ and $a_n \leq (a_{n-1} + \gamma_n r_n)/(1+r_n)$; then $a_n \to 0$.

For that we take $\gamma_n := \epsilon_n + \epsilon_n/r_n + \|x(n) - x(n-1)\|/r_n$ showing that $\|x(n) - x(n-1)\|/r_n \to 0$. In fact, thanks to nonexpansiveness we can obtain the estimate

$$\|x(n) - x(n-1)\| \leq |1 - \frac{r_n}{r_{n-1}}| \; \|x - x(n)\|,$$

and, from Lemma 6.1 (i),

$$\|x(n) - x(n-1)\| \leq |1 - \frac{r_n}{r_{n-1}}| \; 2d(x, S).$$

So we get $\|x_n - x(n)\| \to 0$ and the result since $x(n) \to \mathrm{proj}_S x$. □

Corollary 8.1 *Let T be a maximal monotone operator on X such that $S := T^{-1}(0) \neq \emptyset$. Let $\lambda > 0$. Under the same assumptions on r_n and ϵ_n than in Proposition 8.1, any sequence $\{x_n\}$ generated by the approximate iterative scheme*

$$\|x_n - J^T_\lambda (1 - \lambda r_n) x_{n-1} + \lambda r_n x)\| \leq \epsilon_n, \; n = 1, 2, \ldots,$$

converges in norm to $proj_S x$.

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