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FINITE SYMMETRIC FUNCTIONS WITH NON-TRIVIAL ARITY GAP

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Abstract. Given an n-ary k-valued function f, gap(f) denotes the essential arity gap of f which is the minimal number of essential variables in f which become fictive when identifying any two distinct essential variables in f. In the present paper we study the properties of the symmetric function with non-trivial arity gap ($2 \leq \text{gap}(f)$). We prove several results concerning decomposition of the symmetric functions with non-trivial arity gap with its minors or subfunctions. We show that all non-empty sets of essential variables in symmetric functions with non-trivial arity gap are separable.

Introduction. Given a function f, the essential variables in f are defined as variables which occur in f and affect the values of that function. They are investigated when replacing variables with constants or variables (see, e.g., [1, 2, 6, 9]). If we replace some variables in a function f with constants the result is a subfunction of f and when replacing several variables with other variables, the result is a minor of f.

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The essential arity gap of a finite-valued function f is the minimum decrease in the number of essential variables in identification minors of f. In this paper we investigate functions in k-valued logics with non-trivial arity gap, which are important in theoretical and applied computer science, namely the symmetric functions.

R. Willard proved that if a function $f : A^n \to B$ depends on n variables and $k < n$, where $k = |A|$ then $\text{gap}(f) \leq 2$ [10]. On the other hand it is clear that $\text{gap}(f) \leq n$. Thus in the case we have $\text{gap}(f) \leq \min(n, k)$.

M. Couceiro and E. Lehtonen proposed a classification of functions according to their arity gap [3, 4].

We have proved that if $2 \leq \text{gap}(f) < \min(n, k)$ then f can be decomposed as a sum of functions of a prescribed type (see Theorem 3.4 [8]).

A natural question to ask is which additional properties, of the arity gap are typical of symmetric and linear functions with non-trivial arity gap. We investigate the behavior of subfunctions of symmetric functions with non-trivial arity gap. So, in this paper we consider together the both types of replacement in a function's inputs—with constants (subfunctions) and with variables (minors). We prove that "almost" all subfunctions of a symmetric function f with non-trivial arity gap inherit the property of f concerning the identification of variables. We are interested also in decomposition of symmetric functions as "sums of conjunctions" (following [8]).

We also characterize the relationship between separable sets and subfunctions of symmetric functions with non-trivial arity gap.

1. Preliminaries.

Let k be a natural number with $k \geq 2$. Denote by $K = \{0, 1, \ldots, k-1\}$ the set (ring) of remainders modulo k. An *n-ary k-valued function (operation) on K* is a mapping $f : K^n \to K$ for some natural number n, called *the arity* of f. The set of all n-ary k-valued functions is denoted by P_k^n.

Let $f \in P_k^n$ and $\text{var}(f) = \{x_1, \ldots, x_n\}$ be the set of all variables, which occur in f. We say that the i-th variable $x_i \in \text{var}(f)$ is *essential* in $f(x_1, \ldots, x_n)$, or f *essentially depends* on x_i, if there exist values $a_1, \ldots, a_n, b \in K$, such that

$$f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

The set of all essential variables in the function f is denoted by $\text{Ess}(f)$ and the number of its essential variables is denoted by $\text{ess}(f) := |\,\text{Ess}(f)|$.

Let x_i and x_j be two distinct essential variables in f. The function h is obtained from $f \in P_k^n$ by *the identification of the variable x_i with x_j*, if

$$h(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) := f(a_1, \ldots, a_{i-1}, a_j, a_{i+1}, \ldots, a_n),$$

for all $(a_1, \ldots, a_n) \in K^n$.

Briefly, when h is obtained from f by identification of the variable x_i with x_j, we will write $h = f_{i \leftarrow j}$ and h is called *an identification minor of f*. Clearly, $\mathrm{ess}(f_{i\leftarrow j}) \leq \mathrm{ess}(f)$, because $x_i \notin \mathrm{Ess}(f_{i\leftarrow j})$, even though it might be essential in f. When h is an identification minor of f we shall write $f \vdash h$. The transitive closure of $\vdash$ is denoted by $\models$. $\mathrm{Min}(f) = \{h \mid f \models h\}$ is the set of all minors of f.

Let $f \in P_k^n$ be an n-ary k-valued function. Then *the essential arity gap* (shortly *arity gap* or *gap*) of f is defined by

$$\mathrm{gap}(f) := \mathrm{ess}(f) - \max_{h \in \mathrm{Min}(f)} \mathrm{ess}(h).$$

Let $h \in \mathrm{Min}(f)$ be a minor of f and

$$L_h := \{m \mid \exists\ (h_1, \ldots, h_m) \ \text{ with } f \vdash h_1 \vdash \ldots \vdash h_m = h\}.$$

The number $\mathrm{depth}(h) := \max L_h$ is called *the depth* of h and *the gap index* of f is defined as follows

$$\mathrm{ind}(f) := \max_{h \in \mathrm{Min}(f)} \mathrm{depth}(h).$$

Let $2 \leq p \leq m$. We let $G_{p,k}^m$ denote the set of all k-valued functions which essentially depend on m variables whose arity gap is equal to p, i.e., $G_{p,k}^m = \{f \in P_k^n \mid \mathrm{ess}(f) = m \ \&\ \mathrm{gap}(f) = p\}$.

Let x_i be an essential variable in f and $c \in K$ be a constant from K. The function $g := f(x_i = c)$ obtained from $f \in P_k^n$ by replacing the variable x_i with c is called a *simple subfunction of f*.

When g is a simple subfunction of f we shall write $f \rhd g$. The transitive closure of $\rhd$ is denoted by $\gg$. $\mathrm{Sub}(f) = \{g \mid f \gg g\}$ is the set of all subfunctions of f and $\mathrm{sub}(f) := |\,\mathrm{Sub}(f)|$.

Let $g \in \mathrm{Sub}(f)$ be a subfunction of f and let

$$O_g := \{m \mid \exists\ (g_1, \ldots, g_m) \ \text{ with } f \rhd g_1 \rhd \ldots \rhd g_m = g\}.$$

The number $\mathrm{ord}(g) := \max O_g$ is called *the order* of g.

As usual we denote by S_n the set of all permutations of the set $\{1, \ldots, n\}$. Let $\mathrm{Ess}(f) = \{x_{i_1}, \ldots, x_{i_m}\} \subseteq \{x_1, \ldots, x_n\}$. Let S_f be the set of all permutations of $\{i_1, \ldots, i_m\}$. We say that f is a *symmetric function* if $f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$, for all $\pi \in S_f$.

