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DISPERSIVE ESTIMATES OF SOLUTIONS TO THE WAVE EQUATION WITH A POTENTIAL IN DIMENSIONS TWO AND THREE

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Abstract. We prove dispersive estimates for solutions to the wave equation with a real-valued potential $V \in L^\infty(\mathbf{R}^n)$, $n = 2$ or 3, satisfying $V(x) = O(\langle x\rangle^{-(n+1)/2-\epsilon})$, $\epsilon > 0$.

1. Introduction and statement of results. Let $V \in L^\infty(\mathbf{R}^n)$, $n \geq 2$, be a real-valued function satisfying

$$|V(x)| \leq C\langle x\rangle^{-\delta}, \quad \forall x \in \mathbf{R}^n, \tag{1.1}$$

with constants $C > 0$ and $\delta > (n+1)/2$, where $\langle x\rangle = (1+|x|^2)^{1/2}$. Denote by G_0 and G the self-adjoint realizations of the operators $-\Delta$ and $-\Delta + V(x)$ on $L^2(\mathbf{R}^n)$. It is well known that the absolutely continuous spectrums of the

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operators G_0 and G coincide with the interval $[0,+\infty)$. Moreover, by Kato's theorem the operator G has no strictly positive eigenvalues. When $n \geq 3$ this implies that G has no strictly positive resonances neither. Indeed, it is possible to show in this case that, under the assumption (1.1) with $\delta > 2$, such a resonance is in fact an eigenvalue (e.g. see [5], [6]). When $n = 2$ it follows easily that, under the assumption (1.1) with $\delta > 1$, there exists an $a_0 > 0$ such that G has no resonances in the interval $[a_0,+\infty)$.

Throughout this paper, given $1 \leq p \leq +\infty$, L^p will denote the space $L^p(\mathbf{R}^n)$. Also, given an $a > 0$ denote by $\chi_a \in C^\infty(\mathbf{R})$ a function supported in the interval $[a,+\infty)$, $\chi_a = 1$ on $[a+1,+\infty)$. The purpose of this work is to prove the following

Theorem 1.1 *Assume* (1.1) *fulfilled. If* $n = 3$, *for every* $a > 0$, $2 \leq p < +\infty$, *there exists a constant* $C > 0$ *so that the following estimate holds*

$$\left\| e^{it\sqrt{G}}(\sqrt{G})^{-\alpha(n+1)/2}\chi_a(\sqrt{G})\right\|_{L^{p'}\to L^p} \leq C|t|^{-\alpha(n-1)/2}, \quad \forall t \neq 0, \tag{1.2}$$

where $1/p + 1/p' = 1$, $\alpha = 1 - 2/p$.

If $n = 2$, *the estimate* (1.2) *holds with* $a = a_0$, a_0 *being as above. Moreover, if in addition* G *has no strictly positive resonances, then* (1.2) *holds for any* $a > 0$.

Note that given a smooth, bounded function f supported in the interval $(0,+\infty)$, the operator-valued function $f(\sqrt{G})$ is well defined even if the operator G is not non-negative. In particular, the operator in the LHS of (1.2) is well defined.

It is well known that the estimate (1.2) holds true for the free operator G_0 with $\chi_a \equiv 1$ in all dimensions. For $n = 3$, an analogue of (1.2) (for $2 \leq p \leq 4$) is proved in [5] for non-negative potentials satisfying (1.1) as well as an extra regularity assumption. In [2] an analogue of (1.2) is proved in all dimensions $n \geq 3$ (for $2 \leq p \leq \dfrac{2(n+1)}{n-1}$) for a class of non-negative potentials. Note also the work [1], where an analogue of (1.2) for $n \geq 3$ is proved for all $2 \leq p \leq +\infty$ but with loss of ε-derivatives for potentials belonging to the Schwartz class $S(\mathbf{R}^n)$. Recently, in [4] an analogue of (1.2) with $n = 3$ has been proved for a class of potentials satisfying (1.1) with $4/3 < \delta \leq 2$, but with a weaker decay as $|t| \to +\infty$. Note also the work [3], where a better time decay than that in (1.2) has been obtained on weighted L^p spaces for potentials satisfying (1.1) with $n = 3$ and $\delta > 2$.

To our best knowledge, for the first time in the present paper the estimate (1.2) is proved in the whole range of values of p. It also seems that the case $n = 2$

has not been treated before our work. We believe that the estimate (1.2) holds in all dimensions under the assumption (1.1) only, but the method developed here does not work any more when $n \geq 4$, because in this case the outgoing and incoming free resolvents do not satisfy analogues of the estimates (2.15) and (2.17) below. The reason for this is that the singularity on the diagonal of the kernels of these resolvents (which in turn is determined by the behaviour at zero of the corresponding Hankel functions) is too strong when $n \geq 4$.

It is expected that the function χ_a in (1.2) could be replaced by the characteristic function of the interval $[0, +\infty)$ (the absolutely continuous spectrum of G) if one additionally supposes that the zero is neither an eigenvalue nor a resonance of G. We believe that our method can be modified in a way allowing to prove such a statement. When $n = 3$ this seems not to be very difficult in view of the nice behaviour at zero of both the free and the perturbed resolvents. When $n = 2$, however, these resolvents have a logarithmic singularity at zero, which would make the proof quite technical.

