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PROJECTIVELY NORMAL LINE BUNDLES ON *k*-GONAL CURVES AND RATIONAL SURFACES

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Abstract. Here we prove the projective normality of several special line bundles on a general k-gonal curve. Let X be a k-gonal curve arising as the normalization of a certain nodal curve $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$. We prove the existence of many projectively normal special line bundles on X. We also show the existence of a large set, Φ, of special line bundles on X which are not projectively normal (and not even quadratically normal) and for every $L \in \Phi$ we compute the dimension of the cokernel of the multiplication map $H^0(X, L) \otimes H^0(X, L) \to H^0(X, L^{\otimes 2})$. Let M be the blowing - up either of $\mathbb{P}^2$ or of $\mathbb{P}^1 \times \mathbb{P}^1$ at a general finite set S. We show the projective normality of certain line bundles on M, the case $\mathbb{P}^1 \times \mathbb{P}^1$ being used to prove our results on k-gonal curves.

1. Introduction. Let X be a smooth k-gonal curve of genus g and $R \in \mathrm{Pic}^k(X)$ its degree k pencil. We assume $h^0(X, R^{\otimes t}) = t + 1$ if $0 \le t \le [g/(k-1)]$

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and $h^0(X, R^{\otimes t}) = tk + 1 - g$ if $t > [g/(k-1)]$. By [2] or [5, Theorem 1.1], this is the case when X is a general $k-$gonal curve of genus g.

Definition and Remark 1.1. (i) Fix $L \in \mathrm{Pic}^d(X)$ and set $r := h^0(X, L) - 1$. Hence $h^1(X, L) = g + r - d$. Following [6, Definition 1.3] and [7] we will say that L is of type II if $\omega_X \otimes L^* \cong R^{\otimes(g-d+r-1)}(B)$ with B a possibly empty effective divisor.
(ii) Notice that $h^0(X, \omega_X \otimes L^*) = g - d + r = h^0(X, R^{\otimes(g-d+r-1)})$, i.e. B is the base locus of $\omega_X \otimes L^*$.
(iii) It is easy to see that for fixed integers d, g and r the set of all type II linear series on X with numerical invariants d and r is parametrized by an open subset of the symmetric product of $\deg(B)$ copies of X. Indeed, we equivalently fix the integer $t := g - d + r - 1$ and the integer $b := \deg(B) = 2g - 2 - d - kt$ and we take as B every degree b effective divisor B' with $h^0(X, R^{\otimes t}(B')) = t + 1$. By semicontinuity the set of all such B' is an open subset Ω of the symmetric product and the numerical condition $h^1(X, R^{\otimes t}) = r + 1 + b \geq b + 1$ implies the non-emptiness of Ω. Hence the set of all line bundles on X of type II is parametrized by an irreducible variety.

The first aim of this paper is to study the projective normality of type II special linear systems on k-gonal curves. We will give positive results (a general line bundle of this form on a certain curve X is projectively normal) as well as negative results (a general line bundle of this form on a certain curve X is not projectively normal). In the latter case we will even measure the failure of the projective normality of L (see Theorem 3.9). More precisely, we will compute the dimension of the cokernel of the multiplication map $\mu: H^0(X, L) \otimes H^0(X, L) \to H^0(X, L^{\otimes 2})$, i.e. we will measure the failure for the quadratic normality of L. We will do this for general $B \subset X$ with fixed $\mathrm{card}(B)$. By semicontinuity, the failure for the quadratic normality of L (i.e. $\dim \mathrm{Coker}(\mu)$) for any given B cannot be lower.

First we will give another proof of the following results proved in [3] with a slight improvement. It corresponds to the case $B = \emptyset$.

Theorem 1.2. *Fix integers g, k and t with $g \geq 2k \geq 8$. Let x be the minimal integer with $g \leq kx - x - k + 1$ and assume $0 \leq 2t \leq x - 3$. Let X be a general k-gonal curve of genus g and $R \in \mathrm{Pic}^k(X)$ its degree k pencil. Then $\omega_X \otimes (R^*)^{\otimes t}$ is very ample and normally generated, i.e. the complete linear system associated to $\omega_X \otimes (R^*)^{\otimes t}$ is an embedding and the image curve is projectively normal.*

The next result is an extension of Theorem 1.2 to linear series of type II with $B \neq \emptyset$.

Theorem 1.3. *Fix integers g, k, t and z with $g \geq 2k \geq 8$. Let x be the minimal integer with $g \leq kx - x - k + 1$. Assume $0 \leq 2t \leq x - 3$ and $0 \leq z \leq (k-3)(x-3-2t)$. Let X be a general k-gonal curve of genus g, $R \in \mathrm{Pic}^k(X)$ its degree k pencil and B a general subset of X with $\mathrm{card}(B) = z$. Then $\omega_X \otimes (R^*)^{\otimes t}(-B)$ is quadratically normal.*

In section 4 we will prove the following result.

