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Pliska Stud. Math. Bulgar. **21** (2012), 7–24

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OPERATORS WITH POLYNOMIAL COEFFICIENTS AND GENERALIZED GELFAND-SHILOV CLASSES

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Abstract. We study the problem of the global regularity for linear partial differential operators with polynomial coefficients. In particular for multi-quasi-elliptic operators we prove global regularity in generalized Gelfand-Shilov classes. We also provide counterexamples of globally regular operators which are not multi-quasi-elliptic.

1. Introduction. Aim of this paper is to study the global regularity of the solutions for partial differential equations with polynomial coefficients in $\mathbf{R}^n$

$$Au = f \ ,$$

where

$$A = \sum_{|\alpha|+|\beta|\leq m} a_{\alpha\beta}\, x^\beta\, D^\alpha \ , \quad a_{\alpha\beta} \in \mathbf{C}, \quad D^\alpha = (-i)^{|\alpha|}\partial^\alpha \ . \tag{1}$$

In Nicola-Rodino [21] different sufficient conditions on the symbol

$$a(x,\xi) = \sum_{|\alpha|+|\beta|\leq m} a_{\alpha\beta}\, x^\beta\, \xi^\alpha \tag{2}$$

2010 *Mathematics Subject Classification:* Primary 35S05, 35J60; Secondary 35A20, 35B08, 35B40.

Key words: global hypoellipticity, Gelfand-Shilov spaces, multi-quasi-elliptic operators.

are reviewed, proving global regularity in the Schwartz spaces $S(\mathbf{R}^n)$, $S'(\mathbf{R}^n)$, namely: if $u \in S'(\mathbf{R}^n)$ and $Au \in S(\mathbf{R}^n)$, then $u \in S(\mathbf{R}^n)$. In particular, this type of global regularity is granted assuming Hörmander's property on the polynomial $a(z)$, $z = (x, \xi) \in \mathbf{R}^{2n}$, in (2):

$$|\partial_z^\gamma a(z)| \leq C|a(z)|\,\langle z\rangle^{-\rho|\gamma|}, \quad |z| \geq R, \tag{3}$$

for some ρ with $0 < \rho \leq 1$, $\langle z\rangle = (1 + |z|^2)^{\frac{1}{2}}$, $\gamma \in \mathbf{N}^{2n}$, and C, R positive constants. Relevant classes of polynomial $a(z)$ satisfying (3) are given, with increasing order of generality, by the elliptic, quasi-elliptic, and multi-quasi-elliptic polynomials, cf. Boggiatto-Buzano-Rodino [1]. On the other hand, for elliptic and quasi-elliptic symbol $a(z)$, the regularity in the Schwartz spaces of the operator A in (1), can be improved in terms of Gelfand-Shilov classes, see Cappiello-Gramchev-Rodino [9, 10]. Main subject of the present paper, in the Section 3, will be to obtain a similar improvement of regularity for operators with multi-quasi-elliptic symbols. To this end, we will introduce first a generalization of the standard Gelfand-Shilov classes and then, following the proceeding in Gramchev-Pilipovich-Rodino [17] we shall provide in this functional frame a result of regularity for the more general problem of the iterates. In Section 4 we shall produce an example of operator A in dimension $n = 1$, of the form

$$A = D^m - x^q + ix^t D^r \ , \tag{4}$$

which satisfies (3), but which is not multi-quasi-elliptic, see De Donno-Oliaro [13] for a similar result, in a different contest. Since (3) is verified, the operator (4) is globally regular in the Schwartz space, whereas the corresponding Gelfand-Shilov regularity remains an interesting open problem. In fact, we do not know exactly how relate the parameter ρ in (3) to Gelfand-Shilov regularity. Instead, in the next Section 2 we present a short survey on Gevrey and Gelfand-Shilov classes.

2. Definitions and first properties. Let us begin by recalling the definition of Gevrey classes $G^s(\Omega)$, $1 < s < \infty$, Ω open subset of $\mathbf{R}^n$, and Gelfand-Shilov classes $S^s_r(\mathbf{R}^n)$, with $s > 0$, $r > 0$, $s + r \geq 1$.
A function f belongs to $G^s(\Omega)$ if for every compact subset $K \subset\subset \Omega$ we have

$$\sup_{x\in K} |\partial_x^\alpha f(x)| \leq C^{|\alpha|+1}(\alpha!)^s, \quad \forall \alpha \in \mathbf{N}^n,$$

for a suitable positive constant C independent of the multi-index α. We then define $G^s_0(\Omega) = G^s(\Omega) \cap C^\infty_0(\Omega)$. Passing to L^2-norms in $\mathbf{R}^n$, this is equivalent to say that for f with compact support we have for some $C < \infty$:

$$\|\partial_x^\alpha f\| \leq C^{|\alpha|+1} (\alpha!)^s, \quad \forall \alpha \in \mathbf{N}^n.$$

Willing to find a counterpart of the Schwartz space $\mathcal{S}(\mathbf{R}^n)$, we are then led to the classes of Gelfand-Shilov [15]. Namely, a function f belongs to the Gelfand-Shilov class $S_r^s(\mathbf{R}^n)$, if there exists a constant $C < \infty$ such that

$$\left\| x^\beta \partial_x^\alpha f \right\| \leq C^{|\alpha|+|\beta|+1} (\alpha!)^s (\beta!)^r, \quad \forall \alpha \in \mathbf{N}^n, \forall \beta \in \mathbf{N}^n. \tag{5}$$

According to [11], this definition is equivalent to the following one, seemingly weaker than (5). A function f belongs to the Gelfand-Shilov class $S_r^s(\mathbf{R}^n)$, if $f \in \mathcal{S}(\mathbf{R}^n)$ and there exists a constant $C < \infty$ such that f satisfies the following two conditions

$$\begin{array}{ll} (i) & \|\partial_x^\alpha f\| \leq C^{|\alpha|+1} (\alpha!)^s, \quad \forall \alpha \in \mathbf{N}^n, \\ (ii) & \left\| x^\beta f \right\| \leq C^{|\beta|+1} (\beta!)^r, \quad \forall \beta \in \mathbf{N}^n. \end{array} \tag{6}$$

