PLISKA
STUDIA MATHEMATICA BULGARICA

ПЛИСКА
БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

Pliska Stud. Math. Bulgar. **12** (1998), 21-30

PLISKA
STUDIA MATHEMATICA
BULGARICA

PC POINTS AND THEIR APPLICATION TO VECTOR OPTIMIZATION

Ewa Bednarczuk, Wen Song

In this paper we present some results concerning stability under perturbations and some topological properties of minimal point sets in vector optimization. A common feature of all these results is that they exploit the notion of a point of continuity (PC point) of a subset of a topological vector space. The concept of a PC point plays an important role in investigation of the geometry of Banach spaces.

Keywords: PC points, efficient points, density, connectivity, contractibility, stability, vector optimization.

AMS subject classification: 90C29, 90C48

1 Introduction

Let X be a Hausdorff topological vector space. Let A be a subset of X and $S \subset X$ be a closed convex pointed cone.

A point $a_0 \in A$ is a *minimal point* (an efficient point) of A with respect to S (see [26]), $a_0 \in \mathrm{Min}(A, S)$, if

$$(A - a_0) \cap (-S) = \{0\} .$$

In this paper we present some results concerning stability, density, connectivity and contractibility of the minimal point set. These properties have been investigated by many authors and we give some references in subsequent sections. Our results are formulated with the help of the notion of the point of continuity (PC point) of a set. The use of PC points allows us to relax some strong requirement imposed on the set A, eg. compactness, which appeared in the existing results. The concept of PC points has been extensively used in investigation of the geometry of Banach spaces (see [12], [28], [29] [30], [44], [40], [24], [39], [9] and [4]).

2 PC points and related notions

Let A be a subset of a topological vector space X. $\overline{A}$ ($\overline{A}^w$) denotes the closure (weak closure) of A, $\overline{co}(A)$ denotes the closed convex hull of A, and int A denotes the interior of A.

A point $a \in A$ is called a *point of continuity* (PC point) of A if for any $0-$neighbourhood V

$$a_0 \notin \overline{[A \setminus (a_0 + V)]}^w.$$

Equivalently, $a_0 \in A$ is a PC point for A if a net in A converges (strongly) to a_0 , whenever it converges weakly to a_0. By PC(A) we denote the set of all PC points of A.

A point $a_0 \in A$ is called a *denting point* of A if, for any 0-neighbourhood V,

$$a_0 \notin \overline{co}[A \setminus (a_0 + V)],$$

The concept of PC point was first introduced by Lin-Lin-Troyanski ([29]) in normed spaces. It was used to study the relationship between the topological concept of denting point and the geometric concept of extreme point.

In locally convex spaces, it follows directly from the definitions that every denting point of A is also a PC point of A. For a closed bounded convex set A, in virtue of the Hahn-Banach theorem, every denting point of A is also an extreme point of A (see Lemma 1.2 , 1.3 in [40]). The next result shows that in Banach spaces the converse is also true.

Theorem 1 ([29]) *Let a be an element of a bounded closed convex subset A of a Banach space X. Then the following are equivalent:*

(i) *a is a denting point of A,*

(ii) *a is a PC point of A and a is an extreme point of A.*

For unbounded sets, the equivalence (i) $\Leftrightarrow$ (ii) has been proved by Balder [4] and Benabdellah [9]. When the space X is not complete, (ii) does not imply (i) (for an example see [30]).

PC points can also be used in formulating some global properties of sets. Following [29], we say that a Banach space X has the Kadec property (K) (resp. property (G)) if every point of the unit sphere is a PC point (resp. denting point) of the closed unit ball.

As a corollary of Theorem 1, we obtain

Theorem 2 ([28], [30]) *A Banach space X has the property (G) if and only if X is strictly convex and X has the property (K).*

A bounded closed convex subset K of a Banach space X is said to have the CPCP (convex point of continuity property) if every closed convex subset of K contains at least one PC point. K has Radon-Nikodym property (RNP) (resp. Krein-Milman property (KMP)) if and only if every closed convex subset of K has a denting point (resp. an extreme point). It is shown in [44] (see also [40], [24]) that if K has CPCP then KMP is equivalent to RNP.

