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Serdica Math. J. **31** (2005), 279–290

Serdica
Mathematical Journal

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NECESSARY AND SUFFICIENT CONDITION FOR OSCILLATIONS OF NEUTRAL DIFFERENTIAL EQUATION

E. M. Elabbasy, T. S. Hassan, S. H. Saker

Communicated by I. D. Iliev

Abstract. We obtain necessary and sufficient conditions for the oscillation of all solutions of neutral differential equation with mixed (delayed and advanced) arguments

$$\left[x(t) + \sum_{k=1}^{m} r_k x(t-\mu_k)\right]' + \sum_{i=1}^{n} p_i x(t-\tau_i) = 0,$$

where r_k, μ_k, p_i, $\tau_i \in R$ for $k = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$. Our results extend and improve several known results in the literature and solve an open problem posed by Gyori and Ladas [6].

1. Introduction.

In recent years the literature on the oscillation of neutral delay differential equations is growing very fast. It is a relatively new field with interesting applications in real world life problems. In fact, the neutral delay differential equations appear in modelling of the networks containing lossless

2000 *Mathematics Subject Classification:* 34K15, 34C10.
Key words: Oscillation, Characteristic equation, Neutral delay differential equations.

transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect swithching circuits), in the study of vibrating masses attached to an elastic bar, as the Euler equation in some variational problems, theory of automatic control and in neuromechanical systems in which inertia plays an important role (see [2] and [5]).

In this paper we shall consider the following first order linear neutral differential equation with mixed (delayed and advanced) arguments

$$\left[x(t)+\sum_{k=1}^{m} r_k x(t-\mu_k)\right]' + \sum_{i=1}^{n} p_i x(t-\tau_i) = 0, \tag{1}$$

where $r_k,\ \mu_k,\ p_i,\ \tau_i \in R$ for $k = 1, 2, \dots, m$ and $i = 1, 2, \dots, n$.

As usual a solution $x(t)$ of equation (1) is said to be oscillatory if it has arbitrarily large zeros on $[t_0, \infty)$. Otherwise it is nonoscillatory. The equation (1) is called oscillatory if every solution of this equation is oscillatory.

In the case when $n = m = 1$, $r_1 \in R$, μ_1, τ_1 and $p_1 \geq 0$, Ladas and Stavroulakis [10] proved that the necessary and sufficient condition for the oscillation of (1) is that the characteristic equation has no real roots. For $m = 1$ and $r_1 \in R$, μ_1, p_i and $\tau_i \geq 0$ for $i = 1, 2, \dots, n$ this was proved by Kulenovic, Ladas and Meimaridou [8]. Also for $n = m = 1$ and μ_1, r_1, τ_1 and $p_1 \in R$ the above result was proved by Grove, Ladas and Meimaridou [5]. When $r_k = 0$ or $\mu_k = 0$, the result is due to Tramov [15]. For a simple proof see Ladas, Sficas and Stavroulakis [9].

One of our results establishes the important fact that the oscillatory nature of the solutions of Eq. (1) is determined by the roots of the characteristic equation

$$\lambda + \sum_{k=1}^{m} r_k \lambda e^{-\lambda\mu_k} + \sum_{i=1}^{n} p_i e^{-\lambda\tau_i} = 0.$$

This is in contrast to the stability nature of the solutions of Eq. (1). Snow [13] has shown, for example, that even though the characteristic roots of an NDE may all have negative real parts, it is still possible for some solutions to be unbounded. See also [11].

In the sequel, for convenience, when we write a functional inequality, we shall mean that it holds eventually, that is, for all sufficiently large values of the argument t.

2. Main results. The following lemmas are useful tools in the proof of the main results of this paper.

Lemma 1. *Let $v(t)$ be a positive and continuously differentiable function on some interval $[t_0,\infty)$.*

Assume that there exist positive numbers A and α such that for t sufficiently large,

$$v'(t) \leq 0 \text{ and } v(t-\alpha) < Av(t).$$

Set

$$\Lambda = \left\{\lambda \geq 0 : v'(t) + \lambda v(t) \leq 0, \quad \text{for } t \text{ sufficiently large.}\right\}$$

Then $A > 1$ and $\lambda_0 = \dfrac{\ln(A)}{\alpha} \notin \Lambda$.