Our method consists of reducing (1.2) to *semi-classical* estimates (see Theorem 2.1 below) valid for all $2 \leq p \leq +\infty$. The advantage of such an approach is that one can easily make an interpolation between $L^2 \to L^2$ and $L^1 \to L^\infty$ estimates.

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2. Proof of Theorem 1.1. Given parameters $a > 0$, $0 < h \leq 1$, and a function $\varphi \in C_0^\infty([a, +\infty))$, denote

$$\Phi(t; h) = e^{it\sqrt{G}}\varphi(h\sqrt{G}) - e^{it\sqrt{G_0}}\varphi(h\sqrt{G_0}).$$

It is easy to see that Theorem 1.1 follows from the following

Theorem 2.1. *Assume* (1.1) *fulfilled. If* $n = 3$, *for every* $a > 0$, $2 \leq p \leq +\infty$, *there exists a constant* $C > 0$ *so that the following estimate holds*

$$\|\Phi(t; h)\|_{L^{p'} \to L^p} \leq C h^{1-\alpha n} |t|^{-\alpha(n-1)/2}, \quad \forall t \neq 0,\ 0 < h \leq 1, \tag{2.1}$$

where $1/p + 1/p' = 1$, $\alpha = 1 - 2/p$.

If $n = 2$, the estimate (2.1) *holds with $a = a_0$, a_0 being as in the introduction. Moreover, if in addition G has no strictly positive resonances, then* (2.1) *holds for any $a > 0$.*

Indeed, using the identity

$$\sigma^{-\alpha(n+1)/2}\chi_a(\sigma) = \int_0^1 \varphi(\theta\sigma)\theta^{\alpha(n+1)/2-1}d\theta,$$

where $\varphi(\sigma) = \sigma^{1-\alpha(n+1)/2}\chi_a'(\sigma) \in C_0^\infty([a,+\infty))$, since $0 \le \alpha < 1$, we get

$$\left\| e^{it\sqrt{G}}(\sqrt{G})^{-\alpha(n+1)/2}\chi_a(\sqrt{G}) - e^{it\sqrt{G_0}}(\sqrt{G_0})^{-\alpha(n+1)/2}\chi_a(\sqrt{G_0}) \right\|_{L^{p'}\to L^p}$$

$$\le \int_0^1 \|\Phi(t;\theta)\|_{L^{p'}\to L^p}\,\theta^{\alpha(n+1)/2-1}d\theta \tag{2.2}$$

$$\le C|t|^{-\alpha(n-1)/2}\int_0^1 \theta^{-\alpha(n-1)/2}d\theta \le C|t|^{-\alpha(n-1)/2},$$

which implies (1.2).

P r o o f o f T h e o r e m 2.1. Clearly, it suffices to prove (2.1) for $p = +\infty$, $p' = 1$ and $p = p' = 2$. In what follows in this section we will show that (2.1) holds true with $p = +\infty$, $p' = 1$. We write $\Phi(t;h)$ in terms of the outgoing and incoming resolvents

$$R_0^\pm(\lambda) = (G_0 - \lambda^2 \pm i0)^{-1}, \quad R^\pm(\lambda) = (G - \lambda^2 \pm i0)^{-1},$$

as follows

$$\Phi(t;h) = \frac{1}{\pi i}\int_0^\infty e^{it\lambda}\varphi(h\lambda)\left((R^+(\lambda) - R_0^+(\lambda)) - (R^-(\lambda) - R_0^-(\lambda))\right)\lambda d\lambda$$

$$= \sum_\pm \pm\frac{1}{\pi i}\int_0^\infty e^{it\lambda}\varphi(h\lambda)T^\pm(\lambda)\lambda d\lambda := \sum_\pm \pm\Phi^\pm(t;h), \tag{2.3}$$

where

$$T^\pm(\lambda) = R_0^\pm(\lambda)\left(-V + VR^\pm(\lambda)V\right)R_0^\pm(\lambda).$$

Recall that the kernel of the resolvent $R_0^{\pm}(\lambda)$ is given in terms of the Hankel functions by

$$[R_0^{\pm}(\lambda)](x,y) = \pm\frac{i}{4}\left(\frac{\lambda}{2\pi|x-y|}\right)^{\nu} H_{\nu}^{\pm}(\lambda|x-y|),$$

where $\nu = (n-2)/2$. The perturbed resolvent $R^{\pm}(\lambda)$ is defined by the limit

$$R^{\pm}(\lambda) = \lim_{\varepsilon\to 0^+}(G-\lambda^2 \pm i\varepsilon)^{-1} : \langle x\rangle^{-s}L^2 \to \langle x\rangle^{s}L^2, \quad s > 1/2,$$

which exists in view of the limiting absorption principle.

We need now the following

Proposition 2.2. *Assume* (1.1) *fulfilled. If* $n=3$, *the operator-valued function* $T^{\pm}(\lambda) : L^1 \to L^\infty$ *is* C^1 *in* λ *and satatisfies the estimates*

$$\|T^{\pm}(\lambda)\|_{L^1\to L^\infty} \le C, \quad \lambda \ge a, \tag{2.4}$$

$$\left\|\frac{dT^{\pm}}{d\lambda}(\lambda)\right\|_{L^1\to L^\infty} \le C, \quad \lambda \ge a, \tag{2.5}$$

$\forall a > 0$ *with a constant* $C > 0$ *independent of* λ, *which may depend on* a.