Given a variable x and $c \in K$, x^c is an unary function defined by:

$$x^c = \begin{cases} 1 & \text{if} \quad x = c \\ 0 & \text{if} \quad x \neq c. \end{cases}$$

We use *sums of conjunctions (SC)* for representation of functions in P_k^n. This is the most natural representation of functions in finite algebras. It is based on the so-called operation tables of the functions.

Each function $f \in P_k^n$ can be uniquely represented in SC-form as follows

$$f = a_0 \cdot x_1^0 \ldots x_n^0 \oplus \ldots \oplus a_m \cdot x_1^{c_1} \ldots x_n^{c_n} \oplus \ldots \oplus a_{k^n-1} \cdot x_1^{k-1} \ldots x_n^{k-1}$$

with $m = \Sigma_{i=1}^n c_i k^{n-i}$, and $c_i, a_m \in K$, where "$\oplus$" and "$\cdot$" are the operations of addition and multiplication modulo k in the ring K.

2. Symmetric functions with non-trivial arity gap.

We are going to study the behavior of the symmetric k-valued functions f with non-trivial arity gap, i.e., with $\mathrm{gap}(f) > 1$.

Lemma 2.1. *Let $f \in P_k^n$ be a symmetric function which essentially depends on n variables and let $f \gg g$ then g is a symmetric function and if* $\mathrm{Ess}(g) \neq \emptyset$ *then* $\mathrm{ess}(g) = n - \mathrm{ord}(g)$.

Proof. Without loss of generality let us assume that $\mathrm{ord}(g) = m > 0$ and

$$f \rhd f(x_1 = c_1) \rhd \ldots \rhd f(x_1 = c_1, x_2 = c_2, \ldots, x_m = c_m) = g.$$

It is obvious that g is symmetric.

Clearly, $x_i \in \mathrm{Ess}(g)$ if and only if $x_j \in \mathrm{Ess}(g)$ for all $i, j \in \{m+1, \ldots, n\}$. Hence if $Ess(g) \neq \emptyset$ then $\mathrm{Ess}(g) = X_n \setminus \{x_1, \ldots, x_m\}$. $\square$

Lemma 2.2. *Let $2 \leq p \leq \min(k, n)$. If $f \in G_{p,k}^n$ is a symmetric function, then $p = 2$ or $p = n$.*

Proof. Let us suppose this is not the case. Then $2 < p < n$. Hence there is an identification minor h of f such that $\mathrm{gap}(f) = n - \mathrm{ess}(h)$ and $2 < n - \mathrm{ess}(h) <$

n. Without loss of generality assume that $h = f_{n\leftarrow n-1}$ and $\mathrm{Ess}(h) = \{x_1, \ldots, x_q\}$, where $q = n - p$ such that $0 < q < n - 2$. Then $x_{n-2} \in \mathrm{Ess}(f) \setminus \mathrm{Ess}(h)$. Hence for every n constants $c_1, \ldots, c_{n-3}, c_{n-2}, d_{n-2}, c_{n-1} \in K$ we have

$$f(c_1, \ldots, c_{n-3}, c_{n-2}, c_{n-1}, c_{n-1}) = f(c_1, \ldots, c_{n-3}, d_{n-2}, c_{n-1}, c_{n-1}).$$

Since f is symmetric, Lemma 2.1 implies

$$f(c_{n-2}, \ldots, c_2, c_1, x_{n-1}, x_{n-1}) = f(d_{n-2}, c_{n-3}, \ldots, c_2, c_1, x_{n-1}, x_{n-1}).$$

Hence $x_1 \notin \mathrm{Ess}(h)$, which is a contradiction. □

Lemma 2.3 ([8]). *Let f be a k-valued function which depends essentially on all of its n, $n > 3$ variables and* $\mathrm{gap}(f) = 2$. *Then there exist two distinct essential variables x_u, x_v such that* $\mathrm{ess}(f_{u\leftarrow v}) = n-2$ *and* $x_v \notin \mathrm{Ess}(f_{u\leftarrow v})$. *Moreover,* $\mathrm{ess}(f_{u\leftarrow m}) = \mathrm{ess}(f_{v\leftarrow m}) = n - 2$ *for all m, $1 \le m \le n$ with $m \notin \{u, v\}$.*

Lemma 2.4. *Let $3 < n \le k$. If $f \in G^n_{2,k}$ is a symmetric function then $x_v \notin \mathrm{Ess}(f_{u\leftarrow v})$ for all $1 \le u, v \le n$ with $u \ne v$.*

Proof. From Lemma 2.3, there are $1 \le u, v \le n$ with $u \ne v$ such that $x_v \notin \mathrm{Ess}(f_{u\leftarrow v})$. Without loss of generality, let $u = 1$ and $v = 2$. Further, let $1 \le i < j \le n$ and $a_1, \ldots, a_n, b \in K$. Then we have

$$f_{i\leftarrow j}(a_1, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, a_j, a_{i+1}, \ldots, a_n) =$$

$$f(a_j, a_j, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n) =$$

$$f(b, b, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n) =$$

$$f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n) =$$

$$f_{i\leftarrow j}(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n).$$

This shows that $x_j \notin \mathrm{Ess}(f_{i\leftarrow j})$. □

Remark 2.1. If f is a symmetric function with non-trivial arity gap then all its identification minors are symmetric. In fact, we have $h = f_{2\leftarrow 1} = f(c, c, x_3, \ldots, x_n)$ for all $c \in K$, according to Lemma 2.4. Hence h is the subfunction $h = f(x_1 = c, x_2 = c)$ of f and by Lemma 2.1 it follows that h is symmetric.

Lemma 2.5. *If $f \in G^n_{2,k}$, $n \geq 2$, is a symmetric function then* $1 \leq \operatorname{ind}(f) \leq \dfrac{n}{2}$.

Proof. Clearly if $\operatorname{ess}(f) \geq 2$ then $ind(f) \geq 1$ for all $f \in P^n_k$.

Lemma 2.3 and Lemma 2.4 imply that if $f \vdash h_1 \vdash \ldots \vdash h_m$ with $m = \operatorname{ind}(f)$ then $\operatorname{depth}(h_i) = i$ and $\operatorname{ess}(h_i) = n - 2i$ for $i = 1, \ldots, m$. Hence $\operatorname{ind}(f) \leq \dfrac{n}{2}$. □

Let $f \in G^n_{2,k}$, $n > 2$, be a symmetric function and let $\operatorname{ind}(f) = m < \dfrac{n}{2}$. Then for each minor $h \in \operatorname{Min}(f)$ with $\operatorname{depth}(h) < m$ there is $g \in \operatorname{Min}(f)$ such that $f \models h \models g$ and $\operatorname{depth}(g) = m$.