Lemma 2. *Let $v(t)$ be a positive and continuously differentiable function on some interval $[t_0,\infty)$. Assume that there exist positive numbers A and α such that for t sufficiently large,*

$$v'(t) \geq 0 \text{ and } v(t+\alpha) < Av(t).$$

Set

$$\Lambda = \left\{\lambda \geq 0 : -v'(t) + \lambda v(t) \leq 0, \quad \text{for } t \text{ sufficiently large.}\right\}.$$

Then $A > 1$ and $\lambda_0 = \dfrac{\ln(A)}{\alpha} \notin \Lambda$.

Lemma 3. *Let p and σ be positive constants and let $z(t)$ be an eventually positive solution of the delay differential inequality*

$$z'(t) + pz(t-\sigma) \leq 0.$$

Let $y(t)$ be an eventually positive solution of the advance differential inequality

$$y'(t) - py(t+\sigma) \geq 0.$$

Then for t sufficiently large,

$$z(t-\sigma) \leq \beta z(t),$$

and

$$y(t+\sigma) \leq \beta y(t),$$

where $\beta = \dfrac{4}{(p\sigma)^2}$.

For the proof of these lemmas see [6].

Theorem 1. *Consider the neutral differential equation with mixed (delayed and advanced) arguments* (1). *Assume that* $r_k \in R^+$, μ_k, p_i, $\tau_i \in R$, $p_i p_j \geq 0$ *and* $\tau_i \tau_j \geq 0$ *for* $k = 1, 2, \ldots, m$ *and* $i,\ j = 1, 2, \ldots, n$. *Then a necessary and sufficient condition for the oscillation of* (1) *is that the characteristic equation*

$$\lambda + \sum_{k=1}^{m} r_k \lambda e^{-\lambda \mu_k} + \sum_{i=1}^{n} p_i e^{-\lambda \tau_i} = 0, \tag{2}$$

has no real roots.

Proof. The proof of the necessity part of the theorem is very brief. If it were false, the characteristic equation (2) would have a real root λ_0 and therefore Eq. (1) would have the nonoscillatory solution

$$x(t) = e^{\lambda_0 t}.$$

But this contradicts the hypothesis that every solution of Eq. (1) oscillates.

Now suppose that equation (2) has no real root.

Set

$$F(\lambda) = \lambda + \sum_{k=1}^{m} r_k \lambda e^{-\lambda \mu_k} + \sum_{i=1}^{n} p_i e^{-\lambda \tau_i},$$

and

$$z(t) = x(t) + \sum_{k=1}^{m} r_k x(t - \mu_k).$$

As $F(\lambda) = 0$ has no real roots, we consider the following cases.

Case (1): $\tau_i < 0$ for $i = 1, 2, \ldots, n$.

As $F(-\infty) = -\infty$ it follows that $F(0) = \sum_{i=1}^{n} p_i < 0$ implies $p_i < 0$ for $i = 1, 2, \ldots, n$. Then $F(\infty) = -\infty$ and because equation (2) has no real roots it follows that there exists a positive constant δ such that

$$\lambda + \sum_{k=1}^{m} r_k \lambda e^{-\lambda \mu_k} + \sum_{i=1}^{n} p_i e^{-\lambda \tau_i} \leq -\delta, \quad \text{for } \lambda \in R, \tag{3}$$

where

$$-\delta = \max_{\lambda \in R} F(\lambda).$$

Assume, for the sake of contradiction, that equation (1) has an eventually positive solution $x(t)$. Then eventually

$$z'(t) = -\sum_{i=1}^{n} p_i x(t - \tau_i) > 0.$$

Define the set

$$\Lambda = \left\{ \lambda \geq 0 : -z'(t) + \lambda z(t) \leq 0, \quad \text{for } t \text{ sufficiently large.} \right\}.$$

Clearly $0 \in \Lambda$ and Λ is a non-empty subinterval of R^+.