The Gevrey classes $G^s(\Omega)$ have been generalized in different ways by several authors. Here we address in particular to the multi-anisotropic Gevrey classes, see Bouzar-Chaili [2, 3], Calvo [4], Calvo-Hakobyan [5], Gindikin-Volevich [16], Zanghirati [23, 24].
In short, we fix a complete polyhedron $\mathcal{P} \subset \mathbf{R}_+^n$. Let us denote

$$k(\alpha, \mathcal{P}) = \inf \left\{ t > 0 : t^{-1}\alpha \in \mathcal{P} \right\}, \quad \alpha \in \mathbf{R}_+^n,$$

and let μ be the formal order of $\mathcal{P}$, see the next section 3 for details. We may introduce the multi-anisotropic class with compact support $G_0^{s,\mathcal{P}}(\mathbf{R}^n)$, $s > 1$, of all the functions $f \in C_0^\infty(\mathbf{R}^n)$ satisfying for suitable $C < \infty$

$$\|\partial_x^\alpha f\| \leq C^{|\alpha|+1} k(\alpha, \mathcal{P})^{s\mu k(\alpha,\mathcal{P})}, \quad \forall \alpha \in \mathbf{N}^n. \tag{7}$$

We recapture the standard Gevrey classes $G_0^s(\mathbf{R}^n)$ when $\mathcal{P}$ is the polyhedron of vertices $\{0, me_j,\ j = 1, \ldots, n\}$ for some integer $m \geq 1$. Another relevant example is given by the anisotropic Gevrey classes, when $\mathcal{P}$ is the polyhedron of vertices $\{0, m_j e_j,\ j = 1, \ldots, n\}$ for some integers $m_j \geq 1$, see [23, 24]. In the next section 3 we shall present a Gelfand-Shilov version of the multi-anisotropic Gevrey classes. Namely, taking (7) as a model and fixing a complete polyhedron $\mathcal{P}$ in dimension $2n$, $\mathcal{P} \subset \mathbf{R}_+^{2n}$, we define $S^{\mathcal{P},s}(\mathbf{R}^n)$, $s \geq \frac{1}{2}$, as the subset of $\mathcal{S}(\mathbf{R}^n)$ of all the functions f satisfying

$$\left\| x^\beta \partial_x^\alpha f \right\| \leq C^{|\gamma|+1} k(\gamma, \mathcal{P})^{s\mu k(\gamma,\mathcal{P})}, \quad \forall \gamma = (\alpha, \beta) \in \mathbf{N}^{2n} \tag{8}$$

for some positive constant $C < \infty$. Main result in the following will be to show the equivalence of (8) with suitable estimates of type (6), for $x^\alpha \partial_x^\beta f(x)$; let us address to the next Theorem 1 for a precise statement. We leave to future papers possible applications to partial differential equations in $\mathbf{R}^n$ with polynomial coefficients, cf. Boggiatto-Buzano-Rodino [1], and a discussion of a generalization of the definition (8) to the case when $s < \frac{1}{2}$, which presents difficult problems of non-triviality for the class $S^{s,\mathcal{P}}(\mathbf{R}^n)$. For a different class of multi-anisotropic Gelfand-Shilov classes, we address to [6]. See also the bibliography in [22], about functions of Gevrey type, and in [8], about recent applications of Gelfand-Shilov classes to linear and non-linear partial differential equations.

3. Generalized Gelfand-Shilov classes and main results. To introduce our study of Gelfand-Shilov classes of multi-anisotropic type, we start by describing complete polyhedra and some related properties. For more properties and applications to the theory of partial differential equations, we can refer to [1, 2, 3, 4, 5, 14, 16, 23, 24]. Let $\mathcal{P}$ be a convex polyhedron in $\mathbf{R}^d$, then $\mathcal{P}$ can be obtained as convex hull of a finite set $\mathcal{V}(\mathcal{P}) \subset \mathbf{R}^d$ of convex-linearly-independent points, called the vertices of $\mathcal{P}$ and uniquely determined by $\mathcal{P}$. Moreover, if $\mathcal{P}$ has non-empty interior and the origin belongs to $\mathcal{P}$, there is a finite set $\mathcal{N}(\mathcal{P}) = \mathcal{N}_0(\mathcal{P}) \cup \mathcal{N}_1(\mathcal{P})$, with $|\nu| = 1$, $\forall \nu \in \mathcal{N}_0(\mathcal{P})$, such that

$$\mathcal{P} = \{z \in \mathbf{R}^d | \nu \cdot z \geq 0, \forall \nu \in \mathcal{N}_0(\mathcal{P}), \nu \cdot z \leq 1, \forall \nu \in \mathcal{N}_1(\mathcal{P})\},$$

$\mathcal{N}_1(\mathcal{P})$ is the set of the normal vectors to the faces of $\mathcal{P}$.

Definition 1. *A complete polyhedron is a convex polyhedron $\mathcal{P} \subset \mathbf{R}^d_+$ such that the following properties are satisfied*

1. $\mathcal{V}(\mathcal{P}) \subset \mathbf{N}^d$ *(i.e. all vertices have non-negative integer coordinates);*
2. *the origin* $(0, 0, \ldots, 0)$ *belongs to* $\mathcal{P}$*;*
3. $\mathcal{N}_0(\mathcal{P}) = \{e_1, e_2, \ldots, e_d\}$, *with* $e_j = (0, \ldots, 0, 1_{j-th}, 0, \ldots, 0) \in \mathbf{R}^d$, *for* $j = 1, \ldots, d$*;*
4. *every* $\nu \in \mathcal{N}_1(\mathcal{P})$ *has strictly positive components.*