If $n = 2$, *the operator-valued function* $T^{\pm}(\lambda) : L^1 \to L^\infty$ *is Hölder of order* $1/2$ *and satatisfies the estimates*

$$\|T^{\pm}(\lambda)\|_{L^1\to L^\infty} \le C\lambda^{-1}, \quad \lambda \ge a_0, \tag{2.6}$$

$$\left\|T^{\pm}(\lambda_2) - T^{\pm}(\lambda_1)\right\|_{L^1\to L^\infty} \le C\lambda_1^{-1}|\lambda_2-\lambda_1|^{1/2}, \quad \lambda_2 > \lambda_1 \ge a_0, \tag{2.7}$$

a_0 *being such that* G *has no resonances in the interval* $[a_0, +\infty)$.

Let $n = 3$. Integrating by parts once the integral in (2.3) and using (2.4) and (2.5), one easily gets

$$\left\|\Phi^{\pm}(t;h)\right\|_{L^1\to L^\infty} \le Ch^{-2}|t|^{-1}, \quad \forall t\neq 0,\ 0 < h \le 1,$$

which clearly implies (2.1) with $p=+\infty$, $p'=1$ in this case.

Let $n = 2$. We will first consider the case $|t| \ge 1$. Choose a real-valued function $\phi \in C_0^\infty([1/3,1/2])$, $\phi \ge 0$, such that $\int \phi(\sigma)d\sigma = 1$. Then the function

$$T_\theta^{\pm}(\lambda) = \theta^{-1}\int T^{\pm}(\lambda+\sigma)\phi(\sigma/\theta)d\sigma, \quad 0 < \theta \le 1,$$

is smooth with values in $\mathcal{L}(L^1, L^\infty)$ and, in view of (2.6) and (2.7), satisfies the estimates

$$\|T_\theta^\pm(\lambda)\|_{L^1\to L^\infty} \le C\lambda^{-1}, \tag{2.8}$$

$$\|T_\theta^\pm(\lambda) - T^\pm(\lambda)\|_{L^1\to L^\infty} \le \theta^{-1}\int \|T^\pm(\lambda+\sigma) - T^\pm(\lambda)\|_{L^1\to L^\infty}\phi(\sigma/\theta)d\sigma$$

$$\le C\theta^{-1}\lambda^{-1}\int \sigma^{1/2}\phi(\sigma/\theta)d\sigma \le C\theta^{1/2}\lambda^{-1}, \tag{2.9}$$

$$\left\|\frac{T_\theta^\pm}{d\lambda}(\lambda)\right\|_{L^1\to L^\infty} \le \theta^{-2}\int \|T^\pm(\lambda+\sigma) - T^\pm(\lambda)\|_{L^1\to L^\infty}|\phi'(\sigma/\theta)|d\sigma$$

$$\le C\theta^{-2}\lambda^{-1}\int \sigma^{1/2}|\phi'(\sigma/\theta)|d\sigma \le C\theta^{-1/2}\lambda^{-1}. \tag{2.10}$$

Hence,

$$\left\|\int_0^\infty e^{it\lambda}\varphi(h\lambda)\left(T_\theta^\pm(\lambda) - T^\pm(\lambda)\right)\lambda d\lambda\right\|_{L^1\to L^\infty}$$

$$\le C\theta^{1/2}\int_0^\infty |\varphi(h\lambda)|d\lambda \le C\theta^{1/2}h^{-1}, \tag{2.11}$$

$$\left\|\int_0^\infty e^{it\lambda}\varphi(h\lambda)T_\theta^\pm(\lambda)\lambda d\lambda\right\|_{L^1\to L^\infty}$$

$$= \left\|t^{-1}\int_0^\infty e^{it\lambda}\frac{d}{d\lambda}\left(\varphi(h\lambda)T_\theta^\pm(\lambda)\lambda\right)d\lambda\right\|_{L^1\to L^\infty} \le Ch^{-1}|t|^{-1}\theta^{-1/2}. \tag{2.12}$$

By (2.11) and (2.12),

$$\|\Phi^\pm(t;h)\|_{L^1\to L^\infty} \le Ch^{-1}\left(\theta^{1/2} + |t|^{-1}\theta^{-1/2}\right) \le Ch^{-1}|t|^{-1/2}, \tag{2.13}$$

if we take $\theta = |t|^{-1}$. If $0 < |t| \le 1$, by (2.6) we get

$$\|\Phi^\pm(t;h)\|_{L^1\to L^\infty} \le Ch^{-1}. \tag{2.14}$$

Clearly, (2.1) with $p = +\infty$, $p' = 1$ follows in this case from (2.13) and (2.14). □

P r o o f o f P r o p o s i t i o n 2.2. We will first prove the following

Lemma 2.3. *If $n = 3$, we have for every $\lambda \geq 0$,*

$$\left\| R_0^{\pm}(\lambda)\langle x\rangle^{-1/2-\epsilon} \right\|_{L^2\to L^\infty} \leq C, \tag{2.15}$$

$$\left\| \frac{dR_0^{\pm}}{d\lambda}(\lambda)\langle x\rangle^{-3/2-\epsilon} \right\|_{L^2\to L^\infty} \leq C, \tag{2.16}$$

$\forall 0 < \epsilon \ll 1$, with a constant $C > 0$ independent of λ but depending on ϵ.