We show that Λ has the following contradictory properties

(p_1) There exists positive numbers λ_1 and λ_2 such that

$$\lambda_1 \in \Lambda \quad \text{and} \quad \lambda_2 \notin \Lambda.$$

(p_2) If $\lambda \in \Lambda$ therefor $\lambda + \delta \in \Lambda$ where δ is defined in (3).

We take $\tau^* \geq \tau_i$ for $i = 1, 2, \ldots, n$. Observe that $z(t)$ is increasing $z(t) < x(t)$ and so

$$z'(t) + \sum_{i=1}^{n} p_i z(t - \tau^*) \geq 0. \tag{4}$$

Therefore,

$$z'(t) + \sum_{i=1}^{n} p_i z(t) \geq 0,$$

which implies that $-\sum_{i=1}^{n} p_i \in \Lambda$. From Lemmas 2, 3 and (4) it follows that

$$\lambda_2 = \frac{\ln \left[\frac{4}{\left(\tau^* \left(\sum_{i=1}^{n} p_i\right)\right)^2} \right]}{-\tau^*} \notin \Lambda.$$

We turn to the proof of (p_2). Let $\lambda \in \Lambda$ and set

$$\phi(t) = e^{-\lambda t} z(t).$$

Therefore

$$\phi'(t) = -e^{-\lambda t} \left(-z'(t) + \lambda z(t) \right) \geq 0,$$

which shows that $\phi(t)$ is increasing. Thus

$$\begin{aligned}
-z'(t) + (\lambda + \delta) z(t) &= \sum_{i=1}^{n} p_i z(t - \tau_i) + (\lambda + \delta) z(t) \\
&= \sum_{i=1}^{n} p_i \phi(t - \tau_i) e^{\lambda(t - \tau_i)} + (\lambda + \delta) \phi(t) e^{\lambda t} \\
&\leq \phi(t) e^{\lambda t} \left[\sum_{k=1}^{m} r_k \lambda e^{-\lambda \mu_k} + \sum_{i=1}^{n} p_i e^{-\lambda \tau_i} + \lambda + \delta \right] \\
&\leq \phi(t) e^{\lambda t} (-\delta + \delta) = 0.
\end{aligned}$$

This proves (p_2) and thus the proof of the Theorem is complete in this case.

Case (2): $\tau_i > 0$ for $i = 1, 2, \dots, n$.

As $F(\infty) = \infty$, it follows that $F(0) = \sum_{i=1}^{n} p_i > 0$ implies $p_i > 0$ for $i = 1, 2, \dots, n$. Then $F(-\infty) = \infty$ and because equation (2) has no real roots it follows that there exists a positive constant δ such that

$$\lambda + \sum_{k=1}^{m} r_k \lambda e^{-\lambda \mu_k} + \sum_{i=1}^{n} p_i e^{-\lambda \tau_i} \geq \delta, \quad \text{for } \lambda \in R, \tag{5}$$

where

$$\delta = \min_{\lambda \in R} F(\lambda).$$

Assume, for the sake of contradiction, that equation (1) has an eventually positive solution $x(t)$. Then eventually

$$z'(t) = -\sum_{i=1}^{n} p_i x(t - \tau_i) < 0.$$

Define the set

$$\Lambda = \left\{ -\lambda \geq 0 : \; z'(t) - \lambda z(t) \leq 0, \quad \text{for } t \text{ sufficiently large.} \right\}.$$

As in case (1), Λ is a non-empty subinterval of R^+.

We show that Λ has the following contradictory properties

(p_1) There exists positive numbers λ_1 and λ_2 such that

$$\lambda_1 \in \Lambda \quad \text{and} \quad \lambda_2 \notin \Lambda.$$

(p_2) If $-\lambda \in \Lambda$ therefore $-\lambda + \delta \in \Lambda$ where δ is defined in (5).