Remark. The condition 4 implies that for every $x \in \mathcal{P}$ the set $Q(x) = \{y \in \mathbf{R}^d | 0 \leq y \leq x\}$ is included in $\mathcal{P}$ and if x belongs to a face of $\mathcal{P}$ and $y > x$, then $y \notin \mathcal{P}$ (where for $x, y \in \mathbf{R}^d$, $y \leq x$ means that $y_i \leq x_i$, $i = 1, \ldots, d$;

and $y < x$ means $y \leq x$, $y \neq x$). In the definition of Gelfand-Shilov classes in the sequel, we shall have $d = 2n$, i.e. we shall only need to consider $\mathcal{P}$ in even dimension d. Let us now summarize some notations related to a complete polyhedron $\mathcal{P}$: $k(\gamma, \mathcal{P}) = \inf\{t > 0 : t^{-1}\gamma \in \mathcal{P}\} = \max_{\nu \in \mathcal{N}_1(\mathcal{P})} \nu \cdot \gamma$, $\forall \gamma \in \mathbf{R}^d_+$; $\mu_j(\mathcal{P}) = \max_{\nu \in \mathcal{N}_1(\mathcal{P})} \nu_j^{-1}$; $\mu = \mu(\mathcal{P}) = \max_{j=1,\dots,d} \mu_j$ the formal order of $\mathcal{P}$; $\mu^{(0)} = \mu^{(0)}(\mathcal{P}) = \min_{\gamma \in \mathcal{V}(\mathcal{P}) \setminus \{0\}} |\gamma|$ the minimum order of $\mathcal{P}$; $\mu^{(1)} = \mu^{(1)}(\mathcal{P}) = \max_{\gamma \in \mathcal{V}(\mathcal{P})} |\gamma|$ the maximum order of $\mathcal{P}$. Finally, we define the weight function associated to $\mathcal{P}$:

$$|\xi|_{\mathcal{P}} := \left(\sum_{v \in \mathcal{V}(\mathcal{P})} |\xi^v| \right)^{\frac{1}{\mu}}, \quad \forall \xi \in \mathbf{R}^d. \tag{9}$$

It is a weight function according to the definition of Liess-Rodino [18]. The definition of the previous quantities is clarified by the following result (for the proof we refer to [4]).

Proposition 1. *Let $\mathcal{P}$ be a complete polyhedron in $\mathbf{R}^d$ with vertices $v^l = (v^l_1, \dots, v^l_d)$, for $l = 1, \dots, N(\mathcal{P})$. Then*

1. *for every $j = 1, 2, \dots, d$, there is a vertex v^{l_j} of $\mathcal{P}$ such that $v^{l_j} = v_j^{l_j} e_j$, $v_j^{l_j} = \max_{\gamma \in \mathcal{P}} \gamma_j =: m_j(\mathcal{P})$;*
2. *the boundary of $\mathcal{P}$ has at least one vertex lying outside the coordinate axes if the formal order $\mu(\mathcal{P})$ is greater than the maximum order $\mu^{(1)}(\mathcal{P})$;*
3. *if γ belongs to $\mathcal{P}$, then $|\xi^\gamma| \leq \sum_{l=1}^{N(\mathcal{P})} |\xi^{v^l}|$, $\forall \xi \in \mathbf{R}^d$, where $\xi^\gamma = \prod_{j=1}^d \xi_j^{\gamma_j}$ and $N(\mathcal{P})$ is the number of vertices of $\mathcal{P}$, including the origin;*
4. *$\dfrac{\gamma}{k(\gamma, \mathcal{P})}$, for any $\gamma \in \mathbf{N}^d$, belongs to the boundary of $\mathcal{P}$, and therefore $\gamma = k(\gamma, \mathcal{P}) \sum_{i=1}^m \lambda^i v^{l^i}$, $\lambda^i \geq 0$, $i = 1, \dots, m$, $\sum_{i=1}^m \lambda^i = 1$, where $v^{l^1}, \dots, v^{l^m}$ are the vertices of the face of $\mathcal{P}$ where $\dfrac{\gamma}{k(\gamma, \mathcal{P})}$ lies;*
5. *For all $\xi \in \mathbf{R}^d$, saying $N(\mathcal{P})$ the number of vertices of $\mathcal{P}$, the following inequality is satisfied $N(\mathcal{P})^{j-1} \sum_{v \in \mathcal{V}(\mathcal{P})} |\xi^{v_j}| \leq |\xi|^j_{\mathcal{P}} \leq 2^{N(\mathcal{P})(j-1)} \sum_{v \in \mathcal{V}(\mathcal{P})} |\xi^{v_j}|$, for any $j = 1, 2, \dots$.*

Proposition 2. *For any complete polyhedron $\mathcal{P}$ and any $s \in \mathbf{R}^d_+$, $k(\gamma, \mathcal{P})$ is bounded as follows:*

$$\frac{|\gamma|}{\mu^{(1)}} \leq k(\gamma, \mathcal{P}) \leq \frac{|\gamma|}{\mu^{(0)}}.$$

To clarify our treatment, we give now some examples of complete polyhedra (for more details cf. [4]).

1. Consider the complete polyhedron of vertices $\{0, me_j,\ j = 1, \dots, d\}$. The set $\mathcal{N}_1(\mathcal{P})$ is reduced to the point $\nu = m^{-1} \sum_{j=1}^{d} e_j$, and $m_j(\mathcal{P}) = \mu_j(\mathcal{P}) = \mu^{(0)}(\mathcal{P}) = \mu^{(1)}(\mathcal{P}) = \mu(\mathcal{P}) = m$, for all $j = 1, \dots, d$.

2. Consider the complete polyhedron $\mathcal{P}$ with vertices $\{0, m_j e_j,\ j = 1, \dots, d\}$, where $m_j = m_j(\mathcal{P})$ are fixed integers. The set $\mathcal{N}_1(\mathcal{P})$ is reduced to a point $\nu = \sum_{j=1}^{d} m_j^{-1} e_j$; then $\mu_j(\mathcal{P}) = m_j$, for all $j = 1, \dots, d$, $\mu^{(0)}(\mathcal{P}) = \min_{j=1,\dots,d} m_j$, $\mu(\mathcal{P}) = \mu^{(1)}(\mathcal{P}) = \max_{j=1,\dots,d} m_j$. It is the anisotropic case.