If $n = 2$, we have

$$\left\| R_0^{\pm}(\lambda)\langle x\rangle^{-1/2-\epsilon} \right\|_{L^2\to L^\infty} \leq C\lambda^{-1/2}, \quad \lambda \geq \lambda_0, \tag{2.17}$$

$$\left\| \left(R_0^{\pm}(\lambda_2) - R_0^{\pm}(\lambda_1) \right) \langle x\rangle^{-1-\epsilon} \right\|_{L^2\to L^\infty} \leq C\lambda_1^{-1/2}|\lambda_2-\lambda_1|^{1/2}, \quad \lambda_2 > \lambda_1 \geq \lambda_0, \tag{2.18}$$

$\forall 0 < \epsilon \ll 1$, $\forall \lambda_0 > 0$, with a constant $C > 0$ independent of λ, λ_1 and λ_2, but depending on λ_0 and ϵ.

P r o o f. When $n = 3$, the estimates (2.15) and (2.16) are proved in [5] (Lemma II.2) and [3] (Proposition 2.1) using that

$$H_{1/2}^{\pm}(z) = c^{\pm} e^{\pm iz} z^{-1/2}.$$

That is why we will consider here only the case $n = 2$. Recall that, when $z \to 0$, the function $H_0^{\pm}(z)$ is of the form

$$H_0^{\pm}(z) = H_{0,1}^{\pm}(z) + H_{0,2}^{\pm}(z)\log z, \tag{2.19}$$

with functions $H_{0,j}^{\pm}(z)$ analytic at $z = 0$. For $z \geq 1$ the function $H_0^{\pm}(z)$ is of the form

$$H_0^{\pm}(z) = e^{\pm iz} b_0^{\pm}(z), \tag{2.20}$$

where $b_0^{\pm}(z)$ is a symbol of order $-1/2$, i.e.

$$\left|\partial_z^j b_0^{\pm}(z)\right| \leq C_j z^{-1/2-j}, \quad z \geq 1, \tag{2.21}$$

for all integers $j \geq 0$. Hence, for $\lambda \geq \lambda_0$, we have

$$\left\| R_0^{\pm}(\lambda)\langle x\rangle^{-1/2-\epsilon} \right\|^2_{L^2\to L^\infty} \leq \sup_{x\in\mathbf{R}^2} \int_{\mathbf{R}^2} \left| H_0^{\pm}(\lambda|x-y|)\right|^2 \langle y\rangle^{-1-2\epsilon} dy$$

$$\leq C \sup_{x\in\mathbf{R}^2} \int_{\lambda|x-y|\leq 1} |\log(\lambda|x-y|)|^2 \, dy + C\lambda^{-1} \sup_{x\in\mathbf{R}^2} \int_{\lambda|x-y|\geq 1} \frac{\langle y\rangle^{-1-2\epsilon}}{|x-y|} dy$$

$$\leq C\lambda^{-2} \int_{|\xi|\leq 1} |\log|\xi||^2 \, d\xi + C\lambda^{-1} \sup_{x\in\mathbf{R}^2} \int_{\mathbf{R}^2} \frac{\langle y\rangle^{-1-2\epsilon}}{|x-y|} dy \leq C\lambda^{-1},$$

which implies (2.17). To prove (2.18) we will use the inequality

$$\left|H_0^{\pm}(\sigma\lambda_2) - H_0^{\pm}(\sigma\lambda_1)\right|^2 \leq \sigma \left(\left|H_0^{\pm}(\sigma\lambda_1)\right| + \left|H_0^{\pm}(\sigma\lambda_2)\right| \right) \int_{\lambda_1}^{\lambda_2} \left| \frac{dH_0^{\pm}}{dz}(\sigma\lambda) \right| d\lambda$$

$$\leq \sigma|\lambda_2-\lambda_1| \left(\left|H_0^{\pm}(\sigma\lambda_1)\right|^2 + \left|H_0^{\pm}(\sigma\lambda_2)\right|^2 + |\lambda_2-\lambda_1|^{-1} \int_{\lambda_1}^{\lambda_2} \left| \frac{dH_0^{\pm}}{dz}(\sigma\lambda) \right|^2 d\lambda \right).$$

Thus, in view of (2.19)-(2.21), we have for $\lambda_2 > \lambda_1 \geq \lambda_0$,

$$|\lambda_2-\lambda_1|^{-1} \left\| \left(R_0^{\pm}(\lambda_2) - R_0^{\pm}(\lambda_1)\right) \langle x\rangle^{-1-\epsilon} \right\|^2_{L^2\to L^\infty}$$

$$\leq \sum_{j=1}^{2} \sup_{x\in\mathbf{R}^2} \int_{\mathbf{R}^2} \left|H_0^{\pm}(\lambda_j|x-y|)\right|^2 |x-y|\langle y\rangle^{-2-2\epsilon} dy$$

$$+|\lambda_2-\lambda_1|^{-1} \int_{\lambda_1}^{\lambda_2} \sup_{x\in\mathbf{R}^2} \int_{\mathbf{R}^2} \left| \frac{dH_0^{\pm}}{dz}(\lambda|x-y|) \right|^2 |x-y|\langle y\rangle^{-2-2\epsilon} dy d\lambda$$

$$\leq C \sum_{j=1}^{2} \sup_{x\in\mathbf{R}^2} \int_{\lambda_j|x-y|\leq 1} |\log(\lambda_j|x-y|)|^2 \, |x-y| dy$$

$$+C \sum_{j=1}^{2} \lambda_j^{-1} \sup_{x\in\mathbf{R}^2} \int_{\lambda_j|x-y|\geq 1} \langle y\rangle^{-2-2\epsilon} dy$$