3 Application of PC points to study the density in vector optimization

In this section we present some density results for positive proper efficiency. Since Arrow, Barankin and Blackwell proved a density theorem in 1953, density results have been investigated by many authors (see [22], [11], [27], [35], [13], [15], [17], [19], [16], [33], [42] and [7]). In [17], two density results are p[resented which demonstrate a trade-off between restrictions imposed on the feasible set and the ordering cone. In [7], by using the concept of PC points, we obtained a density result with less restrictive assumptions on the feasible set and / or the ordering cone.

Let X be a Hausdorff topological vector space with topological dual X^*. Let $S \subset X$ be a convex cone. Denote by S^+ the dual cone of S, i.e.,

$$S^+ = \{x^* \in X^* \mid \langle x^*, x\rangle \geq 0,\ \forall x \in S\},$$

where $\langle , \rangle$ is the canonical bilinear form establishing the duality between X^* and X. Let S^{+i} be the set of all strictly positive linear functionals in S^+, i.e.,

$$S^{+i} = \{x^* \in X^* | \langle x^*, x\rangle > 0,\ \forall x \in S \text{ and } x \neq 0\}.$$

A nonempty convex subset B of S is a *base* for S if $0 \notin \overline{B}$ and

$$S = \text{cone}(B) = \{\lambda b \mid b \in B, \lambda \geq 0\}.$$

If a cone has a base, then it is pointed, i.e., $S \cap (-S) = \{0\}$. Moreover, in locally convex spaces, S has a base if and only if $S^{+i} \neq \emptyset$ (see Proposition 3.6 of [37]).

A point $a_0 \in A$ is a *positive proper efficient point* of A if there exists an $h \in S^{+i}$ such that

$$h(a_0) \leq h(x), \quad \text{for all } x \in A.$$

By $\mathrm{Pos(A, S)}$ we denote the set of all positive proper efficient points of A with respect to S. It is easily seen that $\mathrm{Pos(A, S)} \subset \mathrm{Min(A, S)}$.

Theorem 3 ([7]) *Let A be a weakly compact subset of a locally convex space X such that $A + S$ is convex and let $S \subset X$ be a closed convex cone with a base B. If $a_0 \in \mathrm{Min}(A, S)$, then $a_0 \in \overline{\mathrm{Pos(A, S)}}$ under any of the following conditions:*

(*a*) *a_0 is a PC point for A,*

(*b*) *0 is a PC point for S.*

The following density result is a direct consequence of Theorem 3.

Corollary 1 *Let $A \subset X$ be a nonempty weakly compact set such that $A + S$ is convex and let $S \subset X$ be a closed convex cone with a base B. If either of the conditions is satisfied:*

(*a*) $\mathrm{Min}(A,S) \subset \mathrm{PC}(A)$,

(*b*) 0 *is a PC point for* S,

then $\mathrm{Min}(A,S) \subset \overline{\mathrm{Pos(A,S)}}$.

Remark 1 Condition (*a*) of Corollary 1 holds for any compact subset A of X but there exists examples of non-compact sets satisfying condition (*a*). For instance, the unit ball in an infinite-dimensional and uniformly convex Banach space satisfies condition (*a*) (see [33]) or Example in [30]. Therefore, Corollary 1 generalizes Theorem 1 of [16] and refines Corollary 2.8 of [17].

4 Application of PC points to connectivity and contractibility of the efficient point set

Connectivity and contractibility of the efficient point set have been investigated by many authors (eg.[35], [34],[22], [11], [46]), [31], [18],[41], [42] [43] and [8]). In this section we present some results concerning connectivity and contractibility of the efficient point set. By using the concept of PC points we obtain stronger versions of some existing results.

Let U be a topological space and X a Hausdorff topological vector space. A multivalued mapping $\Gamma : U \to X$ is *upper semicontinuous* (u.s.c.) at u_0 (see [10], [3]) if for every open set V with $\Gamma(u_0) \subset V$, there exists a neighborhood U of u_0 such that

$$\Gamma(u) \subset V \text{ for all } u \in U\,.$$

$\Gamma : U \to X$ is *lower semicontinuous* (l.s.c.) at (x_0, u_0) if for each $0-$neighbourhood W in X there exists a neighbourhood U_0 of u_0 such that $(x_0 + W) \cap \Gamma(u) \neq \emptyset$ for $u \in U_0$. Γ is l.s.c. at u_0 if it is l.s.c. at every (x_0, u_0), $x_0 \in \Gamma(u_0)$. If Γ is a single-valued mapping, the definitions of upper and lower semicontinuity both reduce to the continuity of Γ.

