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MAXIMIZATION OF A LINEAR UTILITY FUNCTION OVER THE SET OF THE HOUSING MARKET SHORT-TERM EQUILIBRIA*

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Some generalization of the housing market models published by Herbert and Stevens [4], Gustafsson et al. [2], and Wiesmeth [7] is suggested. The set of short-term equilibria in a housing market in the sense of Wiesmeth [7] is parameterized by Pareto-maximal integral points of some polyhedron. The problem of maximization of a linear utility function over the set of short-term equilibriums is studied. The problem is proved to be reducible (under some natural assumptions) to a linear programming problem (LPP), or to finite number of the LPPs in general case. The possible applications of the results and some related problems are pointed out.

Keywords: housing market, quantity constrained equilibrium, linear programming, unimodularity.

AMS subject classification: 90C05, 90A14.

1 Model

The first model of the considered type was suggested by Herbert and Stevens [4], the advanced variant was included in the survey of Gustafsson et al. [2]. Wiesmeth [7] offered (maybe, independently) the similar model which, nevertheless, somewhat differs from the preceding ones. These three models are the particular cases of the following one. We consider an imaginary market period throughout which the asset and rental prices, and consumers' preferences are invariable, and new consumers and suppliers do not appear.

The information basis of the model is formed by two classifications: groups of households and types of dwellings. These classifications should satisfy the following conditions.

1. The utility function of each household over the set of the dwellings types is well-defined.

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2. The households of the same group which occupy the dwellings of the same type have identical utility functions.

Notations:

H is the set of (numbers of) groups of households;

I is the set of (numbers of) types of dwellings, $0 \in I$;

D_{ih} is the number of h-group households occupying the i-type dwellings at the beginning of the market period, $(i,h) \in I \times H$ (D_{0h} is the number of h-group consumers having no dwelling in the housing market being considered);

S_i is the number of i-type dwellings, which are vacant at the beginning of the market period, $i \in I$ (S_0 is some sufficiently large number).

Each triplet $(i,j,h) \in I^2 \times H$ can be interpreted as a bargain resulting in moving of an h-group household from i-type to j-type dwelling. As the prices and households' preferences are fixed within the market period, the set of all bargains acceptable for their participants is completely defined. Let V be the set of triplets (i,j,h) corresponding to feasible bargains. Without loss of generality we assume that $D_{ih} \neq 0$ for $(i,j,h) \in V$.

Description of some initial situation in housing market includes the parameters $H, I, V, \{D_{ih}\}, \{S_i\}$. Gustafsson et al. [2] assume $V = I^2 \times H$. Wiesmeth [7] defines V as the set $\{(i,j,h)/h \in H,\ i \notin A_h, j \in A_h\}$, where $A_h \subseteq I$ is the given set of (numbers of) types of dwellings acceptable for the h-group households. Our results are true for an arbitrary V.

The model describes "activity" of a market as follows: a bargain $(i,j,h) \in V$ can be carried out if some j-type dwelling is vacant or may be vacated. We assume each household to change its dwelling no more than once for the period.

Notations:

x_{ijh} is the number of h-group households which have exchanged i-type to j-type dwellings during considered period (for $(i,j,h) \in V$), unknown quantity;

$x = (x_{ijh}/(i,j,h) \in V))$;

$T^-_{ih}(x) = \sum_j x_{ijh}$, $T^-_i(x) = \sum_h T^-_{ih}(x)$;

$T^+_{jh}(x) = \sum_i x_{ijh}$, $T^+_j(x) = \sum_h T^+_{jh}(x)$;

$Q = \{x/x_{ijh} \geq 0,\ T^-_{ih}(x) \leq D_{ih},\ T^+_i(x) - T^-_i(x) \leq S_i$ for $(i,j,h) \in V,\ i \in I,\ h \in H\}$;

P is the set of all integral vectors from Q.

To each $x \in Q$ there corresponds some "final situation" in the housing market: the vectors $D(x)$ and $S(x)$ with the components $D_{ih}(x) = D_{ih} + T^+_{ih}(x) - T^-_{ih}(x)$ and $S_i(x) = S_i + T^-_i(x) - T^+_i(x)$. $D(x)$ describes the allocation of households over the dwellings and $S_i(x)$ specifies the number of vacant i-type dwellings at the end of the period if all bargains corresponding to x would be carried out.