$$+C|\lambda_2-\lambda_1|^{-1}\int_{\lambda_1}^{\lambda_2}\lambda^{-2}\sup_{x\in\mathbf{R}^2}\int_{\lambda|x-y|\le 1}\frac{dy}{|x-y|}d\lambda$$

$$+C|\lambda_2-\lambda_1|^{-1}\int_{\lambda_1}^{\lambda_2}\lambda^{-1}\sup_{x\in\mathbf{R}^2}\int_{\lambda|x-y|\ge 1}\langle y\rangle^{-2-2\epsilon}dyd\lambda$$

$$\le C\sum_{j=1}^{2}\lambda_j^{-3}\int_{|\xi|\le 1}|\xi|\,|\log|\xi||^2\,d\xi+C\sum_{j=1}^{2}\lambda_j^{-1}\int_{\mathbf{R}^2}\langle y\rangle^{-2-2\epsilon}dy$$

$$+C\lambda_1^{-2}\int_{|\xi|\le 1}\frac{d\xi}{|\xi|}+C\lambda_1^{-1}\int_{\mathbf{R}^2}\langle y\rangle^{-2-2\epsilon}dy\le C\lambda_1^{-1},$$

which implies (2.18). □

It is easy to see that Proposition 2.2 follows from Lemma 2.3 and the following

Proposition 2.4. *Assume* (1.1) *fulfilled. Then, there exists a constant* $\lambda_0>0$ *so that we have*

$$\left\|\langle x\rangle^{-1/2-\epsilon}R^{\pm}(\lambda)\langle x\rangle^{-1/2-\epsilon}\right\|_{L^2\to L^2}\le C\lambda^{-1}, \tag{2.22}$$

for $\lambda\ge\lambda_0$, $0<\epsilon\ll 1$, *with a constant* $C>0$ *independent of* λ. *Moreover, if* $n=3$, *we have*

$$\left\|\langle x\rangle^{-3/2-\epsilon}\frac{dR^{\pm}}{d\lambda}(\lambda)\langle x\rangle^{-3/2-\epsilon}\right\|_{L^2\to L^2}\le C\lambda^{-1}, \tag{2.23}$$

for $\lambda\ge\lambda_0$, $0<\epsilon\ll 1$, *with a constant* $C>0$ *independent of* λ. *If* $n=2$, *we have*

$$\left\|\langle x\rangle^{-1-\epsilon}\left(R^{\pm}(\lambda_2)-R^{\pm}(\lambda_1)\right)\langle x\rangle^{-1-\epsilon}\right\|_{L^2\to L^2}\le C\lambda_1^{-1}|\lambda_2-\lambda_1|^{1/2}, \tag{2.24}$$

for $\lambda_2>\lambda_1\ge\lambda_0$, $0<\epsilon\ll 1$, *with a constant* $C>0$ *independent of* λ_1 *and* λ_2.

If G *has no strictly positive resonances, the above estimates hold true for any* $\lambda_0>0$ *with a constant* $C>0$ *depending on* λ_0.

P r o o f. The estimate (2.22) is well known to hold for every $\lambda>0$ for the free operator G_0 in all dimensions, i.e. we have

$$\left\|\langle x\rangle^{-1/2-\epsilon}R_0^{\pm}(\lambda)\langle x\rangle^{-1/2-\epsilon}\right\|_{L^2\to L^2}\le C\lambda^{-1}. \tag{2.25}$$

Let us now see that (2.22) holds under the assumption (1.1) with $\delta > 1$ in all dimensions. To this end, we will take advantage of the identity

$$\langle x\rangle^{-s}R^{\pm}(\lambda)\langle x\rangle^{-s_1}\left(1+K^{\pm}(\lambda)\right)=\langle x\rangle^{-s}R_0^{\pm}(\lambda)\langle x\rangle^{-s_1}, \tag{2.26}$$

where

$$K^{\pm}(\lambda)=\langle x\rangle^{s_1}VR_0^{\pm}(\lambda)\langle x\rangle^{-s}.$$

By (2.25), we have with $1/2 < s_1 \le \delta - 1/2$,

$$\left\|K^{\pm}(\lambda)\right\|_{L^2\to L^2}\le C\lambda^{-1},\quad \forall\lambda>0. \tag{2.27}$$

Hence, there exists $\lambda_0 > 0$ so that we have

$$\left\|\left(1+K^{\pm}(\lambda)\right)^{-1}\right\|_{L^2\to L^2}\le Const,\quad \forall\lambda\ge\lambda_0. \tag{2.28}$$

Thus, (2.22) follows from (2.25), (2.26) and (2.28). Moreover, if G has no strictly positive resonances, (2.28) holds for any $\lambda_0 > 0$. Therefore, in this case (2.22) holds for any $\lambda_0 > 0$.