3. If $\mathcal{P} \subset \mathbf{R}^2$ is the polyhedron of vertices $\mathcal{V}(\mathcal{P}) = \{(0,0), (0,3), (1,2), (2,0)\}$, then $\mathcal{P}$ is complete and $\mathcal{N}_1(\mathcal{P}) = \left\{ \nu_1 = \left(\frac{1}{3}, \frac{1}{3} \right),\ \nu_2 = \left(\frac{1}{2}, \frac{1}{4} \right) \right\}$. We have $m_1(\mathcal{P}) = \mu^{(0)}(\mathcal{P}) = 2$, $m_2(\mathcal{P}) = m(\mathcal{P}) = \mu^{(1)}(\mathcal{P}) = 3$, $\mu(\mathcal{P}) = 4$. We observe that in this case the formal order $\mu(\mathcal{P})$ is bigger than the maximum order and $\mathcal{P}$ has a vertex lying outside the coordinate axes (cf. Proposition 1).

Basing on the definition of complete polyhedra, we now introduce the multi-anisotropic version of the standard Gelfand-Shilov classes [15], cf. the Introduction.

Definition 2. *Let $\mathcal{P}$ be a complete polyhedron in $\mathbf{R}^{2n}$. We say that a function f belongs to the Gelfand-Shilov class $S^{\mathcal{P},s}(\mathbf{R}^n)$, for $s \geq \frac{1}{2}$ if there is a constants $C < \infty$ such that*

$$\left\| x^\beta \partial_x^\alpha f \right\| \leq C^{|\gamma|+1} k\left(\gamma, \mathcal{P}\right)^{s\mu k(\gamma,\mathcal{P})}, \quad \forall \gamma = (\alpha, \beta) \in \mathbf{N}^{2n}. \tag{10}$$

We may note that polyhedra $\mathcal{P}$ and $\mathcal{P}'$, which are similar in the sense of the Euclidean geometry, define the same class $S^{\mathcal{P},s}(\mathbf{R}^n)$, since denoting μ and μ' the respective formal orders we have $\mu k\left(\gamma, \mathcal{P}\right) = \mu' k\left(\gamma, \mathcal{P}'\right)$. As first example, consider the polyhedron of vertices $\{0, me_j,\ j = 1, \dots, 2n\}$. By similarity, we may limit ourselves to the case $m = 1$. Since then $\mu = \mu^{(0)} = \mu^{(1)} = 1$, in view of Proposition 2 we have $k\left(\gamma, \mathcal{P}\right) = |\gamma|$, so that (10) reads

$$\left\| x^\beta \partial_x^\alpha f \right\| \leq C^{|\gamma|+1} \left|\gamma\right|^{s|\gamma|}.$$

From (5) and standard factorial estimates we obtain then for such $\mathcal{P}$:

$$S^{\mathcal{P},s}(\mathbf{R}^n) = S^s_s(\mathbf{R}^n), \quad s \geq \frac{1}{2}.$$

Before analysing other examples, it will be convenient to have equivalent definitions of $S^{\mathcal{P},s}(\mathbf{R}^n)$. Let us introduce, for $p \in \mathbf{N}$:

$$|f|_p = \sum_{\gamma=(\alpha,\beta)\,\in\, p\mathcal{P}} \left\| x^\beta \partial_x^\alpha f \right\|, \tag{11}$$

where $\gamma \in p\mathcal{P}$ means that $p^{-1}\gamma \in \mathcal{P}$, i.e. $k(\gamma, \mathcal{P}) \leq p$, and morever

$$|f|_p^* = \sum_{\gamma=(\alpha,\beta)\,\in\, p\mathcal{V}(\mathcal{P})} \left\| x^\beta \partial_x^\alpha f \right\|, \tag{12}$$

where $\gamma \in p\mathcal{V}(\mathcal{P})$ means that $\gamma = pv^l$ for some vertex v^l, $l = 1, \ldots, N(\mathcal{P})$. Our main result is the following.

Theorem 1. *For any $f \in \mathcal{S}(\mathbf{R}^n)$, the following conditions are equivalent:*

i) f belongs to $S^{\mathcal{P},s}(\mathbf{R}^n)$.

ii) There exists a constant $C < \infty$ such that

$$|f|_p \leq C^{p+1}(p!)^{s\mu}, \quad \forall p \in \mathbf{N}. \tag{13}$$

iii) There exists a constant $C < \infty$ such that

$$|f|_p^* \leq C^{p+1}(p!)^{s\mu}, \quad \forall p \in \mathbf{N}. \tag{14}$$

In the proof we shall use the following lemma.

Lemma 1. *There exists a constant $C < \infty$, depending on $\mathcal{P}$, such that for every $p \in \mathbf{N}$ and every $\gamma = (\alpha, \beta) \in p\mathcal{P}$ we have*

$$\left\| x^\beta \partial_x^\alpha f \right\| \leq C^{p+1}\left(\|f\|_p^* + (p!)^{\frac{\mu}{2}} \|f\| \right). \tag{15}$$

Proof. *Of Theorem 1.* First, observe that i) is equivalent to ii). In fact, if i) is satisfied, i.e. the estimates (10) are satisfied, for $\gamma = (\alpha, \beta) \in p\mathcal{P}$, i.e. $k(\gamma, \mathcal{P}) \leq p$, then we have

$$\left\| x^\beta \partial_x^\alpha f \right\| \leq C^{|\gamma|+1} k(\gamma, \mathcal{P})^{s\mu k(\gamma,\mathcal{P})} \leq C^{|\gamma|+1} p^{s\mu p}.$$