Theorem 4 ([43]) *Let X be a locally convex space and let S be a closed convex cone in X such that the set S^{+i} is nonempty. Let A be a weakly compact subset of X such that $A+S$ is convex and let* $\mathrm{Min}(A,S) \subset \mathrm{PC}(A)$. *Then* $\mathrm{Pos}(A,S)$, $\mathrm{Min}(A,S)$ *are connected.*

A set $A \subset X$ is said to be *contractible* if there exists a continuous mapping $\chi\colon A \times [0,1] \to A$ such that $\chi(x,0) = x_0$ and $\chi(x,1) = x$ for all $x \in A$ and some $x_0 \in A$.

Following [21] a set $A \subset X$ is said to be *R_δ-contractible* if there exists an upper semicontinuous set-valued mapping $\Gamma\colon A \times [0,1] \to A$ with R_δ images such that $\Gamma(x,0) = A_0$ and $x \in \Gamma(x,1)$ for all $x \in A$ and some $A_0 \subset A$. Recall that a subset A is R_δ whenever there exists a decreasing sequence A_n of compact contractible sets such that $A = \bigcap\{A_n;\ n = 1,2,\ldots\}$.

Note that any compact convex set is R_δ and any R_δ-contractible set is acyclic and, in particular, connected.

A function $g: X \to R$ is *increasing* (see [31]) if for every $x, y \in X$, $x - y \in S \setminus \{0\}$ implies $g(x) > g(y)$.

g is said to be *quasiconvex* (see [31]) if

$$g(\lambda x + (1-\lambda)y) \leq \max\{g(x), g(y)\} \text{ for all } x, y \in X\,, \lambda \in (0,1).$$

g is said to be *strictly quasiconvex* (see [31]) if

$$g(\lambda x + (1-\lambda)y) < \max\{g(x), g(y)\} \text{ for all } x, y \in X, x \neq y\,, \lambda \in (0,1).$$

It is clear that if g is strictly quasiconvex, then it is also quasiconvex.

Define a set-valued mapping $G: A + S \to X$ by

$$G(x) = (x - S) \cap (A + S) \text{ for } x \in A + S.$$

Lemma 1 *Let X be a locally convex space and let $S \subset X$ be a closed convex pointed cone in X. Let A be a weakly compact subset of X such that $A + S$ is convex and* ri $(A + S) \neq \emptyset$*. Let $g : X \to R$ be an increasing, quasiconvex and continuous function. Assume that* $\mathrm{Min}(A, S) \subset \mathrm{PC}(A)$*. Then the set-valued mapping $H: A + S \to X$ defined by*

$$H(x) = \{y \in G(x) \,|\, g(y) \leq V(x)\},$$

where $V(x) = \inf\{g(z) \,|\, z \in G(x)\}$, is upper semicontinuous at any $x \in ri(A + S) \cup$ and for every $x \in A + S$, $H(x)$ is a compact convex subset of $\mathrm{Min}(A, S)$.

Proof. Since g is increasing, we have

$$V(x) = \inf\{g(y) \,|\, y \in G(x)\} = \inf\{g(y) \,|\, y \in (x - S) \cap A\}.$$

Since g is quasiconvex and continuous, every level set of g is closed and convex, hence weakly closed. It means that g is weakly lower semicontinuous. This, with the weak compactness of A, implies that g attains its minimum on the set $(x - S) \cap A$. Hence, for each $x \in A + S$, $H(x)$ is nonempty.

We show that for all $x \in A + S$, $H(x) \subset \mathrm{Min}(A, S)$. Assume on the contrary that $y \in H(x) \setminus \mathrm{Min}(A, S)$, i.e., there exists $y_1 \in A$ such that $y_1 \in y - S \setminus \{0\}$. Hence $y_1 \in (y - S) \cap A \subset G(x)$. Since g is increasing, $g(y_1) < g(y) \leq V(x)$, a contradiction.

It is easily seen that $H(x)$ is closed and convex. Hence $H(x)$ is weakly compact. Since $H(x) \subset \mathrm{Min}(A, S) \subset \mathrm{PC}(A)$, $H(x)$ is compact.

