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EVERY n-DIMENSIONAL SEPARABLE METRIC SPACE CONTAINS A TOTALLY DISCONNECTED $(n-1)$-DIMENSIONAL SUBSET WITH NO TRUE QUASI-COMPONENTS*

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The quasi-component $Q(x)$ of a point x of a topological space X is by definition the intersection of all open and closed subsets of X, every one of which contains x. If a quasi-component consists of more than one point, it is called a *true quasi-component*. In this note we give a simple construction of (at least) $(n-1)$-dimensional totally disconnected subspace Y of a given n-dimensional separable metric space X such that every quasi-component in Y is a single point.

1. Basic concepts and definitions. Let X be a topological space and $x \in X$ be some point of X. The intersection $Q(x)$ of all closed and open (clopen) subsets of X that contain x is called the *quasi-component* of x. The quasi-component may consists of more than one point even if X does not contain a connected subspace different from a point – for example the Knaster-Kuratowski fan [1].

Suppose, further, that X is a separable metric space and $\dim X = n$. Then X may be regarded as a subset of some Euclidean space $\mathbb{R}^m$ for $m \leq 2n+1$ [2, p. 262].

A topological space X is called totally disconnected if X does not contain a connected subspace different from a point [3].

And so, we suppose below that $X \subset \mathbb{R}^m$ and $\dim X = n$. We call as well that $F \subset X$ is a *separator* in X, if F is a closed subset of X and $X \setminus F$ is not connected.

2. The space Y. Denote by $\mathcal{S}$ the set of all separators of X. It is easy to see that the cardinality card$\mathcal{S}$ of $\mathcal{S}$ is equal to $\mathbf{c}$, the cardinal number of the continuum.

Consider next the set $\mathcal{P}$ of all hyperplanes in $\mathbb{R}^m$ and denote by $\mathcal{P}_0$ the subset of $\mathcal{P}$ which consists of hyperplanes p with equations

$$p: \ a_0 + a_1x_1 + a_2x + \cdots + a_mx_m = 0,$$

such that a_k, $k = 0, 1, \ldots, m$ are rational numbers. Evidently, $\mathcal{P}_0$ is a countable set and, hence, the set $\mathcal{P}_1 = \mathcal{P} \setminus \mathcal{P}_0$ has cardinality $\mathbf{c}$. After that it is obvious that to every separator $F \in \mathcal{S}$ one can attach a hyperplane $p_F \in \mathcal{P}_1$ so that the intersection $F \cap p_F$ is non empty set and, moreover, it is possible to choose $x_F \in F \cap p_F$ in such a way that $x_F \neq x_G$ for every two different elements F and G in $\mathcal{S}$ (we suppose here that $\dim X > 1$.

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Then, the desired set Y is $\{x_F | F \in \mathcal{S}\}$. Clearly, Y is not connected between any pair $x \neq y$ of different points of Y because one can find a hyperplane $p_{xy} \in \mathcal{P}_0$ which separates x and y in $\mathbb{R}^m$ and, evidently, $p_{xy} \cap Y = \emptyset$.

Next, it is easy to see that $\dim Y \geq n-1$, because Y meets every partition in X. Note that if we add a single point $\{*\}$ to Y, the space $Y^* = Y \cup \{*\}$ remains totally disconnected, which is an answer of a question of G. Dimov

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ВСЯКО n-МЕРНО СЕПАРАБЕЛНО МЕТРИЧНО ПРОСТРАНСТВО СЪДЪРЖА НАПЪЛНО НЕСВЪРЗАНО $(n-1)$-МЕРНО ПОДМНОЖЕСТВО С ЕДНОТОЧКОВИ КВАЗИКОМПОНЕНТИ

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Тази бележка съдържа елементарна конструкция на множество с указаните в заглавието свойства. Да отбележим в допълнение, че така полученото множество остава напълно несвързано дори и след като се допълни с краен брой елементи.