Remark 2.2. Let $f \in G^n_{2,k}$, $n > 2$, be a symmetric function and let $h \in \operatorname{Min}(f)$. From Lemma 2.2, we conclude that if $\operatorname{depth}(h) = l < \operatorname{ind}(f)$, then $h \in G^{n-2l}_{2,k}$, else $h \in G^{n-2l}_{n-2l,k}$.

Let k and n, $k \geq n > 1$, be two natural numbers such that $1 < n \leq k$. The set K^n of all $n-$tuples over K is the disjoint union of the following two sets:

$$\mathrm{Eq}^n_k := \{(c_1, \ldots, c_n) \in K^n \mid \ c_i = c_j, \text{ for some } i, j \text{ with } \ i \neq j\},$$

$$\mathrm{Dis}^n_k := \{(c_1, \ldots, c_n) \in K^n \mid \ c_i \neq c_j, \text{ for all } i, j \text{ with } \ i \neq j\}.$$

Theorem 2.1 ([8]). *Let $3 \leq n \leq k$. Then $f \in G^n_{n,k}$, if and only if f can be represented as follows*

$$f = \left[\bigoplus_{\beta \in Dis^n_k} a_\beta . x_1^{d_1} \ldots x_n^{d_n}\right] \oplus a_0 . \left[\bigoplus_{\alpha \in Eq^n_k} x_1^{c_1} \ldots x_n^{c_n}\right], \tag{1}$$

where $\beta = (d_1, \ldots, d_n)$ and $\alpha = (c_1, \ldots, c_n)$, and at least two among the coefficients $a_0, a_\beta \in K$ for $\beta \in \mathrm{Dis}^n_k$, are distinct.

Let $\alpha = (c_1, \ldots, c_n) \in K^n$. We denote

$$S(n, \alpha) := \bigoplus_{\pi \in S_n} x_1^{c_{\pi(1)}} \ldots x_n^{c_{\pi(n)}}.$$

Let $\alpha = (c_1, \ldots, c_n) \in K^n$ and $\beta = (d_1, \ldots, d_m) \in K^m$ with $m \leq n$.

We shall write $\beta \leq \alpha$ if there are $1 \leq i_1, \ldots, i_m \leq n$ such that $c_{i_j} = d_j$ and $c_s \neq d_j$ for all $s \notin \{i_1, \ldots, i_m\}$ and all $j \in \{1, \ldots, m\}$.

Example 2.1. Let $k = 5$. Then $(0,1,1) \leq (0,1,2,1,4)$, but $(0,1) \not\leq (0,1,2,1,4)$ and $(0,2,3) \not\leq (0,1,2,1,4)$. Let $\alpha = (1,2,4)$. Then

$$S(3,\alpha) = x_1^1 x_2^2 x_3^4 \oplus x_1^1 x_2^4 x_3^2 \oplus x_1^2 x_2^1 x_3^4 \oplus x_1^2 x_2^4 x_3^1 \oplus x_1^4 x_2^1 x_3^2 \oplus x_1^4 x_2^2 x_3^1.$$

Theorem 2.2. *Let $f \in G_{n,k}^n$, $3 \leq n \leq k$. Then f is a symmetric function if and only if it can be represented in the following form:*

$$f = a_0 \left[\bigoplus_{\alpha \in Eq_k^n} x_1^{c_1} x_2^{c_2} \dots x_n^{c_n} \right] \oplus \left[\bigoplus_{\beta \in Dis_k^n} b_\beta S(n,\beta) \right], \tag{2}$$

where $\alpha = (c_1, c_2, \dots, c_n) \in Eq_k^n$, and at least two among the coefficients $a_0, b_\beta \in K$, for $\beta \in Dis_k^n$ are distinct.

Proof. Let $f \in G_{n,k}^n$, $2 < n \leq k$ be a symmetric function and $\beta = (d_1, \dots, d_n) \in Dis_k^n$. Let us set $b_\beta = f(\beta)$. Since f is a symmetric function, it follows that $f(d_{\pi(1)}, d_{\pi(2)}, \dots, d_{\pi(n)}) = b_\beta$, for each $\pi \in S_n$.

Let $\alpha \in \mathrm{Eq}_k^n$. Then (1) implies $f(\alpha) = f(0,0,\dots,0) = a_0$, which proves that f is represented in the form (2). Clearly, if f is represented as in (2), then it is a symmetric function with arity gap equal to n. □

Corollary 2.1. *There are $k^{\binom{k}{n}+1} - k$ different symmetric functions in $G_{n,k}^n$.*

Proof. There exists $\binom{k}{n}$ ways to choose β in (2). Thus there are $\binom{k}{n} + 1$ coefficients in (2), including a_0 taken from K. On the other hand we have to exclude all k cases when $a_0 = b_\beta$ for $\beta \in Dis_k^n$. □

We are interested in an explicit representation of the symmetric functions f with $\mathrm{gap}(f) = 2$ in the case when $\mathrm{ess}(f) = 3$. The case $\mathrm{gap}(f) = 2$ and $\mathrm{ess}(f) = 3$ is really special and is deeply discussed in [8] where we decomposed $f \in G_{2,k}^3$ for $k = 3$ (see Theorem 5.1 [8]). In a similar way one can prove the following more general result.

Theorem 2.3. *Let $f \in G_{2,k}^3$, $k \geq 3$. Then f is a symmetric function if and only if it can be represented in one of the following forms:*

$$f = \bigoplus_{i=0}^{k-1} a_i \left[x_1^i x_2^i x_3^i \oplus \left[\bigoplus_{\alpha \in Eq_k^3,\ (i) \leq \alpha} x_1^{c_1} x_2^{c_2} x_3^{c_3} \right] \right] \oplus \left[\bigoplus_{\delta \in Dis_k^3} b_\delta S(3,\delta) \right] \tag{3}$$

or

$$(4)\qquad f = \bigoplus_{i=0}^{k-1} a_i \left[x_1^i x_2^i x_3^i \oplus \left[\bigoplus_{\alpha \in Eq_k^3,\ (ii) \leq \alpha} x_1^{c_1} x_2^{c_2} x_3^{c_3} \right] \right] \oplus \left[\bigoplus_{\delta \in Dis_k^3} b_\delta S(3,\delta) \right],$$

where $\alpha = (c_1, c_2, c_3)$ *and at least two among the coefficients* $a_i \in K$*, for* $i = 0, \ldots, k-1$ *are distinct.*

Theorem 2.4. *Let* $f \in P_k^n$ *be a symmetric function with non-trivial arity gap. Then*

(i) If $\mathrm{gap}(f) = n$ *or* n*,* $n \geq 2$*, is an even natural number or* $\mathrm{ind}(f) < \dfrac{n-1}{2}$ *then* $f(c_1, \ldots, c_1) = f(c_2, \ldots, c_2)$ *for all* $c_1, c_2 \in K$*;*

(ii) If n*,* $3 \leq n \leq k$*, is an odd natural number,* $\mathrm{gap}(f) = 2$ *and* $\mathrm{ind}(f) = \dfrac{n-1}{2}$ *then there exist at least two values* $c_1, c_2 \in K$ *such that* $f(c_1, \ldots, c_1) \neq f(c_2, \ldots, c_2)$*.*

P r o o f. (i) We have to consider three cases:

Case A. Let $\mathrm{gap}(f) = n$.