2 Short-term equilibriums

Notations:
$E(M) = \{x/x \in M$ and $(y \in M \& y \geq x \to y = x)\}$ for $M \subseteq R^n$; $E = E(P)$.

The vector $x \in Q$ is efficient and the final situation $(D(x), S(x))$ is a short-term equilibrium if $x \in E$, i.e., no bargain from V is possible after all bargains described by x were carried out. We assume (with some idealization) the households to have the perfect information on supply of housing. Then any initial situation over the considered period will be transformed into some equilibrium.

In this section we shall describe the elements of Q which belong to E (Corollary 2).

Theorem 1 *The matrix U of constraints describing the polyhedron Q is totally unimodular.*

PROOF. Let's denote by a_{ih} and b_i the vectors of coefficients of linear forms $T_{ih}^-(x)$ and $T_i^+(x)$ respectively, $c_i = b_i - \sum_h a_{ih}$. The matrix U consists of all rows a_{ih} and c_i.

For any $B \subseteq H$ let $c_i(B) = b_i - \sum_{h \in B} a_{ih}$; clearly $c_i = c_i(H)$. Let v_{ijh} be the column of U corresponding to variable x_{ijh}. Notice that non-zero a_{ih} is not equal to any $c_j(B)$. We set $M(D) = \{(i,h)/\exists B(h \in B$ and the rows a_{ih} and $c_i(B)$ are both in $D)\}$ for arbitraty matrix D containing the rows of the types a_{ih} and $c_i(B)$. An arbitrary square submatrix D of U can be transformed now as follows.

Let $D_0 = D$. Assume the matrix D_k $(k \geq 0)$ to be already defined and $\det(D_k) = \det(D)$. If $M(D_k) = \emptyset$, then the transformation is finished.

Otherwise, after choosing any pair $(i,h) \in M(D_k)$ and replacing in D_k the corresponding row $c_i(B)$ by $c_i(B) + a_{ih} = c_i(B \setminus \{h\})$ we shall obtain the matrix D_{k+1} with $\det(D_{k+1}) = \det(D_k) = \det(D)$.

Described procedure is obviously finite, and we shall obtain the matrix D^* such that $M(D^*) = \emptyset$ and $\det(D^*) = \det(D)$. For proving the total unimodularity of D^*, let us divide the rows of D^* into subsets M_1 (including all rows of the type $c_i(B)$) and M_2 (including all rows of the type a_{ih}). For each i there is no more than one non-zero row of the type $c_i(B)$ in D^*, denote it (if any) by r_i. If $i = j$ then the column v_{ijh} contains no more then two non-zero elements: 1 in the rows r_i and a_{ih}. If $i \neq j$ then the column v_{ijh} contains no more then three non-zero elements: 1 in the rows r_j and a_{ih}, -1 in r_i. From $M(D^*) = \emptyset$ it follows that each column of D^* contains no more than two non-zero elements. Consider some column of D^* containing exactly two non-zero elements. If these elements are identical in sign then the corresponding rows belong to the distinct sets M_n $(n \in \{1,2\})$, and if the elements are opposite in sign then the corresponding rows belong to the same set. D^* is totally unimodular by the theorem of Heller and Tompkins [3]. So, $\det(D) = \det(D^*) \in \{-1, 0, 1\}$. □

Corollary 1 *All corner points of Q belong to P.*

PROOF. The right sides of the inequalities describing Q are integers, so all corner points of Q are integral by the theorem of Hoffman and Kruskal [5] and, therefore, belong to P. □

Obviously, every integral element of $E(Q)$ belongs to E. The less evident reverse inclusion is also true.