To prove (2.23) we will use that it holds for G_0 in all dimensions, i.e.

$$\left\|\langle x\rangle^{-3/2-\epsilon}\frac{dR_0^{\pm}}{d\lambda}(\lambda)\langle x\rangle^{-3/2-\epsilon}\right\|_{L^2\to L^2}\le C\lambda^{-1}, \tag{2.29}$$

for $\forall\lambda > 0$, $0 < \epsilon \ll 1$, with a constant $C > 0$ independent of λ. Let now $n = 3$. Then (1.1) is fulfilled with $\delta > 2$. Differentiating (2.26) leads to the identity

$$\langle x\rangle^{-s}\frac{dR^{\pm}}{d\lambda}(\lambda)\langle x\rangle^{-s_1}\left(1+K^{\pm}(\lambda)\right)$$

$$=\langle x\rangle^{-s}\frac{dR_0^{\pm}}{d\lambda}(\lambda)\langle x\rangle^{-s_1}-\langle x\rangle^{-s}R^{\pm}(\lambda)\langle x\rangle^{-s_1+1}\frac{d}{d\lambda}\langle x\rangle^{-1}K^{\pm}(\lambda). \tag{2.30}$$

Let $3/2 < s_1 \le \delta - 1/2$. As above, we conclude that (2.28) still holds. Therefore, (2.23) follows from (2.30) together with the estimates (2.22) and (2.29).

Let now $n = 2$. Then (1.1) is fulfilled with $\delta > 3/2$. We will first show that (2.24) holds for the free operator G_0, i.e.

$$\left\|\langle x\rangle^{-1-\epsilon}\left(R_0^{\pm}(\lambda_2)-R_0^{\pm}(\lambda_1)\right)\langle x\rangle^{-1-\epsilon}\right\|_{L^2\to L^2}\le C\lambda_1^{-1}|\lambda_2-\lambda_1|^{1/2}. \tag{2.31}$$

To this end, fix $\lambda_2 > \lambda_1$ and consider the operator-valued function

$$M^\pm(z) = |\lambda_2 - \lambda_1|^{-z}\langle x\rangle^{-1/2-z-\epsilon}\left(R_0^\pm(\lambda_2) - R_0^\pm(\lambda_1)\right)\langle x\rangle^{-1/2-z-\epsilon}, \quad z \in \mathbf{C}.$$

In view of (2.25), $M^\pm(z)$ is analytic in z for $\operatorname{Re} z \geq 0$ with values in $\mathcal{L}(L^2)$ and satisfies the bounds

$$\|M^\pm(z)\|_{L^2\to L^2} \leq C' e^{A\operatorname{Re} z}, \quad 0 \leq \operatorname{Re} z \leq 1, \tag{2.32}$$

with some constants $C', A > 0$ depending on λ_1 and λ_2. Moreover, by (2.25) and (2.29) we have

$$\|M^\pm(z)\|_{L^2\to L^2} \leq C\lambda_1^{-1}, \tag{2.33}$$

on $\operatorname{Re} z = 0$ and $\operatorname{Re} z = 1$, with a constant $C > 0$ independent of z, λ_1 and λ_2. By (2.32), (2.33) and the Phragmèn-Lindelöf principle we conclude that (2.33) holds for $0 \leq \operatorname{Re} z \leq 1$. In particular, it holds for $z = 1/2$, which is equivalent to (2.31).

In view of (2.26) we have the identity

$$\langle x\rangle^{-s}\left(R^\pm(\lambda_2) - R^\pm(\lambda_1)\right)\langle x\rangle^{-s_1}\left(1 + K^\pm(\lambda_2)\right)$$

$$= \langle x\rangle^{-s}\left(R_0^\pm(\lambda_2) - R_0^\pm(\lambda_1)\right)\langle x\rangle^{-s_1}$$

$$-\langle x\rangle^{-s}R^\pm(\lambda_1)\langle x\rangle^{-s_1+1/2}\left(\langle x\rangle^{-1/2}K^\pm(\lambda_2) - \langle x\rangle^{-1/2}K^\pm(\lambda_1)\right). \tag{2.34}$$

Let $1 < s_1 \leq \delta - 1/2$. Then we have (2.28) with $\lambda = \lambda_2$, which together with (2.22), (2.31) and (2.34) imply (2.24). □

3. Semi-classical L^2 estimates. In this section we will prove (2.1) with $p = p' = 2$. In fact, we will show that this estimate holds for all dimensions and for a larger class of potentials. More precisely, we will prove the following

Theorem 3.1. *Assume* (1.1) *fulfilled with* $\delta > 1$. *Then, there exist constants* $C, a > 0$ *such that the following estimate holds*

$$\|\Phi(t;h)\|_{L^2\to L^2} \leq Ch, \quad \forall t,\ 0 < h \leq 1. \tag{3.1}$$

Moreover, if in addition G *has no strictly positive resonances, then* (3.1) *holds for every* $a > 0$.

Proof. We begin by proving the following

Lemma 3.2. *For every* $\varphi \in C_0^\infty((0,+\infty))$*, we have*

$$\left\|\varphi(h\sqrt{G})-\varphi(h\sqrt{G_0})\right\|_{L^2\to L^2}\le Ch^2. \tag{3.2}$$

Proof. Define the function $\psi \in C_0^\infty((0,+\infty))$ by $\psi(\sigma^2)=\varphi(\sigma)$. To prove (3.2) we will make use of the Helffer-Sjöstrand formula

$$\psi(h^2G)=\frac{1}{\pi}\int_{\mathbf{C}}\frac{\partial\widetilde{\psi}}{\partial\overline{z}}(z)(h^2G-z)^{-1}L(dz), \tag{3.3}$$

where $L(dz)$ denotes the Lebesgue measure on $\mathbf{C}$, and $\widetilde{\psi}\in C_0^\infty(\mathbf{C})$ is an almost analytic continuation of ψ supported in a small complex neighbourhood of $\operatorname{supp}\psi$ and satisfying

$$\left|\frac{\partial\widetilde{\psi}}{\partial\overline{z}}(z)\right|\le C_N|\operatorname{Im}z|^N,\quad \forall N\ge 1.$$