Then $f \in G_{n,k}^n$ and from Theorem 2.1 it follows $f(c_1, \ldots, c_1) = f(c_2, \ldots, c_2)$ for all $c_1, c_2 \in K$.

Case B. Let n, $n \geq 2$ be an even natural number and $\mathrm{gap}(f) = 2$.

Let $c_1, c_2 \in K$ be two constants with $c_1 \neq c_2$. From Lemma 2.4 it follows that $x_v \notin \mathrm{Ess}(f_{u \leftarrow v})$ for all $1 \leq u, v \leq n$ with $u \neq v$. Then we obtain

$$\begin{aligned}
& f(c_2, c_2, \ldots, c_2) \\
&= f(c_1, c_1, c_2, c_2, c_2, \ldots, c_2) && \text{because } x_2 \notin \mathrm{Ess}(f_{1 \leftarrow 2}) \\
&= f(c_1, c_1, c_1, c_1, c_2, \ldots, c_2) && \text{because } x_3 \notin \mathrm{Ess}(f_{4 \leftarrow 3}) \\
&= f(c_1, c_1, c_1, c_1, c_1, c_1, c_2, \ldots, c_2) && \text{because } x_5 \notin \mathrm{Ess}(f_{6 \leftarrow 5}) \\
& \ldots \quad \ldots && \ldots \quad \ldots \\
&= f(c_1, c_1, \ldots, c_1, c_1, c_2, c_2) && \text{because } x_{n-3} \notin \mathrm{Ess}(f_{n-2 \leftarrow n-3}) \\
&= f(c_1, c_1, \ldots, c_1, c_1, c_1, c_1) && \text{because } x_{n-1} \notin \mathrm{Ess}(f_{n \leftarrow n-1}).
\end{aligned}$$

Case C. Let $\mathrm{gap}(f) = 2$, n be odd and $\mathrm{ind}(f) < \dfrac{n-1}{2}$.

Let $\text{ind}(f) = \dfrac{n-m}{2} < \dfrac{n-1}{2}$, for some odd natural number m, $n-2 \geq m \geq 3$. Let $h \in \text{Min}(f)$ be a minor of f with $\text{depth}(h) = \dfrac{n-m}{2}$. Since f is symmetric and $gap(f) = 2$ we have $x_v \notin \text{Ess}(f_{u\leftarrow v})$ for all $1 \leq u, v \leq n$, $u \neq v$. Hence from Lemma 2.1 it follows that

$$\begin{aligned} h = \quad & [\dots[f_{2\leftarrow 1}]_{4\leftarrow 3}\dots]_{n-m\leftarrow n-m-1} = \\ & f(x_1, x_1, x_3, x_3, \dots, x_{n-m-1}, x_{n-m-1}, x_{n-m+1}, \dots, x_n) = \\ & f(c_1, \dots, c_1, x_{n-m+1}, \dots, x_n) \end{aligned}$$

for an arbitrary constant $c_1 \in K$. Since $\text{depth}(h) = \dfrac{n-m}{2}$ and $m \leq n-2$ it follows that $\text{Ess}(h) = \emptyset$. Consequently, $h = f(c_1, \dots, c_1) = f(c_2, \dots, c_2)$ for all $c_1, c_2 \in K$.

(ii) Let n, $3 \leq n \leq k$ be an odd natural number, $\text{gap}(f) = 2$ and $ind(f) = \dfrac{n-1}{2}$.

First, let $n = 3$. Then from (3) and (4) it follows that $f(i,i,i) = a_i$ and there are a_i, a_j, $i, j \in K$ with $a_i \neq a_j$. Hence $f(i,i,i) \neq f(j,j,j)$.

Second, let $n > 3$ and $\text{ind}(f) = \dfrac{n-1}{2}$. Let $g \in \text{Min}(f)$ be a minor of f for which $\text{depth}(g) = \text{ind}(f)$ and as above we can write

$$g = [\dots[f_{2\leftarrow 1}]_{4\leftarrow 3}\dots]_{n-1\leftarrow n-2}.$$

Let h be a minor of f with $\text{depth}(h) = \dfrac{n-3}{2} < \text{ind}(f)$ such that $g = h_{n-1\leftarrow n-2}$, i.e., $x_{n-2}, x_{n-1} \in \text{Ess}(h)$ and by the symmetry of f we have $\{x_{n-2}, x_{n-1}, x_n\} = \text{Ess}(h)$.

Then there is a ternary function $t \in P_k^3$ such that

$$t(x_{n-2}, x_{n-1}, x_n) = h(a_1, \dots, a_{n-3}, x_{n-2}, x_{n-1}, x_n)$$

for all $(a_1, \dots, a_{n-3}) \in K^{n-3}$ and t is symmetric (see Remark 2.1).