Suppose on the contrary that there exists an $x_0 \in \mathrm{ri}(A + S) \cup \mathrm{Min}(A, S)$ such that H is not upper semicontinuous at x_0. There exist an open set Q containing $H(x_0)$ and a net $y_\alpha \in H(x_\alpha) \setminus Q$, where $x_\alpha \to x_0$. We have $g(y_\alpha) = V(x_\alpha)$, and

$$y_\alpha = x_\alpha - c_\alpha = a_\alpha + c^1_\alpha\,,$$

where $a_\alpha \in A$, c_α, $c^1_\alpha \in S$. Since $y_\alpha \in \mathrm{Min}(A, S)$, $y_\alpha = a_\alpha$.

By the weak compactness of A, the net $\{y_\alpha\}$ tends weakly to some $y_0 \in A$. Hence c_α tends weakly to $x_0 - y_0$. By the weak closedness of the cone S, we have $x_0 - y_0 \in S$, and thus $y_0 \in G(x_0)$. By Theorem 4 of [8] (see also Lemma 3.1 of [32]), $G(x)$ is lower semicontinuous at every point of the set $\mathrm{ri}(A+S) \cup \mathrm{Min}(A,S)$. Since g is continuous, by Theorem 1 of [10] (pp. 115), V is upper semicontinuous at every point of the set $\mathrm{ri}(A+S) \cup \mathrm{Min}(A,S)$. Hence, $V(x_0) \geq \limsup V(x_\alpha) = \limsup g(y_\alpha)$. On the other hand, since g is weakly lower semicontinuous,

$$\liminf g(y_\alpha) \geq g(y_0)$$

and consequently,

$$g(y_0) \leq V(x_0).$$

This, together with $y_0 \in G(x_0)$, implies that $y_0 \in H(x_0)$.

Since, by assumption, $y_0 \in \mathrm{PC}(A)$, y_α must tend strongly to y_0, contradictory to the choice $y_\alpha \notin Q$. □

Lemma 2 ([8]) *Let X be a locally convex space and let $S \subset X$ be a closed convex pointed cone in X. Let A be a weakly compact subset of X such that $A+S$ is convex and* ri $(A+S) \neq \emptyset$*. Let $g: X \to R$ be an increasing, strictly quasiconvex and continuous function. Assume that* $\mathrm{Min}(A,S) \subset \mathrm{PC}(A)$*. Then the mapping $H: A+S \to X$ defined in Lemma 1 is single-valued and continuous at any $x \in ri(A+S) \cup \mathrm{Min}(A,S)$.*

Proof. From Lemma 1, we need only to show that $H(x)$ consists a single point. Indeed, let t_0 be the minimal value of g on $G(x)$ and let $y, z \in G(x)$ with $g(y) = g(z) = t_0$. Since $A+S$ is convex, $G(x)$ is a convex set. Hence $\frac{y+z}{2} \in G(x)$. If $y \neq z$, by the strictly quasiconvexity of g, $g(\frac{y+z}{2}) < t_0$. This is a contradiction. □

If X is a strictly convex normed space and S is a convex cone with a bounded base, a function g satisfying the assumptions of Lemma 2 was constructed by Luc (see [32]).

Theorem 5 *Let X be a locally convex space and let $S \subset X$ be a closed convex pointed cone in X. Let A be a weakly compact subset of X, let $A+S$ be convex and let* ri $(A+S) \neq \emptyset$*. If there exists an increasing, quasiconvex and continuous real-valued function defined on X, and* $\mathrm{Min}(A,S) \subset \mathrm{PC}(A)$, *then* $\mathrm{Min}(A,S)$ *is R_δ-contractible.*

Proof. Define a set-valued mapping $\Gamma: \mathrm{Min}(A,S) \times [0,1] \to \mathrm{Min}(A,S)$ by

$$\Gamma(x,t) = H(tx + (1-t)a),$$

where a is a fixed element in $\mathrm{ri}(A+S)$. By Lemma 1, it is clear that Γ is upper semicontinuous in both variables with compact convex images and for every $x \in \mathrm{Min}(A,S)$, $\Gamma(x,0) = H(a) \subset \mathrm{Min}(A,S)$. For every $x \in \mathrm{Min}(A,S)$, note that $G(x) = \{x\}$, we have $H(x) = \{x\}$, and hence $\Gamma(x,1) = H(x) = \{x\}$. Therefore $\mathrm{Min}(A,S)$ is R_δ-contractible. □