Lemma 1 *If $M = \{x/x \in R^n,\ A \cdot x \leq b\}$ and A is a totally unimodular matrix then for any integral $y \in M \setminus E(M)$ there exists an integral $z \in M$ such that $y \leq z$, $y \neq z$.*

PROOF. Let $b_1 = A \cdot y$. If y is integral and $y \in M \setminus E(M)$ then b_1 is integral (because all elements of A are integers); $b_1 \leq b$. From $y \notin E(M)$ it follows that there exists some d, $d \geq 0$, $d \neq 0$, $A \cdot (y + d) \leq b$. Consider the polyhedron $M_1 = \{x/x \in R^n, A \cdot x \leq b - b_1, x \geq 0\}$. $d \in M_1$, therefore M_1 has some non-zero corner point z_1. Vector $b - b_1$ is integral, hence all corner points of M_1 are integral. Therefore, $z = y + z_1$ is the desired vector. □

Corollary 2 *E is exactly the set of all integral points of $E(Q)$.*

PROOF. If $y \in E$ then y is integral and $y \in Q$. If $y \notin E(Q)$ then, by Lemma 1, there exists an integral $z \in Q$ such that $y \leq z$, $y \neq z$. Existence of such $z \in P$ contradicts the choice of y. Inversely, let y be an integral point of $E(Q) \setminus E$. Then some z exists in $P \subseteq Q$ such that $y \leq z$, $y \neq z$, in contradiction with the choice of y. □

3 Maximization of a linear utility function

Each short-term equilibrium corresponds to some vector $x \in E$. We cannot predict the specific equilibrium which will be realized at the end of the period in a given initial situation. Nevertheless, the equilibriums created by the vectors maximizing some social utility function are of especial interest for (at least) two reasons:

(a) when knowing the optimal $x \in E$, a planning body (e.g., the local administration) can support and promote the corresponding bargains (all the more, it does not contradict the consumers' preferences);

(b) having the method for calculating the maximum of some objective function at any initial situation, it is possible to look for modification of the initial situation in order to increase the maximum.

Let us assume the utility function $g(x)$ to be linear with respect to variables x_{ijh}: $g(x) = c \cdot x$, $c \geq 0$. Our problem is:

$$\max \{c \cdot x / x \in E\}, c \geq 0. \tag{1}$$

Any solution of this problem creates some equilibrium maximizing the utility function $c \cdot x$.

Theorem 2 *If $c > 0$, x^* is a corner point of Q, and $x^* \in \operatorname{Argmax}_Q(c \cdot x)$ then $x^* \in \operatorname{Argmax}_E(c \cdot x)$.*

PROOF. If x^* is a corner point of Q and $x^* \in \mathrm{Argmax}_Q(c \cdot x)$ then x^* is integral by Corollary 1. Thus $x^* \in E(Q)$, as Charnes and Cooper proved [1, Theorem 1, p. 301]. By Corollary 2, $x^* \in E \subseteq Q$, therefore $x^* \in \mathrm{Argmax}_E(c \cdot x)$. $\square$

From Theorem 2 it follows that the problem (1) in the case of $c > 0$ can be reduced to the LPP $\max_Q(c \cdot x)$. Note, that Wiesmeth [7] reduced the problem $\max_E \sum_{i,j,h} x_{ijh}$ to a nonlinear programming problem.

For $c \in R^n_+$ let $I(c) = \{i / c_i = 0\}$, and for any real α let us define the vector $c(\alpha)$: $c_i(\alpha) = c_i$, if $i \notin I(c)$; $c_i(\alpha) = \alpha$, if $i \in I(c)$.

Lemma 2 *If D is a polytop in R^n_+, then there exists some real $\alpha_0 > 0$ such that* $\mathrm{Argmax}_D\ (c(\alpha) \cdot x) \subseteq \mathrm{Argmax}_D\ (c \cdot x)$ *for any $\alpha \in (0, \alpha_0)$.*

PROOF. Without loss of generality we assume

$$\max_D x_i > 0 \quad \text{for each } i, \tag{2}$$

since otherwise the variable x_i can be excluded from the description of D (by fixing $x_i = 0$) and we can consider D and c in R^{n-1}_+. If $I(c) = \emptyset$ then $c(\alpha) = c$ and the lemma is true. Assume now $I(c) \neq \emptyset$.