We take $\tau_i \geq \tau^*$ for $i = 1, 2, \dots, n$. Observe that $x(t)$ is decreasing, $z(t) < x(t)$ and so

$$z'(t) + \sum_{i=1}^{n} p_i z(t - \tau^*) \leq 0. \tag{6}$$

Therefore

$$z'(t) + \sum_{i=1}^{n} p_i z(t) \leq 0,$$

which implies that $\sum_{i=1}^{n} p_i \in \Lambda$. From Lemmas 1, 3 and (6) it follows that

$$\lambda_2 = \frac{\ln\left[\dfrac{4}{\left(\tau^*\left(\sum_{i=1}^{n} p_i\right)\right)^2}\right]}{\tau^*} \notin \Lambda.$$

We turn to the proof of (p_2).

Let $-\lambda \in \Lambda$ and set

$$\phi(t) = e^{-\lambda t} z(t).$$

Therefore

$$\phi'(t) = e^{-\lambda t}\left(z'(t) - \lambda z(t)\right) \leq 0,$$

which shows that $\phi(t)$ is non-increasing. Thus

$$\begin{aligned} z'(t) + (-\lambda+\delta)z(t) &= -\sum_{i=1}^{n} p_i z(t-\tau_i) + (-\lambda+\delta)z(t) \\ &= -\sum_{i=1}^{n} p_i \phi(t-\tau_i) e^{\lambda(t-\tau_i)} + (-\lambda+\delta)\phi(t)e^{\lambda t} \\ &\leq \phi(t)e^{\lambda t}\left[-\sum_{k=1}^{m} \lambda r_k e^{-\lambda\mu_k} - \sum_{i=1}^{n} p_i e^{-\lambda\tau_i} - \lambda + \delta\right] \\ &\leq \phi(t)e^{\lambda t}(-\delta+\delta) = 0. \end{aligned}$$

This proves (p_2) and therefore the proof of the theorem is complete in this case. □

Remark. Consider equation (1) and assume that $p_{g_i} > 0$, $i = 1, 2, \ldots, l$ and that $p_{h_j} \leq 0$, $j = 1, 2, \ldots, s$ with $l + s = n$. Let $q_{h_j} = -p_{h_j}$, $j = 1, 2, \ldots, s$. Then equation (1) takes the form

$$\left[x(t) + \sum_{k=1}^{m} r_k x(t-\mu_k)\right]' + \sum_{i=1}^{l} p_{g_i} x(t-\tau_{g_i}) - \sum_{j=1}^{s} q_{h_j} x\left(t-\tau_{h_j}\right) = 0, \tag{7}$$

or simply,

$$\left[x(t) + \sum_{k=1}^{m} r_k x(t-\mu_k)\right]' + \sum_{i=1}^{l} p_i x(t-\tau_i) - \sum_{j=1}^{s} q_j x(t-\sigma_j) = 0, \tag{8}$$

where $r_k \in R^-$, μ_k, p_i, τ_i, q_j and $\sigma_j \in R^+$, $k = 1, 2, \ldots, m$, $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, s$ with $l + s = n$, $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_l$ and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_s$.

Theorem 2. *Consider the neutral delay differential equation* (8)*. Then*

$$\sum_{i=1}^{l} p_i > \sum_{j=1}^{s} q_j, \quad \text{and} \quad \tau_1 \geq \sigma_1, \tag{9}$$

are necessary conditions for the oscillation of all solutions of equation (8) *while*

$$lp_i > \sum_{j=1}^{s} q_j, \ \tau_i \geq \sigma_1 \quad \text{for all } i = 1, 2, \dots, l, \tag{10}$$

$$\sum_{i=1}^{l} \left[1 + \sum_{k=1}^{m} r_k - \sum_{j=1}^{s} q_j \left(\tau_i - \sigma_j \right) \right] \geq 0, \tag{11}$$

and

$$\sum_{i=1}^{l} \left[lp_i - \sum_{j=1}^{s} q_j \right] > \frac{1}{e} \left[\sum_{i=1}^{l} \left[1 + \sum_{k=1}^{m} r_k - \sum_{j=1}^{s} q_j \left(\tau_i - \sigma_j \right) \right] \right], \tag{12}$$

where e is natural number, are sufficient conditions for the oscillation.