Theorem 1.4. *Fix integers a, z with $a \geq 1$ and $0 \leq z \leq a(a-1)/2$ and a general $S \subset \mathbb{P}^2$ with $\mathrm{card}(S) = z$. Let $u : U \to \mathbb{P}^2$ be the blowing up of $\mathbb{P}^2$ at S and E_i, $1 \leq i \leq z$, the exceptional divisors of u. Set $L := u^*(\mathcal{O}_{\mathbb{P}^2}(a)) - \sum_{1 \leq i \leq z} E_i \in Pic(U)$. Then L is normally generated, i.e. L is very ample and for all integers $t \geq 2$ the natural map $\sigma(S,t) : S^t(H^0(U,L)) \to H^0(U, L^{\otimes t})$ is surjective.*

Remark 1.5. (i) We believe that Theorem 1.4 is not optimal. The main point is however its extension to more general varieties instead of $\mathbb{P}^2$ and to the case $L := u^*(\mathcal{O}_{\mathbb{P}^2}(a)) - \sum_{1 \leq i \leq z} e_i E_i$ with $e_i > 0$. For the very ampleness of L with weaker assumptions on z (i.e. for every $z \leq a(a+1)/2 - 5$), see [8, Theorem 2.3].
(ii) The main tool for our proofs of Theorem 1.2 and 1.3 will be the case of $\mathbb{P}^1 \times \mathbb{P}^1$ instead of $\mathbb{P}^2$ (see Theorems 3.6, 3.8 and 3.9). Indeed, our smooth k-gonal curve X will be the normalization of a nodal curve $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$.
(iii) For general ideas and tools for proving such type of results, see [1], [10] and references therein.

2. Line bundles on a blowing-up of a quadric surface. Set $Q := \mathbb{P}^1 \times \mathbb{P}^1$. We choose homogeneous coordinates z_0, z_1 and w_0, w_1 for the two factors of Q. For all non negative integers c, d the vector space $H^0(Q, \mathcal{O}_Q(c,d))$ may be identified with the set of all polynomials, f, in the variables z_0, z_1, w_0, w_1 which are bihomogeneous of type (c,d), i.e. each monomial occurring in f with non-zero coefficient is homogeneous of degree c in the variables z_0, z_1 and homogeneous of degree d in the variables w_0, w_1.

Remark 2.1. Fix integers a, b with $a \geq b$; by the Künneth formula we have $h^0(Q, \mathcal{O}_Q(a,b)) = 0$ if $b < 0$, $h^0(Q, \mathcal{O}_Q(a,b)) = (a+1)(b+1)$ if $b \geq -1$. We also have $h^1(Q, \mathcal{O}_Q(a,b)) = 0$ if $b \geq -1$ or $a \leq -1$, $h^1(Q, \mathcal{O}_Q(a,b)) = -(b+1)(a+1)$ if $a \geq -1$ and $b \leq -1$. We have $h^i(Q, \mathcal{O}_Q(c,d)) = h^i(Q, \mathcal{O}_Q(d,c))$, $0 \leq i \leq 2$, for all integers c, d.

Remark 2.2. Fix non-negative integers a, b and z and a general $S \subset Q$ with $\mathrm{card}(S) = z$. Then we have $h^0(Q, \mathcal{I}_S(a, b)) = \max\{0, (a+1)(b+1) - z\}$.

Remark 2.3. For all non-negative integers a, b, c and d the multiplication map $\mu : H^0(Q, \mathcal{O}_Q(a, b)) \otimes H^0(Q, \mathcal{O}_Q(c, d)) \to H^0(Q, \mathcal{O}_Q(a + c, b + d))$ is surjective. Indeed every monomial in the variables z_0, z_1, w_0 and w_1 which is bihomogeneous of type $(a + c, b + d)$ is in the image of μ.

Lemma 2.4. *Fix $P \in Q$. We have*
(i) $h^0(Q, \mathcal{I}_{\{P\}}(1, 1)) = 3$ *and* $h^0(Q, (\mathcal{I}_{\{P\}})^2(2, 2)) = 6$.
(ii) *The multiplication map*

$$\mu : H^0(Q, \mathcal{I}_{\{P\}}(1, 1)) \otimes H^0(Q, \mathcal{I}_{\{P\}}(1, 1)) \to H^0(Q, (\mathcal{I}_{\{P\}})^2(2, 2))$$

is surjective.
(iii) *The symmetrized multiplication map*

$$\sigma : S^2(H^0(Q, \mathcal{I}_{\{P\}}(1, 1))) \to H^0(Q, (\mathcal{I}_{\{P\}})^2(2, 2))$$

is bijective.