By applying Lemma 2, we obtain:

Theorem 6 ([8]) *Let X be a locally convex space and let $S \subset X$ be a closed convex pointed cone in X. Let A be a weakly compact subset of X such that $A+S$ is convex and* ri $(A+S) \neq \emptyset$. *If there exists an increasing, strictly quasiconvex and continuous real-valued function defined on X, and $\mathrm{Min}(A,S) \subset \mathrm{PC}(A)$, then $\mathrm{Min}(A,S)$ is contractible.*

5 Application of PC points to stability in vector optimization

Lower continuity of minimal point set has been investigated by several authors (see [14], [2], [36], [20], [5] and others). In this section the starting point is a general result concerning lower semicontinuity of minimal point set. The crucial requirement for this result to hold is the density of strictly minimal points (see [6]) in the set of minimal points. Sufficient conditions assuring this density can be expressed in terms of points of continuity (PC points).

Let S be a closed convex pointed cone in a Hausdorff topological vector space X. Following [36] we say that Γ is *$S-$lower continuous* ($S-$l.c.) at u_0 if for every $x_0 \in \Gamma(u_0)$ and $0-$neighbourhood W there exists a neighbourhood U_0 of u_0 such that $\Gamma(u) \cap (x_0 + W - S) \neq \emptyset$ for $u \in U_0$. We say that Γ is *upper Hausdorff continuous* (u.H.c.) at u_0 if for each $0-$neighbourhood W of X there exists a neighbourhood U_0 of u_0 such that $\Gamma(u) \subset \Gamma(u_0) + W$ for $u \in U_0$.

Let A be a subset of X. The *domination property* (DP) holds for A if $A \subset \mathrm{Min}(A,S)+S$.

Following Bednarczuk [6] we say that $a_0 \in A$ is a *strict minimal point*, $a_0 \in \mathrm{SMin}(A,S)$, if for every $0-$neighbourhood W there exists a $0-$neighbourhood V such that

$$[A - a_0] \cap [V - S] \subset W.$$

Clearly, each strict minimal point is minimal, and not conversely.

Let $\Gamma : U \to X$ be a multivalued mapping. By $\mathcal{M} : U \to X$ we denote the multivalued mapping defined by $\mathcal{M}(u) = \mathrm{Min}(\Gamma(u), S)$. Exploiting the notion of strict minimality we can prove the following theorem.

Theorem 7 ([6]) *Let S be a closed convex pointed cone in X, and $x_0 \in \mathrm{Min}(\Gamma(u_0), S)$. Assume that*

(i)

$$x_0 \in cl(\mathrm{SMin}(\Gamma(u_0), S)) \tag{4}$$

(ii) (DP) holds for all $\Gamma(u)$ in a certain neighbourhood U_0 of u_0,

(iii) Γ is $S-$l.c. and u.H.c. at u_0,

Then $\mathcal{M}$ is l.s.c. at (x_0, u_0).

By using the concept of PC points we can formulate the following sufficient conditions for lower semicontinuity of $\mathrm{Min}(\Gamma(u), S)$.

Theorem 8 *Let S be a closed convex pointed cone in X and let $\Gamma(u_0) \subset X$ be a weakly compact subset of X. Assume that $\mathrm{Min}(\Gamma(u_0, S) \neq \emptyset$. Suppose that*

(i) either $\mathrm{Min}(\Gamma(u_0), S) \subset \mathrm{PC}(\Gamma(u_0))$ or $0 \in \mathrm{PC}(S)$,

(ii) (DP) holds for all $\Gamma(u)$ in a certain neighbourhood U_0 of u_0,

(iii) Γ is $S-$l.c. and u.H.c. at u_0.

Then $\mathcal{M}$ is l.s.c. at u_0.