Thus we have $t(x_{n-2}, x_{n-1}, x_n) = f(c_1, c_1, \dots c_1, c_1, x_{n-2}, x_{n-1}, x_n)$ for an arbitrary $c_1 \in K$. Hence $f(c, \dots, c) = t(c, c, c)$ for all $c \in K$. If $x_u, x_v \in \text{Ess}(h)$ then $x_v \notin \text{Ess}(h_{u\leftarrow v})$, else $x_v \in \text{Ess}(f_{u\leftarrow v})$ which is impossible, according to Lemma 2.3. If we suppose that $\text{Ess}(h_{u\leftarrow v}) = \emptyset$, then by the symmetry of f it follows that $\text{depth}(h) = \text{ind}(f)$, which is a contradiction. Again, by the symmetry of f it follows that $\text{Ess}(h_{u\leftarrow v}) = \text{Ess}(t_{u\leftarrow v}) = \text{Ess}(t) \setminus \{x_u, x_v\}$ and hence $t \in$

$G^3_{2,k}$. According to Theorem 2.3 it follows that there exist $c_1, c_2 \in K$ such that $t(c_1, c_1, c_1) \neq t(c_2, c_2, c_2)$ (see case $n = 3$, $\text{gap}(f) = 2$) and hence $f(c_1, \dots, c_1) \neq f(c_2, \dots, c_2)$. $\square$

Theorem 2.5. *Let* $3 < min(n, k)$. *If* $f \in G^n_{2,k}$ *is a symmetric function then*

$$f = \bigoplus_{i=1}^{n-1} \bigoplus_{j=i+1}^{n} \bigoplus_{m=0}^{k-1} x_i^m x_j^m g(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n) \oplus$$

$$\oplus\ h(x_1, \dots, x_n),$$

where g *and* h *are symmetric functions such that:* $h(\alpha) = 0$ *for all* $\alpha \in Eq^n_k$ *and*

$$g \in \begin{cases} G^{n-2}_{2,k} & \text{if} \quad \text{ind}(f) > 2 \\ G^{n-2}_{n-2,k} & \text{if} \quad \text{ind}(f) = 2. \end{cases}$$

Proof. The conjunctions in SC-form of any function $f \in P^n_k$ can be reordered so that

$$f = \bigoplus_{i=1}^{n-1} \bigoplus_{j=i+1}^{n} \bigoplus_{m=0}^{k-1} x_i^m x_j^m g_{ijm} \ \oplus\ h(x_1, \dots, x_n),$$

$\text{var}(g_{ijm}) = \{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n\}$ and $h(\alpha) = 0$ for all $\alpha \in \text{Eq}^n_k$.

Let $f \in G^m_{2,k}$ be a symmetric function with $n > 2$. Since h might assume non-zero values on the set Dis^n_k, only, it follows that h has to be a symmetric function.

Then we obtain

$$f_{2\leftarrow 1} = \left[\bigoplus_{m=0}^{k-1} x_1^m x_1^m g_{12m}\right] \ \oplus\ \left[\bigoplus_{i=3}^{n-1} \bigoplus_{j=i+1}^{n} \bigoplus_{m=0}^{k-1} x_i^m x_j^m [g_{ijm}]_{2\leftarrow 1}\right] \oplus$$

$$\oplus \bigoplus_{i=3}^{n} \bigoplus_{m=0}^{k-1} x_i^m g_{1im}(x_2 = m) \ \oplus\ \bigoplus_{i=3}^{n} \bigoplus_{m=0}^{k-1} x_i^m g_{2im}(x_1 = m).$$

Since $x_v \notin \text{Ess}(f_{u\leftarrow v})$ for $1 \leq u, v \leq n$, $u \neq v$ it follows that $g_{12m} = g_{12s}$ for all $s, m \in K$. By the symmetry of f it follows that $g_{ijm} = g_{ijs}$ for all

$s, m \in K$ and $1 \le i < j \le n$. Hence the index m is redundant and we might write g_{ij} instead of g_{ijm}, i.e., $g_{ij} := g_{ijm}$ for $m \in K$. The symmetry of f implies $g_{ij}(\alpha) = g_{uv}(\alpha)$ for each $\alpha \in K^{n-2}$, i.e., the functions g_{ij} are identical, considered as mappings of K^{n-2} to K. Hence there is an $(n-2)$-ary function $g \in P_k^{n-2}$ which maps each $\alpha \in K^{n-2}$ as follows $g(\alpha) = g_{ij}(\alpha)$. Consequently $g_{ij} = g(x_1, x_2 \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ for $1 \le i < j \le n$.

Suppose that g is not a symmetric function. Without loss of generality assume that g_{ij} is not symmetric with respect to x_1, x_2 and $3 \le i < j \le n$. Then there exist $n-2$ constants $c_1, c_2, c_3, \dots, c_{n-2} \in K$ such that $g_{ij}(c_1, c_2, c_3, \dots, c_{n-2}) \ne g_{ij}(c_2, c_1, c_3, \dots, c_{n-2})$. Clearly $c_1 \ne c_2$. If $d_1, d_2 \in K$ with $d_1 \ne d_2$ then

$$f(x_1 = d_1, x_2 = d_2) = \bigoplus_{i=3}^{n-1} \bigoplus_{j=i+1}^{n} \bigoplus_{m=0}^{k-1} x_i^m x_j^m g_{ij}(x_1 = d_1, x_2 = d_2) \oplus$$

$$\oplus\ h(x_1 = d_1, x_2 = d_2).$$

Since h is symmetric, it follows $h(x_1 = d_1, x_2 = d_2) = h(x_1 = d_2, x_2 = d_1)$ and hence $f(x_1 = c_1, x_2 = c_2) \ne f(x_1 = c_2, x_2 = c_1)$, which is a contradiction.

Hence g_{ij} is a symmetric $(n-2)$-ary function which essentially depends on all of its variables. Since $\text{ess}(f_{2\leftarrow 1}) = n-2$ it follows that $x_1 \notin \text{Ess}([g_{ij}]_{2\leftarrow 1})$ and hence $\text{gap}(g_{ij}) > 1$. According to Lemma 2.2 we have $\text{gap}(g_{ij}) = 2$ or $\text{gap}(g_{ij}) = n-2$.

Let $\text{ind}(f) > 2$. Then $\text{ess}([f_{2\leftarrow 1}]_{4\leftarrow 3}) > 0$ implies $\text{Ess}([g_{ij}]_{2\leftarrow 1}) \ne \emptyset$. By the symmetry of f and g_{ij} it follows that $\text{Ess}([g_{ij}]_{2\leftarrow 1}) = \{x_3, \dots, x_n\} \setminus \{x_i, x_j\}$. Hence $g_{ij} \in G_{2,k}^{n-2}$ for $1 \le i < j \le n$.

Let $\text{ind}(f) = 2$. Then $\text{ess}([f_{2\leftarrow 1}]_{4\leftarrow 3}) = 0$ implies $\text{Ess}([g_{ij}]_{2\leftarrow 1}) = \emptyset$. Hence $g_{ij} \in G_{n-2,k}^{n-2}$ for $1 \le i < j \le n$. $\square$

Theorem 2.2, Theorem 2.3 and Theorem 2.5 provide decompositions of the symmetric functions with non-trivial arity gap in the basis $\langle \oplus, \cdot\ , \{x^\alpha\}_{\alpha \in K} \rangle$ of the algebra P_k^n.