On the other hand $|\gamma| \leq \mu^{(1)} k(\gamma, \mathcal{P}) \leq \mu^{(1)} p$ by Proposition 2, and by standard factorial estimates we obtain for a new constant $C < \infty$:

$$\left\| x^\beta \partial_x^\alpha f \right\| \leq C^{p+1} (p\,!)^{s\mu}.$$

By observing that the number of the terms in the sum in (11) can be estimated by C^p for a constant $C < \infty$, we obtain ii). To prove ii) $\Rightarrow$ i), given $\gamma = (\alpha, \beta)$, take the integer p such that $p - 1 < k(\gamma, \mathcal{P}) \leq p$. Then $\gamma \in p\mathcal{P}$ and from (13) we have

$$\begin{aligned} \left\| x^\beta \partial_x^\alpha f \right\| &\leq C^{p+1} (p!)^{s\mu} \leq C_1^{p+1} (p-1)!^{s\mu} \\ &\leq C_1^{p+1} (p-1)^{s\mu(p-1)} \leq C_1^{p+1} k(\gamma, \mathcal{P})^{s\mu k(\gamma, \mathcal{P})} \end{aligned}$$

for a constant C_1 independent of p. Hence i) is satisfied. Let us now prove that ii) is equivalent to iii). That ii) $\Rightarrow$ iii) is obvious, since $\mathcal{V}(\mathcal{P}) \subset \mathcal{P}$. Assume that iii) is satisfied. Given $\gamma \in p\mathcal{P}$, we apply (15) in Lemma 1. Combining with (14), we have for a new constant C:

$$\left\| x^\beta \partial_x^\alpha f \right\| \leq C^{p+1} \left((p!)^{s\mu} + (p\,!)^{\frac{\mu}{2}} \|f\| \right).$$

At this moment we use the assumption $s \geq \dfrac{1}{2}$. Summing up in (11) for $\gamma \in p\mathcal{P}$, we obtain ii). Theorem 1 is proved.
□ The proof of Lemma 1 is omitted for brevity. A corresponding result in the case of standard Gelfand-Shilov semi-norms is in [7], Lemma 2.2; see also [17], Proposition 4.1. The proof of Lemma 1 follows the lines of [7], by using 3, 4, 5 in the preceding Proposition 1. Since the number of the vertices in $\mathcal{V}(\mathcal{P})$ is finite, from iii) in Theorem 1 we may obtain for the classes $S^{\mathcal{P},s}(\mathbf{R}^n)$ the following counterpart of the result of [11] for standard Gelfand-Shilov classes.

Corollary 1. *We have $f \in S^{\mathcal{P},s}(\mathbf{R}^n)$, $s \geq \dfrac{1}{2}$, if and only if there exists a constant $C < \infty$ such that*

$$\|x_1^{p\beta_1} \dots x_n^{p\beta_n} \partial_{x_1}^{p\alpha_1} \dots \partial_{x_n}^{p\alpha_n} f\| \leq C^{p+1} (p\,!)^{s\mu}, \quad \forall p \in \mathbf{N},$$

for every vertex $v = (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) \in \mathcal{V}(\mathcal{P})$, $v \neq 0$. As before, μ denotes the formal order of $\mathcal{P}$.

As a first example, consider the polyhedron $\mathcal{P}$ with vertices

$$\{0, m_1 e_1, \dots, m_n e_n, M_1 e_{n+1}, \dots, M_n e_{2n}\}$$

in $\mathbf{R}^{2n}$. The formal order is $\mu = \max\{m_1, \dots, m_n, M_1, \dots, M_n\}$. By Corollary 1, and after easy computations, we have that the function f belongs to the corresponding spaces $S^{\mathcal{P},s}(\mathbf{R}^n)$ if and only if for every $j = 1, \dots, n$:

$$\|\partial_{x_j}^p f\| \leq C^{p+1}(p\,!)^{\frac{s\mu}{m_j}}, \quad \forall p \in \mathbf{N}, \tag{16}$$

$$\left\| x_j^p f \right\| \leq C^{p+1}(p\,!)^{\frac{s\mu}{M_j}}, \quad \forall p \in \mathbf{N}. \tag{17}$$

We then recapture the anisotropic classes of Gelfand-Shilov [15]. In particular, under the assumptions $s, r \in \mathbf{Q}$, $r \geq s \geq \dfrac{1}{2}$, we obtain the classes $S_r^s(\mathbf{R}^n)$ defined in (5), by taking $m_1 = \cdots = m_n = m$, $M_1 = \cdots = M_n = M$, with m and M positive integers such that $\dfrac{r}{s} = \dfrac{m}{M}$. In the case when $\mathcal{P}$ has at least one vertex lying outside the coordinate axes, estimates (16) and (17) are not sufficient to characterize the class $S^{\mathcal{P},s}(\mathbf{R}^n)$. For example, consider as before the polyhedron of vertices $\mathcal{V}(\mathcal{P}) = \{(0,0), (0,3), (1,2), (2,0)\}$, with formal order $\mu = 4$. From Corollary 1 we have that the corresponding space $S^{\mathcal{P},s}(\mathbf{R})$, $s \geq \dfrac{1}{2}$, is defined by the estimates

$$\|f^{(p)}\| \leq C^{p+1}(p\,!)^{2s}, \quad \forall p \in \mathbf{N},$$

$$\|x^p f\| \leq C^{p+1}(p\,!)^{\frac{4s}{3}} \quad \forall p \in \mathbf{N},$$

to which we add the further condition

$$\|x^{2p} f^{(p)}\| \leq C^{p+1}(p\,!)^{4s}, \quad \forall p \in \mathbf{N}.$$

Let us now present our result of regularity for operators with polynomial coefficients. We write the symbol in the form

$$a(z) = \sum_{|\gamma| \leq m} a_\gamma\, z^\gamma\ , \quad z = (x, \xi) \in \mathbf{R}^{2n}\ , \quad \gamma \in \mathbf{N}^{2n}.$$

Consider the Newton Polyhedron $\mathcal{P}$ of $a(z)$, i.e. the convex hull of $\mathcal{Q} \bigcup \{0\}$ with

$$\mathcal{Q} = \left\{ \gamma \in \mathbf{N}^{2n}\ , \quad a_\gamma \neq 0 \right\}.$$

Definition 3. *We say that $a(z)$ is multi-quasi-elliptic if the corresponding Newton Polyhedron is complete, cf. Definition 1, and if*

$$|z|_{\mathcal{P}} \leq C\,|a(z)|\,, \quad |z| \geq R,$$

where $|z|_{\mathcal{P}}$ is defined as in (9), with C and R positive constants.