P r o o f. The characteristic equation of equation (8) is

$$F(\lambda) = \lambda + \sum_{k=1}^{m} r_k \lambda e^{-\lambda \mu_k} + \sum_{i=1}^{l} p_i e^{-\lambda \tau_i} - \sum_{j=1}^{s} q_j e^{-\lambda \sigma_j} = 0. \tag{13}$$

Assume that all solutions of equation (8) are oscillatory. Consequently the characteristic equation (13) has no real roots. As $F(\infty) = \infty$, it follows that

$$F(0) = \sum_{i=1}^{l} p_i - \sum_{j=1}^{s} q_j > 0, \tag{14}$$

which implies that

$$\sum_{i=1}^{l} p_i > \sum_{j=1}^{s} q_j. \tag{15}$$

Also, we have $\tau_i \geq \sigma_i$ since if $\tau_i < \sigma_i$, then we get $F(-\infty) = -\infty$, which means that equation (13) has a real root. On the other hand, assume that equation (8) has nonoscillatory solution and then the characteristics equation (13) has a real root λ_0,

$$F(\lambda_0) = \lambda_0 + \sum_{k=1}^{m} r_k \lambda_0 e^{-\lambda_0 \mu_k} + \sum_{i=1}^{l} p_i e^{-\lambda_0 \tau_i} - \sum_{j=1}^{s} q_j e^{-\lambda_0 \sigma_j} = 0. \tag{16}$$

But for all $\lambda \in R$, one can write for every $i = 1, 2, \ldots, l$

$$\begin{aligned}
& \lambda\left[1+\sum_{k=1}^{m} r_k e^{-\lambda\mu_k} - \sum_{j=1}^{s} q_j e^{-\lambda\sigma_j} \int_0^{\tau_i-\sigma_j} e^{-\lambda s} ds\right] \\
(17) \qquad = & \left[\lambda+\sum_{k=1}^{m} r_k \lambda e^{-\lambda\mu_k} - \sum_{j=1}^{s} q_j e^{-\lambda\sigma_j}\right] + \left(\sum_{j=1}^{s} q_j\right) e^{-\lambda\tau_i}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \sum_{i=1}^{l} \lambda\left[1+\sum_{k=1}^{m} r_k e^{-\lambda\mu_k} - \sum_{j=1}^{s} q_j e^{-\lambda\sigma_j} \int_0^{\tau_i-\sigma_j} e^{-\lambda s} ds\right] \\
(18) \qquad = & \; l\left[\lambda+\sum_{k=1}^{m} r_k \lambda e^{-\lambda\mu_k} - \sum_{j=1}^{s} q_j e^{-\lambda\sigma_j}\right] + \left(\sum_{j=1}^{s} q_j\right) \sum_{i=1}^{l} e^{-\lambda\tau_i},
\end{aligned}$$

and consequently $F(\lambda)$ can be written in the form

$$\begin{aligned}
F(\lambda) \; = & \; \frac{1}{l}\left\{\sum_{i=1}^{l} \lambda\left[1+\sum_{k=1}^{m} r_k e^{-\lambda\mu_k} - \sum_{j=1}^{s} q_j e^{-\lambda\sigma_j} \int_0^{\tau_i-\sigma_j} e^{-\lambda s} ds\right]\right. \\
(19) \qquad & \left. - \left(\sum_{j=1}^{s} q_j\right) \sum_{i=1}^{l} e^{-\lambda\tau_i}\right\} + \sum_{i=1}^{l} p_i e^{-\lambda\tau_i}.
\end{aligned}$$

Then for all $\lambda \geq 0$, we get

$$F(\lambda) > \frac{1}{l}\left\{\sum_{i=1}^{l}\left[l p_i - \sum_{j=1}^{s} q_j\right] e^{-\lambda\tau_i} + \sum_{i=1}^{l} \lambda\left[1+\sum_{k=1}^{m} r_k - \sum_{j=1}^{s} q_j(\tau_i - \sigma_j)\right]\right\} > 0.$$