Thus we have

$$\left\|\psi(h^2G)-\psi(h^2G_0)\right\|_{L^2\to L^2}$$

$$\le O(h^2)\int_{\mathbf{C}}\left|\frac{\partial\widetilde{\psi}}{\partial\overline{z}}(z)\right|\left\|(h^2G_0-z)^{-1}V(h^2G-z)^{-1}\right\|_{L^2\to L^2}L(dz)$$

$$\le O_N(h^2)\int_{\mathbf{C}_\psi}|\operatorname{Im}z|^N\left\|(h^2G_0-z)^{-1}\right\|_{L^2\to L^2}\left\|(h^2G-z)^{-1}\right\|_{L^2\to L^2}L(dz), \tag{3.4}$$

where $\mathbf{C}_\psi=\operatorname{supp}\widetilde{\psi}$. Now (3.2) follows from (3.4) and the trivial bounds

$$\left\|(h^2G_0-z)^{-1}\right\|_{L^2\to L^2}+\left\|(h^2G-z)^{-1}\right\|_{L^2\to L^2}\le 2|\operatorname{Im}z|^{-1}. \tag{3.5}$$

□

We will derive now (3.1) from Lemma 3.2 and the following

Proposition 3.3. *Assume* (1.1) *fulfilled with* $\delta>1$ *and let* $\varphi\in C_0^\infty([a,+\infty))$*. Then, for every* $s>1/2$ *there exist constants* $C,a>0$ *such* *that the following estimates hold*

$$\int_{-\infty}^{\infty} \left\| \langle x\rangle^{-s} e^{it\sqrt{G_0}} \varphi(h\sqrt{G_0}) f \right\|_{L^2}^2 dt \le C\|f\|_{L^2}^2, \quad \forall f \in L^2,\ 0 < h \le 1, \tag{3.6}$$

$$\int_{-\infty}^{\infty} \left\| \langle x\rangle^{-s} e^{it\sqrt{G}} \varphi(h\sqrt{G}) f \right\|_{L^2}^2 dt \le C\|f\|_{L^2}^2, \quad \forall f \in L^2,\ 0 < h \le 1. \tag{3.7}$$

Using Duhamel's formula

$$e^{it\sqrt{G}}\varphi(h\sqrt{G}) - e^{it\sqrt{G_0}}\varphi(h\sqrt{G}) = i\frac{\sin\left(t\sqrt{G_0}\right)}{\sqrt{G_0}}\left(\sqrt{G}\varphi(h\sqrt{G}) - \sqrt{G_0}\varphi(h\sqrt{G})\right)$$

$$-\int_0^t \frac{\sin\left((t-\tau)\sqrt{G_0}\right)}{\sqrt{G_0}} V e^{i\tau\sqrt{G}}\varphi(h\sqrt{G})d\tau, \tag{3.8}$$

we obtain

$$e^{it\sqrt{G}}\varphi(h\sqrt{G}) - e^{it\sqrt{G_0}}\varphi(h\sqrt{G_0})$$

$$= \left(\varphi_1(h\sqrt{G}) - \varphi_1(h\sqrt{G_0})\right) e^{it\sqrt{G}}\varphi(h\sqrt{G})$$

$$+\varphi_1(h\sqrt{G_0})e^{it\sqrt{G_0}}\left(\varphi(h\sqrt{G}) - \varphi(h\sqrt{G_0})\right)$$

$$-i\varphi_1(h\sqrt{G_0})\sin\left(t\sqrt{G_0}\right)\left(\varphi(h\sqrt{G}) - \varphi(h\sqrt{G_0})\right) \tag{3.9}$$

$$+i\widetilde{\varphi}_1(h\sqrt{G_0})\sin\left(t\sqrt{G_0}\right)\left(\widetilde{\varphi}(h\sqrt{G}) - \widetilde{\varphi}(h\sqrt{G_0})\right)$$

$$-h\int_0^t \widetilde{\varphi}_1(h\sqrt{G_0})\sin\left((t-\tau)\sqrt{G_0}\right) V e^{i\tau\sqrt{G}}\varphi(h\sqrt{G})d\tau,$$

where $\varphi_1 \in C_0^\infty([a,+\infty))$ is such that $\varphi_1\varphi \equiv \varphi$, $\widetilde{\varphi}(\sigma) = \sigma\varphi(\sigma)$, $\widetilde{\varphi}_1(\sigma) = \sigma^{-1}\varphi_1(\sigma)$. For all nontrivial $f, g \in L^2$, in view of Lemma 3.2, Proposition 3.3 and (3.9), we have with $0 < s - 1/2 \ll 1$, $\forall\gamma > 0$,

$$|\langle \Phi(t;h)f, g\rangle| \le O(h^2)\|f\|_{L^2}\|g\|_{L^2}$$

$$+O(h)\int_{-\infty}^{\infty}\left|\left\langle \langle x\rangle^{s}Ve^{i\tau\sqrt{G}}\varphi(h\sqrt{G})f, \langle x\rangle^{-s}\sin\left((t-\tau)\sqrt{G_0}\right)\widetilde{\varphi}_1(h\sqrt{G_0})g\right\rangle\right|d\tau$$

$$\leq O(h^2)\|f\|_{L^2}\|g\|_{L^2}+O(h)\gamma\int_{-\infty}^{\infty}\left\|\langle x\rangle^{-s}e^{i\tau\sqrt{G}}\varphi(h\sqrt{G})f\right\|_{L^2}^2 d\tau$$