Multi-quasi-elliptic polynomials satisfy the Hörmander's estimates (3), see Boggiatto-Buzano-Rodino [1].

Theorem 2. *Let $a(z)$ be multi-quasi-elliptic, $z = (x,\xi) \in \mathbf{R}^{2n}$, and write A for the corresponding partial differential operator with polynomial coefficients in $\mathbf{R}^{2n}$. Let $\mathcal{P}$ be its complete Newton polyhedron and let $S^{\mathcal{P},s}(\mathbf{R}^n), s \geq \dfrac{1}{2}$, the generalized Gelfand-Shilov-classes as in Definition 2. Then $u \in S'(\mathbf{R}^n)$, $Au \in S^{\mathcal{P},s}(\mathbf{R}^n)$ imply $u \in S^{\mathcal{P},s}(\mathbf{R}^n)$. In particular all the solutions $u \in S'(\mathbf{R}^n)$ of $Au = 0$ belong to $S^{\mathcal{P},\frac{1}{2}}(\mathbf{R}^n)$.*

Theorem 2 will be a consequence of the following more general result, concerning the so-called problem of the iterates.

Theorem 3. *Let $a(z)$, A, $\mathcal{P}$, $S^{\mathcal{P},s}(\mathbf{R}^n), s \geq \dfrac{1}{2}$, be as in Theorem 2, and let be $\mu = \mu(\mathcal{P})$ the formal order of $\mathcal{P}$. Then $u \in S^{\mathcal{P},s}(\mathbf{R}^n)$ if and only if for some positive constant C, we have*

$$\|A^p u\| \leq C^{p+1}(p\,!)^{s\mu}, \quad \forall p \in \mathbf{N}. \tag{18}$$

In fact, if $Au = f$, where $f \in S^{\mathcal{P},s}(\mathbf{R}^n)$ then

$$\|A^p u\| = \|A^{p-1} f\| \leq C^{p+1}|f|_p \leq \tilde{C}^{p+1}(p\,!)^{s\mu},$$

in view of Theorem 1, *ii*), hence (18) is satisfied. Therefore Theorem 3 implies Theorem 2. In turn, to prove Theorem 3 we use the following two propositions. For $\mathcal{P}$ as before, we define $|f|^*_p$ as in (12), and $k(\gamma,\mathcal{P})$, $\gamma = (\alpha,\beta) \in \mathbf{N}^{2n}$ as in Definition 1 and sequel.

Lemma 2. *There exist a positive constant C such that for any given $p \in \mathbf{N}$, for every $\gamma = (\alpha,\beta) \in \mathbf{N}^{2n}$ with $p < k = k(\gamma,\mathcal{P}) < p+1$, and for every $\epsilon > 0$:*

$$\|x^\alpha D^\beta u\| \leq \epsilon\,|u|^*_{p+1} + C^p\,\epsilon^{-\frac{k-p}{n+1-k}}\,|u|^*_p + C^k\,k^{k\frac{\mu}{2}}\,\|u\|. \tag{19}$$

The proof is omitted for brevity. The counterpart of (19) in the elliptic case is proved in Calvo-Rodino [7], Proposition 2.1.

Lemma 3. *Let A be an operator with multi-quasi-elliptic symbol. Then there exists a positive constant C such that for every $v \in S(\mathbf{R}^n)$*

$$\sum_{\gamma=(\theta,\eta)\,\in\,\mathcal{V}(\mathcal{P})} \|x^\theta\, D^\eta\, v\| \le C\,(\,\|Av\| + \|v\|\,). \tag{20}$$

For the proof we address to Boggiatto-Buzano-Rodino [1].

P r o o f o f, T h e o r e m 3. We shall limit ourselves to a sketch of the proof. Note first that, if $u \in S^{\mathcal{P},s}(\mathbf{R}^n)$, then the estimates (18) are obviously satisfied, since as before we apply Theorem 1, *ii*). In the opposite direction, let us assume formulas (18) and prove that $u \in S^{\mathcal{P},s}(\mathbf{R}^n)$. In view of Theorem 1, *iii*), it will be sufficient to check the boundedness of the sequence

$$\sigma_p(u,\lambda) = (p\,\mu)!\,\lambda^{-p}\,|u|^*_p\,, \quad p = 0,1,\ldots$$

for λ sufficiently large. The basic step is to prove the recurrence estimate

$$\sigma_{p+1}(u,\lambda) \le [(p\,\mu+1)\cdots(p\,\mu+\mu)]^{-s}\sigma_p(Au,\lambda) + \sigma_p(u,\lambda) + \sigma_{p-1}(u,\lambda) + \sigma_0(u,\lambda).$$

This is obtained by applying to each term $x^\delta D_x^\gamma u$, $\gamma = (\alpha,\beta) \in (p+1)\,\mathcal{V}(\mathcal{P})$, the estimates in Lemma 3. Namely, we take $(\gamma,\delta) \in p\,\mathcal{V}(\mathcal{P})$ so that $(\alpha-\gamma,\beta-\delta) \in \mathcal{V}(\mathcal{P})$, and then apply (20) to $v = x^\delta\, D_x^\gamma u$, with $\theta = \beta-\delta$, $\eta = \alpha-\gamma$. We now write $Av = x^\delta\, D^\gamma\, Au + [A,\, x^\delta\, D^\gamma]u$ and estimate finally the terms in the commutators by Lemma 2. At this moment the proceeding is the same as in Calvo-Rodino [7] and Gramchev-Pilipovic-Rodino [17], so we omit further details. □

4. A hypoelliptic polynomial, which is not multi-quasi-elliptic.

This section regards with the global regularity in Schwartz space for the operator, in dimension $n = 1$,

$$A = D^m - x^q + i x^t D^r\,, \tag{21}$$

where $m,q,r,t \in \mathbf{N}$, $m \ge 1$, $1 \le q \le m$, $1 \le r+t \le m$.