As usual we shall say that a k-valued function $f \in P_k^n$ is *linear* if $f = a_1x_1 \oplus a_2x_2 \oplus \ldots \oplus a_nx_n \oplus c$, where $a_1, a_2, \ldots a_n, c \in K$. Clearly, $x_i \in \text{Ess}(f)$ if and only if $a_i \ne 0$.

Theorem 2.6. *The set P_k^n, $k, n \ge 2$, contains a linear function with non-trivial arity gap if and only if k is an even natural number.*

Proof. Let $f = \left[\bigoplus_{i=1}^{n} a_i x_i\right] \oplus c$ with $c, a_i \in K$. Without loss of generality let us consider the identification minor $f_{2\leftarrow 1} = (a_1 \oplus a_2)x_1 \oplus \left[\bigoplus_{i=3}^{n} a_i x_i\right] \oplus c$. Clearly $\mathrm{ess}(f_{2\leftarrow 1}) \geq \mathrm{ess}(f) - 2$, i.e., $\mathrm{gap}(f) \leq 2$.

Let k be an even natural number and $k = 2m$ for some $m \in N$. Then let us consider the following linear (and symmetric) function

$$f = m(x_1 \oplus x_2 \oplus \ldots \oplus x_n) \oplus c,$$

for some $c \in K$. Clearly,

$$f_{i\leftarrow j} = m(x_1 \oplus \ldots \oplus x_{j-1} \oplus x_{j+1} \oplus \ldots x_{i-1} \oplus x_{i+1} \oplus \ldots \oplus x_n) \oplus c.$$

Hence $f \in G^n_{2,k}$.

Let k be an odd natural number and let $f = a_1x_1 \oplus \ldots \oplus a_nx_n \oplus c$, for some $c \in K$, be a linear k-valued function. First assume that there are i and j, $1 \leq i, j \leq n$, such that $i \neq j$ and $a_i = a_j \neq 0$. Without loss of generality let us assume $(j, i) = (1, 2)$. Then we have $a_1 \oplus a_2 = 2a_1$ and

$$f_{2\leftarrow 1} = 2a_1x_1 \oplus a_3x_3 \oplus \ldots \oplus a_nx_n \oplus c.$$

Since k is odd it follows that $2a_1 \neq 0 \pmod{k}$. Hence $\mathrm{Ess}(f_{2\leftarrow 1}) = \{x_1, \ldots, x_n\} \setminus \{x_2\}$ and $f \notin G^m_{2,k}$. Second, let $a_i \neq a_j$ for all i and j, $1 \leq i < j \leq n$. Then we have $a_1 + a_2 \neq k$ or $a_1 + a_3 \neq k$. Without loss of generality assume that $a_1 + a_2 \neq k$. Hence

$$f_{2\leftarrow 1} = (a_1 + a_2)x_1 \oplus a_3x_3 \oplus \ldots \oplus x_n \oplus c.$$

Since $k \neq a_1 + a_2 < 2k$ it follows that $a_1 + a_2 \neq 0$ $(mod\ k)$ which implies $f \notin G^n_{2,k}$. $\square$

One can prove that if f is a linear function with non-trivial arity gap then f is symmetric.

3. Subfunctions of symmetric functions with non-trivial arity gap. In this section, we shall study the subfunctions of the symmetric k-valued functions f with non-trivial arity gap.

Let $c \in K$ be a constant from K and $f \in P_k^n$ be a symmetric function. We say that c is a *dominant* of f if $f(c_1, \ldots, c_{n-1}, c) = f(d_1, \ldots, d_{n-1}, c)$ for every $c_1, \ldots, c_{n-1}, d_1, \ldots, d_{n-1} \in K$. $\mathrm{Dom}(f)$ denotes the set of all dominants of f.

Clearly if $c \in \mathrm{Dom}(f)$ then $\mathrm{Ess}(f(x_1, \ldots, x_{n-1}, c)) = \emptyset$, i.e., the subfunctions of f of order 1 obtained by dominants of f are always constant functions. If $f \in G_{n,k}^n$ then $c \in \mathrm{Dom}(f)$ if and only if $f(c_1, \ldots, c_{n-1}, c) = f(0, \ldots, 0)$ for all $c_1, \ldots, c_{n-1} \in K$, according to Theorem 2.1.

A constant $c \in K$ is called a *weak dominant* of f if it is a dominant of an identification minor of f.

If f is a symmetric function then the weak dominants of f are dominants of all identification minors of f. $\mathrm{Wdom}(f)$ denotes the set of all weak dominants of f.

Theorem 3.1. *Let $f \in G_{n,k}^n$ be a symmetric function with $2 \leq k$, $2 < n$ and let $g = f(x_i = c)$ for some x_i, $1 \leq i \leq n$ and for some constant $c \in K$ be a subfunction of f. If $c \notin \mathrm{Dom}(f)$ then g is a symmetric function which belongs to the class $G_{n-1,k}^{n-1}$.*

P r o o f. We shall consider the non-trivial case $n > 2$ (else the subfunctions of f will depend on at most one essential variable). Hence $k > 2$ because $2 < n = \mathrm{gap}(f) \leq k$.

By symmetry we may assume that $g = f(x_n = c)$. Since $c \notin \mathrm{Dom}(f)$ it follows that $\mathrm{Ess}(g) \neq \emptyset$. Lemma 2.1 implies that $\mathrm{Ess}(g) = \{x_1, \ldots, x_{n-1}\}$. Thus we obtain

$$g_{2 \leftarrow 1} = g(x_1, x_1, x_3, \ldots, x_{n-1}) = f(x_1, x_1, x_3, \ldots, x_{n-1}, c).$$

Theorem 2.2 implies that for every $n-2$ constants $c_1, \ldots, c_{n-2} \in K$ we have

$$g(c_1, c_1, c_2, \ldots, c_{n-2}) = f(c_1, c_1, c_2, \ldots, c_{n-2}, c) = f(0, \ldots, 0)$$

because $(c_1, c_1, c_2, \ldots, c_{n-2}, c) \in \mathrm{Eq}_k^n$. Consequently g is symmetric and $g \in G_{n-1,k}^{n-1}$. $\square$

Let us denote $\mathrm{range}(f) = |\{f(\alpha) \mid \alpha \in K^n\}|$ for $f \in P_k^n$ and

$$sub_k^n = \binom{k}{1} + \binom{k}{2} + \ldots + \binom{k}{n-1}.$$

Lemma 3.1. *If $f \in G_{n,k}^n$, $n \leq k$ is a symmetric function, then* $\mathrm{sub}(f) \leq \mathrm{sub}_k^n + \mathrm{range}(f)$.