(20)

Consequently, $F(\lambda)$ has no nonnegative real roots and then $\lambda_0 < 0$. Using (16) and (18), we get

$$\sum_{i=1}^{l} \lambda_0\left[1+\sum_{k=1}^{m} r_k e^{-\lambda_0\mu_k} - \sum_{j=1}^{s} q_j e^{-\lambda_0\sigma_j} \int_0^{\tau_i-\sigma_j} e^{-\lambda_0 s} ds\right]$$

$$= l\left[\lambda_0+\sum_{k=1}^{m}r_k\lambda_0 e^{-\lambda_0\mu_k}-\sum_{j=1}^{s}q_j e^{-\lambda_0\sigma_j}\right]+\left(\sum_{j=1}^{s}q_j\right)\sum_{i=1}^{l}e^{-\lambda_0\tau_i}$$

$$(21)\qquad = -\sum_{i=1}^{l}\left[lp_i-\sum_{j=1}^{s}q_j\right]e^{-\lambda_0\tau_i}<0.$$

Since $\lambda_0<0$, from equation (21), one can write

$$0 < \sum_{i=1}^{l}\left[1+\sum_{k=1}^{m}r_k e^{-\lambda_0\mu_k}-\sum_{j=1}^{s}q_j e^{-\lambda_0\sigma_j}\int_0^{\tau_i-\sigma_j}e^{-\lambda_0 s}ds\right]$$

$$(22)\qquad < \sum_{i=1}^{l}\left[1+\sum_{k=1}^{m}r_k-\sum_{j=1}^{s}q_j\left(\tau_i-\sigma_j\right)\right].$$

Using equations (21) and (22), we obtain

$$\lambda_0\sum_{i=1}^{l}\left[1+\sum_{k=1}^{m}r_k-\sum_{j=1}^{s}q_j\left(\tau_i-\sigma_j\right)\right]$$

$$< \sum_{i=1}^{l}\lambda_0\left[1+\sum_{k=1}^{m}r_k e^{-\lambda_0\mu_k}-\sum_{j=1}^{s}q_j e^{-\lambda_0\sigma_j}\int_0^{\tau_i-\sigma_j}e^{-\lambda_0 s}ds\right]$$

$$(23)\qquad = -\sum_{i=1}^{l}\left[lp_i-\sum_{j=1}^{s}q_j\right]e^{-\lambda_0\tau_i}<0,$$

and consequently,

$$(24)\qquad \lambda_0+\frac{\sum_{i=1}^{l}\left[lp_i-\sum_{j=1}^{s}q_j\right]e^{-\lambda_0\tau_i}}{\sum_{i=1}^{l}\left[1+\sum_{k=1}^{m}r_k-\sum_{j=1}^{s}q_j\left(\tau_i-\sigma_j\right)\right]}<0.$$

Thus, the equation

$$(25)\qquad \lambda+\frac{\sum_{i=1}^{l}\left[lp_i-\sum_{j=1}^{s}q_j\right]e^{-\lambda\tau_i}}{\sum_{i=1}^{l}\left[1+\sum_{k=1}^{m}r_k-\sum_{j=1}^{s}q_j\left(\tau_i-\sigma_j\right)\right]}=0,$$

has a negative real root, which is a contradiction with (12). □

Example. The following delay differential equation

$$(x(t)-0.1x(t-0.2)-0.1x(t+0.3))'+4x(t-2.6)-x(t-2.5)-0.6x(t-1.5)=0,$$

is oscillatory because all necessary and sufficient conditions for that are satified.

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Department of Mathematics
Faculty of Science
Mansoura University
Mansoura, 35516, Egypt
e-mail: emelabbasy@mans.edu.eg
e-mail: tshassan@mans.edu.eg
e-mail: shsaker@mans.edu.eg

Received June 29, 2005