Proof. According to Theorem 7, it is enough to show that $\mathrm{Min}(\Gamma(u_0), S) \subset \mathrm{cl}(\mathrm{SMin}(\Gamma(u_0), S))$. We show that under our assumptions

$$\mathrm{Min}(\Gamma(u_0), S) \subset \mathrm{SMin}(\Gamma(u_0), S).$$

Let $a_0 \in \mathrm{Min}(\Gamma(u_0), S)$ and suppose that $a_0 \notin \mathrm{SMin}(\Gamma(u_0), S)$. By definition, there exists a $0-$neighbourhood $\overline{W}$ such that for any $0-$neighbourhood V one can find $q_v \in V$, $x_v \in \Gamma(u_0)$, $c_v \in S$, satisfying

$$x_v - a_0 = q_v - c_v \notin \overline{W},$$

where the net $\{q_v\}$ tends to 0. Since $\Gamma(u_0)$ is weakly compact, the net $\{x_v\}$ tends weakly to some $a \in \Gamma(u_0)$, and since $a_0 \in \mathrm{Min}(\Gamma(u_0), S)$, we have $a = a_0$.

(i) If $a_0 \in \mathrm{PC}(\Gamma(u_0))$, then $x_v \to a_0$, contradictory the choice $x_v - a_0 \notin \overline{W}$. (ii) If $0 \in \mathrm{PC}(S)$, since $\{c_v\} = q_v - x_v + a_0$ tends weakly to 0, $\{c_v\}$ tends strongly to 0. Consequently, $x_v \to a_0$, contradictory to the choice $x_v - a_0 \notin \overline{W}$. □

REFERENCES

[1] K. J. Arrow, E. W. Barankin, D. Blackwell. Admissible points of convex sets. In: Contributions to the Theory of Games, (II) (eds. H. W. Huhn and A. W. Tucker), Princeton University Press, Princeton, New Jersey, 1953, pp. 87-92

[2] H. Attouch, H. Riahi. Stability results for Ekeland's ε-variational principle and cone extremal solutions. *Math. Oper. Res.* **18** (1993), 173-201

[3] J. P. Aubin, I. Ekeland. Applied Nonlinear Analysis. Wiley-Intersciences, New York, 1984.

[4] E. J. Balder. From weak to strong convergence in L_1-spaces via K-convergence. *Ann. Mat. Pura. Appl.* **165** (1993), 337-349.

[5] E. M. Bednarczuk. Berge-type theorems for vector optimization problems. *Optimization* **32** (1995), 373-384

[6] E. Bednarczuk. A note on lower continuity of minimal points (submitted).

[7] E. Bednarczuk, W. Song. Some more density results for proper efficiencies. *J. Math. Anal. Appl.* (to appear).

[8] E. Bednarczuk, W. Song. On the contractibility of the efficient set (submitted).

[9] H. Benabdellah. Extrémalité et entaillabilité sur des convexes fermés non nécessairement bornes d'un espace de Banach. Caracteristations dans le cas des espaces integraux. *Sem. Anal. Convex.* **21** (1991) Exp No 5.

[10] C. Berge. Topological Spaces. Macmillan Co., New York, 1963.

[11] G. R. Bitran, T. L. Magnanti. The structure of admissible points with respect to cone dominance. *J. Optim. Theory Appl.* **29** (1979), 573-614.

[12] J. Bourgain. Dentability and finite-dimensional decompositions. *Studia Math.* **67** (1979), 135-148.

[13] J. P. Dauer, R. J. Gallagher. Positive proper efficient points and related cone results in vector optimization theory. *SIAM J. Cont. Optim.* **28** (1990), 158-172.

[14] S. Dolecki, C. Malivert. Stability of efficient sets: continuity of mobile polarities. *Nonlinear Anal.* **12** (1988), 1461-1484.

[15] F. Ferro. General form of the Arrow-Barankin-Blackwell theorem in normed spaces and the l^∞-case. *J. Optim. Theory Appl.* **79** (1993), 127-138.

[16] W. Fu. On the density of proper efficient points. *Proc. Amer. Math. Soc.* **124** (1996), 1213-1217.

[17] R. J. Gallagher, O. A. Saleh. Two generalizations of a theorem of Arrow, Barankin, and Blackwell. *SIAM J. Control Optim.* **31** (1993), 247-256.

[18] X. H. Gong. Connectedness of efficient solutions sets for set-valued maps in normed spaces. *J. Optim. Theory and Appl.* **83** (1994), 83-96.

[19] X. H. Gong. Density of the set of positive proper minimal points in the set of minimal points. *J. Optim. Theory Appl.* **86** (1995), 609-630.