P r o o f. Let $f \gg g$ and $\mathrm{ord}(g) = m > 1$. Without loss of generality let us assume $g = f(x_1 = c_1, \ldots, x_m = c_m)$.

Let $\alpha = (c_1, \ldots, c_m) \in \mathrm{Eq}_k^m$. Then Theorem 2.2 implies that $g = f(0, \ldots, 0)$, i.e., g is a constant. So, g can be obtained in two ways, only: when $\alpha \in \mathrm{Eq}_k^m$ or $m = n$. Then it is clear that the number of all constant subfunctions is equal to $\mathrm{range}(f)$.

Let $\alpha = (c_1, \ldots, c_m) \in \mathrm{Dis}_k^m$. Since $|\,\mathrm{Dis}_k^m\,| = \binom{k}{m}.m!$, the symmetry of f implies that there exist at most $\binom{k}{m}$ subfunctions of order m, $1 \le m \le n-1$. Thus, if $f \in G_{n,k}^n$, $n \le k$ is a symmetric function then the number of all its subfunctions is equal to at most sub_k^n. Hence $\mathrm{sub}(f) \le \mathrm{sub}_k^n + \mathrm{range}(f)$. □

Remark 3.1.

(*i*) Note that Lemma 2.1 and Theorem 3.1 imply that if $g \in \mathrm{Sub}(f)$ with $\mathrm{ess}(g) = l > 1$ then $g \in G_{l,k}^l$.

(*ii*) Let f be a function represented as in (2) with $a_0 = 0$ and let $b_\beta \in K$ be non-zero integers for all $\beta \in \mathrm{Dis}_k^n$. Let $(c_{m+1}, \ldots, c_n) \in \mathrm{Dis}_k^{n-m}$ and $m < n$. Then we have $f(x_{m+1} = c_{m+1}, \ldots, x_n = c_n) = \bigoplus_{\gamma \in \mathrm{Dis}_k^m} b_\alpha S(m, \gamma)$, where $\alpha = (d_1, \ldots, d_m, c_{m+1}, \ldots, c_n) \in \mathrm{Dis}_k^n$ and $\gamma = (d_1, \ldots, d_m) \in \mathrm{Dis}_k^m$. Since $b_\beta \neq 0$, it follows that $f(x_{m+1} = c_{m+1}, \ldots, x_n = c_n)$ depends essentially on all its m variables. Consequently, $f(x_{m+1} = c_{m+1}, \ldots, x_n = c_n) = f(x_{m+1} = a_{m+1}, \ldots, x_n = a_n)$ for $a_{m+1}, \ldots, a_n \in K$ if and only if $\{c_{m+1}, \ldots, c_n\} = \{a_{m+1}, \ldots, a_n\}$. This implies that $\mathrm{sub}(f) = \mathrm{sub}_k^n + \mathrm{range}(f)$, i.e., the function f reach the upper bound for $\mathrm{sub}(f)$, obtained in Lemma 3.1.

(*iii*) The next example will show that $\mathrm{sub}(f) < \mathrm{sub}_k^n + \mathrm{range}(f)$ can happen. Let $k = 4$, $n = 3$ and $f = S(3, (0, 1, 3)) \oplus S(3, (0, 2, 3)) \pmod 4$. Clearly, $f \in G_{3,4}^3$ and $f(x_1 = 0, x_2 = 1) = f(x_1 = 0, x_2 = 2) = x_3^3$, $f(x_1 = 1, x_2 = 3) = f(x_1 = 2, x_2 = 3) = x_3^0$, and $f(x_1 = 1, x_2 = 2) = 0$, which shows that $\mathrm{sub}(f) = 4 + 3 + 3 = 10$ and $\mathrm{sub}_4^3 = \binom{4}{1} + \binom{4}{2} + \mathrm{range}(f) = 4 + 6 + 3 = 13$.

Theorem 3.2. *Let $f \in G_{2,k}^n$, $3 \le \min(n, k)$ be a symmetric function and $c \in K$. Then*

(*i*) *$t = f(x_i = c, x_j = c) \in G_{2,k}^{n-2}$ for all i, j, $1 \le i, j \le n$, $i \neq j$ if $\mathrm{ind}(f) > 2$;*

(*ii*) *$t = f(x_i = c, x_j = c) \in G_{n-2,k}^{n-2}$ for all i, j, $1 \le i, j \le n$, $i \neq j$ if $\mathrm{ind}(f) = 2$;*

(iii) $t = f(x_i = c) \in G^{n-1}_{n-1,k}$ *for all* i, $1 \le i \le n$ *if* $c \in \mathrm{Wdom}(f)$;

(iv) $t = f(x_i = c) \in G^{n-1}_{2,k}$ *for all* i, $1 \le i \le n$ *if* $c \notin \mathrm{Wdom}(f)$.

Proof. Let $f \in G^n_{2,k}$, $3 \le \min(n,k)$ be a symmetric function and $c \in K$. By the symmetry of f we may consider the pair $(1,2)$ instead (i,j).

(i) From Lemma 2.5 and $\mathrm{ind}(f) > 2$ it follows that $n \ge 6$. Then we have $t(a_1,\dots,a_{n-2}) = h(c,c,a_1,\dots,a_{n-2})$ for all $(a_1,\dots,a_n-2) \in K^{n-2}$, where $h = f_{2\leftarrow 1}$ and $\mathrm{depth}(h) = 1 < \mathrm{ind}(f)$. From Lemma 2.3 it follows that t and h depends on $n-2$ variables, i.e., $\mathrm{Ess}(h) = \{x_3,\dots,x_n\}$. Then $g = h_{4\leftarrow 3} = [f_{2\leftarrow 1}]_{4\leftarrow 3}$ is a minor of f with $\mathrm{depth}(g) = 2 < \mathrm{ind}(f)$. Hence it follows that $\mathrm{Ess}(g) \ne \emptyset$ and by the symmetry of f we have $\mathrm{Ess}(g) = \mathrm{Ess}(t_{4\leftarrow 3}) = \{x_5,\dots,x_n\}$. Hence $t \in G^{n-2}_{2,k}$.

(ii) Let $\mathrm{ind}(f) = 2$ and t, h and g are as in *(i)*. Now, $\mathrm{depth}(g) = 2 = \mathrm{ind}(f)$ implies that $\mathrm{Ess}(g) = \mathrm{Ess}(t_{4\leftarrow 3}) = \emptyset$. By the symmetry of f it follows that all identification minors of t do not depend on any of its variables. Hence $t \in G^{n-2}_{n-2,k}$.