Let

$$a(x,\xi) = \xi^m - x^q + i x^t \xi^r, \quad (x,\xi) \in \mathbf{R}^2, \tag{22}$$

be the symbol associated to the differential operator A with polynomial coefficients, in (21). In order to check the Hörmander's conditions (3) for the symbol in (22), we consider the following equivalent conditions listed by Hörmander in [19]:

1) $\forall \epsilon > 0, \quad \dfrac{|\partial_z^\gamma a(z)|}{1+|a(z)|} < \epsilon, \; z=(x,\xi) \in \mathbf{R}^{2n}, \; |z| > R, \forall \gamma \in \mathbf{N}^{2n}, \; R = R(\epsilon) > 0;$

2) $|\partial_z^\gamma \, a(z)| \le C|a(z)| \, \langle z \rangle^{-\rho|\gamma|}, \; |z| \ge R,$ for some $\rho, \; 0 < \rho \le 1, \; C > 0, \; R > 0.$

In order to obtain the condition 1), Hörmander showed in [19, 20] that it suffices to consider only the first order derivatives of the symbol a; see also an alternative proof in De Donno [12]. Then, in the case of the symbol $a(x,\xi)$ in (22), the property 1) is equivalent to the conditions:

$$(23) \qquad i) \; \frac{|a_\xi(x,\xi)|^2}{|a(x,\xi)|^2} < \epsilon \quad \text{and} \quad ii) \; \frac{|a_x(x,\xi)|^2}{|a(x,\xi)|^2} < \epsilon \,, \quad x^2+\xi^2 \ge R.$$

Now, we shall prove the global regularity in Schwartz space of the operator (21) by proving the two conditions in (23) . The conditions $i)$ and $ii)$ in (23) will be studied separately in the following three regions of the plane $\Pi_{x,\xi}$ of axes x, ξ:

$$\begin{aligned} \text{I)} \quad & c|x|^q < |\xi|^m < C|x|^q, \\ \text{II)} \quad & |\xi|^m \ge C|x|^q, \\ \text{III)} \quad & |\xi|^m \le c|x|^q, \end{aligned}$$

where $C > 2$ and $c < \dfrac{1}{2}$. Let us limit attention, for simplicity, to the cases $x \ge 0$, and $\xi \ge 0$.

We start to prove the condition $i)$ in (23) regarding the first derivative with respect to ξ:

$$\frac{|a_\xi(x,\xi)|^2}{|a(x,\xi)|^2} = \frac{m^2\xi^{2(m-1)} + r^2x^{2t}\xi^{2(r-1)}}{(\xi^m - x^q)^2 + x^{2t}\xi^{2r}}, \quad r \ge 1, \quad t \ge 0.$$

By using the inequality $(\xi^m - x^q)^2 + x^{2t}\xi^{2r} \ge \xi^{2r}x^{2t}$, and the second part of I), we obtain:

$$(24) \quad \begin{aligned} \frac{m^2\xi^{2(m-1)} + r^2x^{2t}\xi^{2(r-1)}}{(\xi^m - x^q)^2 + x^{2t}\xi^{2r}} &\le m^2 \frac{\xi^{2(m-1)}}{x^{2t}\xi^{2r}} + \frac{r^2}{\xi^2} \\ &< \text{const} \, \frac{\xi^{2(m-1)}}{\xi^{2r+2\frac{mt}{q}}} + \frac{r^2}{\xi^2} \longrightarrow 0, \qquad \xi \to \infty, \end{aligned}$$

provided $r + \dfrac{mt}{q} > m-1$, i.e. $qr + mt > q(m-1)$, for all $r \geq 1$ and $t \geq 0$. We have set const $= \dfrac{m^2}{C^{\frac{2t}{q}}}$. Here and in the next pages we use const for all the constants in the formulas. Formula (24) is satisfied also for $r = 0, (t \geq 1)$.

In the region II), we get $(\xi^m - x^q)^2 + x^{2t}\xi^{2r} \geq \left(1 - \dfrac{2}{C}\right)\xi^{2m} + x^{2t}\xi^{2r}$, so we have:

$$(25) \quad \frac{m^2\xi^{2(m-1)} + r^2x^{2t}\xi^{2(r-1)}}{(\xi^m - x^q)^2 + x^{2t}\xi^{2r}} \leq \frac{m^2\xi^{2(m-1)}}{(1-\frac{2}{C})\xi^{2m} + x^{2t}\xi^{2r}} + \frac{r^2x^{2t}\xi^{2(r-1)}}{(1-\frac{2}{C})\xi^{2m} + x^{2t}\xi^{2r}};$$

by removing $x^{2t}\xi^{2r}$ in the first part at the right-hand side of (25) and ξ^m in the second part, we may further estimate by:

$$\text{const}\,\frac{1}{\xi^2} \to 0, \quad \xi \to \infty, \quad \forall r \geq 1, \quad \forall t \geq 0.$$

The conclusion remains valid for $r = 0, (t \geq 1)$, too.

In the region III) we have $(\xi^m - x^q)^2 + x^{2t}\xi^{2r} \geq (1-2c)x^{2q} + x^{2t}\xi^{2r}$, and we can estimate as:

$$(26) \quad \frac{m^2\xi^{2(m-1)} + r^2x^{2t}\xi^{2(r-1)}}{(\xi^m - x^q)^2 + x^{2t}\xi^{2r}} \leq \frac{m^2\xi^{2(m-1)}}{(1-2c)x^{2q} + x^{2t}\xi^{2r}} + \frac{r^2x^{2t}\xi^{2(r-1)}}{(1-2c)x^{2q} + x^{2t}\xi^{2r}}.$$

By using again inequality III) at the numerator in the first part of the right-hand side of (26), and factoring out x^{2t} at the denominator in the second part, we further estimate by:

$$\frac{\text{const}\,x^{2q\frac{m-1}{m}}}{(1-2c)x^{2q} + x^{2t}\xi^{2r}} + r^2\frac{\xi^{2(r-1)}}{(1-2c)x^{2(q-t)} + \xi^{2r}},$$

and hence by

$$(27) \quad \text{const}\,\frac{1}{x^{2\frac{q}{m}}} + r^2\frac{\xi^{2(r-1)}}{(1-2c)x^{2(q-t)} + \xi^{2r}} \to 0,$$
$$x \to \infty, \quad \forall r \geq 1, \quad t \geq 0, \quad t < q.$$