[20] V. V. Gorokhovik, N. N. Rachkovski. On stability of vector optimization problems. *Wesci ANB* **2**, (1990), 3-8 (in Russian)

[21] L. Górniewicz. Topological approach to differential inlusions. Preprint, Gdańsk University, 1994.

[22] R. Hartley. On cone efficiency, cone convexity, and cone compactness. *SIAM J. Appl. Math.* **34** (1978), 211-222.

[23] J. B. Hiriart-Urruty. Images of connected set by semicontinuous multifunctions. *J. Math. Anal. Appl.* **111** (1985), 407-422.

[24] R. C. James. KMP, RNP and PCP for Banach spaces. *Contemp. Math.* **85** (1989), 281-317.

[25] G. Jameson. Ordered Linear Spaces. Springer-Verlag, New York Berlin, 1970.

[26] J. Jahn. Mathematical Vector Optimization in Partially Order Linear Spaces. Lang, Frankfurt, 1986.

[27] J. Jahn. A generalization of a theorem of Arrow, Barankin, and Blackwell. *SIAM J. Control Optim.* **26** (1988), 999-1005.

[28] B. L. Lin, P. K. Lin, S. L. Troyanski. Some geometric and topological properties of the unit sphere in a Banach space. *Math. Ann.* **274** (1986), 613-616.

[29] B. L. Lin, P. K. Lin, S. L. Troyanski. Characterizations of denting points. *Proc. Amer. Math. Soc.* **102** (1988), 526-528.

[30] B. L. Lin, P. K. Lin, S. L. Troyanski. Some geometric and topological properties of the unit sphere in a normed linear space. *Contemp. Math.* **85** (1989), 339-344.

[31] D. T. Luc. Theory of Vector Optimization. Springer-Verlag, 1989.

[32] D. T. Luc. Contractibility of efficient points sets in normed spaces. *Nonlinear Anal.* **15** (1990), 527-535.

[33] E. K. Makarov, N. N. Rachkowski. Density theorems for generalized Henig proper efficiency. *J. Optim. Theory Appl.* **91** (1996), 419-437.

[34] P. H. Naccache. Connectedness of the set of nondominated outcomes in multicriteria optimization. *J. Optim. Theory and Appl.* **25** (1978), 459-467.

[35] B. Peleg. Topological properties of the efficient point set. *Proc. Amer. Math. Soc.* **35** (1972), 531-536.

[36] J.-P. Penot, A. Sterna-Karwat. Parametrised multicriteria optimisation: continuity and closedness of optimal multifunctions. *J. Math. Anal. Appl.* **120**, (1986), 150-168.

[37] A. L. Peressini. Ordered Topological Vector Spaces. Harper and Row, 1967.

[38] M. Petschke. On a theorem of Arrow, Barankin, and Blackwell. *SIAM J. Control Optim.* **28** (1990), 395-401.

[39] R. R. Phelps. Convex functions, differentiability and monotone operators. In: Lecture Notes in Math., vol. **1364**, Springer-Verlag, New-York, 1989.

[40] H. Rosenthal. On the structure of non-dentable closed bounded convex sets. *Adv. Math.* **70** (1988), 1-58.

[41] W. Song. A note on connectivity of efficient point sets. *Arch. Math.* **65** (1995), 540-545.

[42] W. Song. Connectivity of efficient solution sets in vector optimization of set-valued mappings. *Optimization* **39** (1997), 1-11.

[43] W. Song. On the connectivity of efficient point sets. *Appl. Math.* **25** (1998), 121-127.

[44] W. Schachermayer. The Radon-Nikodym property and the Krein-Milman property are equivalent for strongly regular sets. *Trans. Amer. Math. Soc.* **303** (1987), 673-687.

[45] H. H. Schaefer. Topological Vector Spaces. Springer-Verlag, New York, Heidelberg, Berlin, 1971.

[46] A. R. Warburton. Quasiconcave vector maximization: connectedness of the sets of Pareto-optimal and weak Pareto-optimal alternatives. *J. Optim. Theory and Appl.* **40** (1983), 537-557.

Ewa Bednarczuk
Systems Research Institute
01-447 Warsaw, Newelska 6
Poland
e-mail: `bednarcz@ibis2.ibspan.waw.pl`

Wen Song
Harbin Normal University
Harbin, China
and
Systems Research Institute
01-447 Warsaw, Newelska 6
Poland