(iii) Let $c \in \mathrm{Wdom}(f)$ and $t = f(c,x_2,\dots,x_n)$. Without loss of generality, assume that c is a dominant of $f_{n\leftarrow n-1}$, i.e., $f(c,x_2,\dots,x_{n-2},c_1,c_1)$ does not depend essentially on any variable for all $c_1 \in K$. Then Lemma 2.3 implies $t_{3\leftarrow 2} = f(c,c,c,x_4,\dots,x_n) = f(c,x_2,\dots,x_{n-2},c_1,c_1)$. Hence $\mathrm{Ess}(t_{3\leftarrow 2}) = \emptyset$, i.e., $t \in G^{n-1}_{n-1,k}$.

(iv) Let $c \in K$ and $c \notin \mathrm{Wdom}(f)$ and $t = f(c,x_2,\dots,x_n)$. Then $t_{3\leftarrow 2} = f(c,c,c,x_4,\dots,x_n)$ depends on at least one variable (else $c \in \mathrm{Wdom}(f)$). From Lemma 2.1 it follows that $\mathrm{Ess}(t_{3\leftarrow 2}) = \{x_4,\dots,x_n\}$ and hence $t \in G^{n-1}_{2,k}$. □

Corollary 3.1. *If* $f \in P^n_k$, *is a symmetric function with non-trivial arity gap, then its every subfunction* $g = f(x_n = c)$ *with* $c \notin \mathrm{Dom}(f)$ *has non-trivial arity gap.*

Proof. If $f \in G^n_{n,k}$ we are done by Theorem 3.1 and if $f \in G^n_{2,k}$ by Theorem 3.2. □

4. Separable sets of symmetric functions with non-trivial arity gap.

Definition 4.1. *A set* M *of essential variables in* f *is called separable in* f *if there is a subfunction* g *of* f *such that* $M = \mathrm{Ess}(g)$.

$\mathrm{Sep}(f)$ *denotes the set of all separable sets in* f *and* $\mathrm{sep}(f) = |\,\mathrm{Sep}(f)|$.

Note that the constants in the range $V(f) = \{c \in K \mid \exists \alpha \in K^n,\ \ f(\alpha) = c\}$ of f form subfunctions of f, which do not depend on any essential variable.

So, the empty set, we will include in Sep(f).

The numbers sep(f) and sub(f) are important complexity measures of a function $f \in P_k^n$. The separable sets and the valuations sep(f) and sub(f) are studied in work by many authors: O. Lupanov [5], K. Chimev [1, 2], A. Salomaa [6], S. Shtrakov and K. Denecke [9], etc.

If $f \gg g$ with $\text{ord}(g) = m > 0$ then g uniquely determines an m-element set M, $M = \text{Ess}(g) \subseteq \text{Ess}(f)$, which is separable in f. It is possible for the same M to be the set of essential variables of another subfunction t, $f \gg t$ of f, i.e., $\text{Ess}(g) = \text{Ess}(t)$, but $g \neq t$. Consequently, $\text{sep}(f) \leq \text{sub}(f)$. Theorem 3.1 and Theorem 3.2 show that if f is a symmetric function with non-trivial arity gap then its subfunctions determined by constants outside Dom(f) have non-trivial arity gap. Lemma 3.1 gives an upper bound of sub(f).

In this section, we prove that the complexity measure sep(f) assumes its maximum value on the symmetric functions with non-trivial arity gap.

Theorem 4.1. *If f is a symmetric function with non-trivial arity gap, then each set of essential variables in f is separable in f.*

Proof. Let $f \in G_{n,k}^n$, $n \leq k$ and let $\text{Ess}(f) = \{x_1, \ldots, x_n\}$. Without loss of generality let us prove that $M = \{x_1, \ldots, x_m\}$, $m < n$ is a separable set in f. According to (1) there are constants $c_1, \ldots, c_n \in K$ such that $f(c_1, \ldots, c_n) \neq a_0$, where $a_0 = f(d_1, \ldots, d_n)$ for all $(d_1, \ldots, d_n) \in Eq_k^n$. We must show that if $f_1 := f(x_{m+1} = c_{m+1}, \ldots, x_n = c_n)$ then $M = \text{Ess}(f_1)$. Let $x_t \in M$ be an arbitrary variable from M, i.e., $1 \leq t \leq m$. Again from (1) it follows that

$$f(c_1, \ldots, c_{t-1}, c_n, c_{t+1}, \ldots, c_m, \ldots, c_n) = a_0.$$

Hence $x_t \in \text{Ess}(f_1)$, which implies $M = \text{Ess}(f_1)$.

Let $f \in G_{2,k}^n$, $n \leq k$ be a symmetric function. Without loss of generality let us assume that $M = \{x_1, \ldots, x_m\}$, $m < n$ is a set of essential variables in f. We have to prove that M is a separable set in f. Since $x_1 \in \text{Ess}(f)$ by Theorem 1.2 [2], there is a chain of subfunctions

$$f = f_n \rhd f_{n-1} \ldots \rhd f_2 \rhd f_1$$

such that $\text{Ess}(f_1) = \{x_1\}$ and $\text{Ess}(f_j) = \{x_1, x_{i_2}, \ldots, x_{i_j}\}$ for all $j = 2, 3, \ldots, n$. Without loss of generality we may assume that $i_l = l$ for $l = 2, \ldots, j$ and there are constants $c_{m+1}, \ldots, c_n$ for the variables in $\text{Ess}(f) \setminus Ess(f_m)$ such that

$$f_m = f(x_{m+1} = c_{m+1}, x_{m+2} = c_{m+2}, \ldots, x_n = c_n).$$

Consequently, $f(x_{m+1} = c_{m+1}, x_{m+2} = c_{m+2}, \ldots, x_n = c_n)$ is a function which depends essentially on the variables $x_1, \ldots, x_m$, i.e., M is a separable set in f. $\square$

Corollary 4.1. *If $f \in P_k^n$ is a symmetric function with non-trivial arity gap then* $\mathrm{sep}(f) = 2^n$.

Corollary 4.2. *If $f \in P_k^n$ is a symmetric function with non-trivial arity gap then $sub(f) \geq 2^n$.*

Lemma 3.1 implies that if $n \leq k$ and $f \in G_{n,k}^n$ then

$$2^n = \mathrm{sep}(f) \leq \mathrm{sub}(f) \leq \sum_{i=1}^{n} \binom{k}{i}$$

and if $k = n$ then $2^n = \mathrm{sep}(f) = \mathrm{sub}(f)$.

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