To handle the second term in (27) we have used the following lemma:

Lemma 4. *For all $\alpha,\beta,\gamma,\delta \in \mathbf{N}$, with $\gamma,\delta \neq 0$, $x+\xi \to \infty$, $\xi \geq 0$, $x \geq 0$, we have:*

$$\frac{x^\alpha \xi^\beta}{x^{2\gamma}+\xi^{2\delta}} \to 0 \Leftrightarrow (2\gamma-\alpha)(2\delta-\beta) > \alpha\beta.$$

The proof is direct and we omit it. Formula (27) holds for $r = 0, (t \geq 1)$, too.

Now we study the condition *ii)* in (23) involving the derivative with respect x of the symbol $a(x,\xi)$. By starting from region I) we have as above:

$$\frac{q^2x^{2(q-1)}+t^2x^{2(t-1)}\xi^{2r}}{(\xi^m-x^q)^2+x^{2t}\xi^{2r}} < \text{const}\,\frac{x^{2(q-1)}}{x^{2r\frac{q}{m}+2t}}+\frac{t^2}{x^2} \longrightarrow 0,\ x\to\infty$$

provided $t+\dfrac{rq}{m} > q-1$, i.e. $qr+mt > m(q-1)$, for $r+t \geq 1$, which is less restrictive than what required for formula (24), since $m \geq q$. For region II) we get:

$$(28)\qquad \begin{aligned} \frac{q^2x^{2(q-1)}+t^2x^{2(t-1)}\xi^{2r}}{(\xi^m-x^q)^2+x^{2t}\xi^{2r}} &\leq \text{const}\,\frac{\xi^{2m\frac{q-1}{q}}}{(1-\frac{2}{C})\xi^{2m}+x^{2t}\xi^{2r}}+t^2\frac{x^{2(t-1)}\xi^{2r}}{(1-\frac{2}{C})\xi^{2m}+x^{2t}\xi^{2r}} \\ &\leq \text{const}\,\frac{1}{\xi^{2\frac{m}{q}}}+t^2\frac{x^{2(t-1)}}{(1-\frac{2}{C})\xi^{2(m-r)}+x^{2t}} \longrightarrow 0, \\ &\qquad x+\xi\to\infty \end{aligned}$$

provided $r < m$, and $r+t \geq 1$. For $r = m$, and therefore $t = 0$, the second part of formula (28) vanishes, so the result is true for $s = 0$, too.

In the region III) we get:

$$\frac{q^2x^{2(q-1)}+t^2x^{2(t-1)}\xi^{2r}}{(\xi^m-x^q)^2+x^{2t}\xi^{2r}} \leq \frac{q^2x^{2(q-1)}+t^2x^{2(t-1)}\xi^{2r}}{(1-2c)x^{2q}+x^{2t}\xi^{2r}} \leq \text{const}\,\frac{1}{x^2} \to 0,\ x\to\infty.$$

Summing up, $a(x,\xi)$ satisfies the estimates (22) if:

$$(29)\qquad \begin{cases} rq+mt > q(m-1) \\ t < q \end{cases}.$$

It is easy to see that for $r = 0$, by (27) and the first of (29), $a(x,\xi)$ is hypoelliptic if $t \geq q$. For $t = 0$ we obtain hypoellipticity only for $r = m$. One can also easily check that the previous conditions are necessary for hypoelliptcity. Let $r+t = p$, from formula (29) by replacing r with $p-t$ we then obtain:

$$(30)\qquad \frac{q}{m-q}(m-1-p) < t < q,\quad m > q.$$

If $m = q$, from (29) we obtain $r + t > m - 1$, then there is hypoellipticity only for $r + t = m$.

Remark. Let $p \leq q - 1$, we then obtain from the first part of the formula (30):

$$t > \frac{q}{m-q}(m - 1 - p) \geq \frac{q}{m-q}(m - 1 - q + 1) = q,$$

contradicting the second part, so we have hypoellipticity only for $r + t = p$, where $p \geq q$. Similar computations, shows that there is hypoellipticity for some couple (r, t) on the straight line $p = r + t = q + \alpha$, $\alpha = 0, \dots, m - q$, if and only if:

$$\frac{m}{q} < \alpha + 2.$$

More precisely there are at least β values of t, $\beta = 1, \dots, q - 1$, for hypoellipticity on the straight line $p = q + \alpha$, $\alpha = 0, \dots, m - q$, if and only if:

$$\frac{m}{q} < \frac{\alpha + \beta + 1}{\beta}.$$

In particular we obtain all the $q - 1$ values of t for having hypoellipticity, on the straight line $p = q$, if $\frac{m}{q} < \frac{q}{q-1}$, and $m \geq q$, which imply $q = m - 1$. It is convenient to distinguish two regions, in the set of all the possible couples (r, t) giving hypoellipticity:

$$q(m - 1) < rq + mt \leq qm, \tag{31}$$

and,

$$rq + mt > qm, \quad t < q. \tag{32}$$

In the case when (31) is valid with $rq + mt = qm$, or (32) is satisfied, the polynomial (22) is multi-quasi-elliptic, cf. Boggiatto-Buzano-Rodino [1]. In the follow we shall be mainly interested in non multi-quasi-elliptic polynomials.

Remark. We find hypoellipticity on straight line $p = q + \alpha$ in the region (31) if and only if:

$$\alpha + 1 < \frac{m}{q} < \alpha + 2 \ .$$

More precisely There are at least β values of t, $\beta = 1, \ldots, q-1$, for having hypoellipticity on the straight line $p = q+\alpha$, $\alpha = 0, \ldots, m-q$, in the region (31), if and only if:

$$\alpha + 1 < \frac{m}{q} < \frac{\alpha+\beta+1}{\beta}.$$

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