$$+O(h)\gamma^{-1}\int_{-\infty}^{\infty}\left\|\langle x\rangle^{-s}\sin\left(\tau\sqrt{G_0}\right)\widetilde{\varphi}_1(h\sqrt{G_0})g\right\|_{L^2}^2 d\tau$$

$$\leq O(h^2)\|f\|_{L^2}\|g\|_{L^2}+O(h)\gamma\|f\|_{L^2}^2+O(h)\gamma^{-1}\|g\|_{L^2}^2$$

$$\leq O(h)\|f\|_{L^2}\|g\|_{L^2}, \tag{3.10}$$

if we choose $\gamma=\|g\|_{L^2}/\|f\|_{L^2}$, which clearly implies (3.1). □

Proof of Proposition 3.3. Without loss of generality we may suppose that the function φ is real-valued. Denote by $\mathcal{H}$ the Hilbert space $L^2(\mathbf{R};L^2)$. Clearly, (3.7) is equivalent to the fact that the operator $\mathcal{A}_h: L^2\to\mathcal{H}$ defined by

$$(\mathcal{A}_h f)(x,t)=\langle x\rangle^{-s}e^{it\sqrt{G}}\varphi(h\sqrt{G})f$$

is bounded uniformly in h. Observe that the adjoint $\mathcal{A}_h^*:\mathcal{H}\to L^2$ is defined by

$$\mathcal{A}_h^* f=\int_{-\infty}^{\infty}e^{-i\tau\sqrt{G}}\varphi(h\sqrt{G})\langle x\rangle^{-s}f(\tau,x)d\tau,$$

so we have, $\forall f,g\in\mathcal{H}$,

$$\langle\mathcal{A}_h\mathcal{A}_h^* f,g\rangle_{\mathcal{H}}=\int_{-\infty}^{\infty}\langle\rho(t,\cdot),g(t,\cdot)\rangle_{L^2}\,dt, \tag{3.11}$$

where

$$\rho(t,x)=\int_{-\infty}^{\infty}\langle x\rangle^{-s}e^{i(t-\tau)\sqrt{G}}\varphi^2(h\sqrt{G})\langle x\rangle^{-s}f(\tau,\cdot)d\tau.$$

Hence, for the Fourier transform, $\hat{\rho}(\lambda,x)$, of $\rho(t,x)$ with respect to the variable t we have

$$\hat{\rho}(\lambda,x)=Q(\lambda)\hat{f}(\lambda,x), \tag{3.12}$$

where $Q(\lambda)$ is the Fourier transform of the operator

$$\langle x\rangle^{-s}e^{it\sqrt{G}}\varphi^2(h\sqrt{G})\langle x\rangle^{-s}.$$

On the other hand, the formula

$$e^{it\sqrt{G}}\varphi^2(h\sqrt{G})=\frac{1}{\pi i}\int_{-\infty}^{\infty}e^{it\lambda}\varphi^2(h\lambda)\left(R^+(\lambda)-R^-(\lambda)\right)\lambda d\lambda$$

shows that

$$Q(\lambda)=(\pi i)^{-1}\lambda\varphi^2(h\lambda)\langle x\rangle^{-s}\left(R^+(\lambda)-R^-(\lambda)\right)\langle x\rangle^{-s}. \tag{3.13}$$

By (2.22) and (3.13) we conclude

$$\|Q(\lambda)\|_{L^2\to L^2}\le C \tag{3.14}$$

with a constant $C>0$ independent of λ and h. By (3.12) and (3.14),

$$\|\hat\rho(\lambda,\cdot)\|_{L^2}\le C\|\hat f(\lambda,\cdot)\|_{L^2}, \tag{3.15}$$

which together with (3.11) leads to

$$|\langle\mathcal{A}_h\mathcal{A}_h^*f,g\rangle_{\mathcal{H}}|=\left|\int_{-\infty}^{\infty}\langle\hat\rho(\lambda,\cdot),\hat g(\lambda,\cdot)\rangle_{L^2}\,d\lambda\right|$$

$$\le C\int_{-\infty}^{\infty}\|\hat f(\lambda,\cdot)\|_{L^2}\|\hat g(\lambda,\cdot)\|_{L^2}d\lambda$$

$$\le C\gamma\int_{-\infty}^{\infty}\|\hat f(\lambda,\cdot)\|_{L^2}^2d\lambda+C\gamma^{-1}\int_{-\infty}^{\infty}\|\hat g(\lambda,\cdot)\|_{L^2}^2d\lambda$$

$$=C\gamma\|f\|_{\mathcal{H}}^2+C\gamma^{-1}\|g\|_{\mathcal{H}}^2=2C\|f\|_{\mathcal{H}}\|g\|_{\mathcal{H}}, \tag{3.16}$$

if we take $\gamma=\|g\|_{\mathcal{H}}/\|f\|_{\mathcal{H}}$, with a constant $C>0$ independent of h, f and g. It follows from (3.16) that the operator $\mathcal{A}_h\mathcal{A}_h^*:\mathcal{H}\to\mathcal{H}$ is bounded uniformly in h, and hence so is the operator $\mathcal{A}_h:L^2\to\mathcal{H}$. This proves (3.7). Clearly, (3.6) is treated in the same way using (2.25) instead of (2.22). $\square$

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