Let $d = \max_D \sum_{I(c)} x_i$; let $p_1 < p_2 < \cdots < p_r$ be the values of $c \cdot x$ at the corner points of D. $d > 0$ by (2), and $p_r = \max_D\ (c \cdot x)$. If $r = 1$ then $c \cdot x$ is constant over D and the lemma is true. Assume $r > 1$ and consider some a, $0 < \alpha < \alpha_0 = (p_r - p_{r-1})/d$. Let $y \in \mathrm{Argmax}_D(c(\alpha) \cdot x)$, then $c \cdot y \leq p_r$. If $c \cdot y < p_r$, then $c \cdot y \leq p_{r-1}$, $c(\alpha) \cdot y \leq$ $\leq \alpha \cdot \sum_{I(c)} y_i + c \cdot y < \alpha_0 \cdot \sum_{I(c)} y_i + c \cdot y \leq p_r - p_{r-1} + c \cdot y \leq p_r$.

On the other hand, for $x \in \mathrm{Argmax}_D\ (c \cdot x)$ we have $c(\alpha) \cdot x = \alpha \cdot \sum_{I(c)} x_i + c \cdot x \geq p_r$ in contradiction with the choice of y. So, $c \cdot y = p_r$ and $y \in \mathrm{Argmax}_D\ c \cdot x$. $\square$

The algorithm for solving the problem (1) in the case of $I(c) \neq \emptyset$ can now be described as follows.

Let $\alpha_n = 2^{-n}$; let us assume that $x(n) \in \mathrm{Argmax}_Q\ (c(\alpha_n) \cdot x)$ and $x(n)$ is a corner point of Q for $n > 0$. By Lemma 2, $x(n) \in \mathrm{Argmax}_Q\ (c \cdot x)$ for some sufficiently large n. The criterion for detecting such n is: $\max_Q\ (c \cdot x) = c \cdot x(n)$. $x(n) \in E$ by Theorem 2. So, $x(n)$ solves the problem (1).

The algorithm described above requires us to solve a finite number of the LPPs $\max_Q(c(\alpha_n) \cdot x)$. When $c \geq 0$ is integral, the procedure can be simplified as follows.

Let $a = \max\ \{\max\ \{D_{ih}\}, \max\ \{S_i\}\}$, $n = |V|$, $m_1 = |\{(i,h)\ /\exists j((i,j,h) \in \in V)\}|$, $m_2 = |\{i\ /\exists j \exists h((i,j,h) \in V)\}|$, $m = m_1 + m_2$. If A is the $m \times (n+m)$ matrix of the LPP $\max_Q\ (c \cdot x)$ in the canonical form then $\max |a_{ij}| = 1$. Let $M = (n+m) \cdot (1+a) \cdot m^{2m+3}$, $\alpha^* = (Mn)^{-1}$.

Theorem 3 *If vector c is integral nonnegative, $I(c) \neq \emptyset$, and $\alpha < \alpha^*$ then* $\mathrm{Argmax}_Q(c(\alpha) \cdot x) \subseteq \mathrm{Argmax}_Q\ (c \cdot x)$.

PROOF. It is enough to prove that $\alpha^* \leq \alpha_0$ (Lemma 2). $\alpha_0 = (p_r - p_{r-1})/d$ (see the proof of Lemma 2), $p_r = c \cdot x_1$, $p_{r-1} = c \cdot x_2$, where x_1, x_2 are some distinct corner points of Q (both integral by Corollary 1). So, $p_r - p_{r-1}$ is the positive integer, and $\alpha_0 \geq 1/d$.

$d = \max_Q \sum_{I(c)} x_{ijh} \leq M \cdot |I(c)|$ in consequence of the result of Papadimitriou and Steiglitz [6, Theorem 13.5]. Hence $d \leq Mn$ and $\alpha_0 \geq (Mn)^{-1} = \alpha^*$. $\square$

If $\alpha < \alpha^*$ and x^* from $\text{Argmax}_Q\ (c(\alpha) \cdot x)$ is the corner point of Q then $x^* \in E$ by Theorem 2 and x^* solves the problem (1). So, for an integral c the problem (1) is reducible to LPP $\max_Q\ (c(\alpha) \cdot x)$.

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