Proof. Every curve $D \subset Q$ of type $(1, 1)$ with $P \in D$ is of the form $M \cap Q$, where M is a hyperplane in $\mathbb{P}^3$ such that $P \in M$ and conversely. Hence $h^0(Q, \mathcal{I}_{\{P\}}(1, 1)) = 3$ and $\dim(S^2(H^0(Q, \mathcal{I}_{\{P\}}(1, 1))) = 6$. It is easy to see the existence of curves D, D' and D'' of type $(2, 2)$ on Q with the following properties:

(i) $P \notin D$.

(ii) $P \in D'$ but D' is not tangent at P to the line, A, of type $(1, 0)$ through P.

(iii) $P \in D''$, D'' is smooth at P and tangent to A at P.

Therefore $h^0(Q, (\mathcal{I}_{\{P\}})^2(2, 2)) = 6 = h^0(Q, \mathcal{O}_Q(2, 2)) - 3$.

Fix a hyperplane $H \subset \mathbb{P}^3$ with $P \notin H$. For every hyperplane $M \subset \mathbb{P}^3$ with $P \in M$ the set $M \cap H$ is a line in H. Viceversa, for every line B in H the span of P and B is a plane through P. Hence the surjectivity of μ follows from the surjectivity of the multiplication map $H^0(H, \mathcal{O}_H(1)) \otimes H^0(H, \mathcal{O}_H(1)) \to H^0(H, \mathcal{O}_H(2))$ and the equality $h^0(Q, (\mathcal{I}_{\{P\}})^2(2, 2)) = h^0(H, \mathcal{O}_H(2)) = 6$. Since μ is symmetric, σ is surjective. Since $\dim S^2(H^0(Q, \mathcal{I}_{\{P\}}(1, 1))) = h^0(Q, (\mathcal{I}_{\{P\}})^2(2, 2))$, σ is bijective. $\square$

Lemma 2.5. *Fix integers b, z with $b \geq z \geq 0$ and a general $S \subset Q$ with* $\mathrm{card}(S) = z$. *We have*
(i) $h^0(Q, \mathcal{I}_S(1, b)) = 2b + 2 - z$ *and* $h^0(Q, (\mathcal{I}_S)^2(2, 2b)) = 3(2b + 1) - 3z$.
(ii) *The multiplication map*

$$\mu : H^0(Q, \mathcal{I}_S(1, b)) \otimes H^0(Q, \mathcal{I}_S(1, b)) \to H^0(Q, (\mathcal{I}_S)^2(2, 2b))$$

is surjective.

P r o o f. The equality $h^0(Q, \mathcal{I}_S(1,b)) = 2b+2-z$ is obvious by Remark 2.2. One can easily check that $h^0(Q, (\mathcal{I}_S)^2(2,2b)) = 3(2b+1) - 3z$ for general S by double induction on z and b, the case $z = 0$ being obvious; alternatively, see Lemma 2.6. Now we will check the surjectivity of μ. The case $z = 0$ follows from Remark 2.3. The case $b = z = 1$ is just Lemma 2.4. Hence we may assume $b \geq 2$ and $z > 0$ and use induction on b. First assume $b = z$. Fix $P \in S$ and set $A := S \backslash \{P\}$. Let D be the line of type $(0,1)$ containing P. By the generality of S we may assume $D \cap A = \emptyset$. Apply the inductive assumption to the integer $b-1$ and the set A. For every curve E of type $(1, b-1)$ containing A the curve$E \cup D$ is a curve of type $(1,b)$ containing S. Hence by the inductive assumption we obtain that $\mathrm{Im}(\mu)$ contains the equations of all curves $C + 2D$ with C a curve of type $(2, 2b-2)$ containing A and singular at every point of A. Such equations form a linear subspace, $V(P)$, of $H^0(Q, (\mathcal{I}_S)^2(2,2b))$ with codimension 3. Do the same for all points of S. Note that $\mathrm{card}(S) = z \geq 2$. For $P, P' \in S$ with $P \neq P'$ it is easy to check that $V(P') \cap V(P)$ has codimension 3 in $V(P)$. Hence the linear subspaces $V(P), P \in S$, span $H^0(Q, (\mathcal{I}_S)^2(2,2b))$. Hence μ is surjective. Now assume $z < b$. Apply the case $b' := z$ just proved and then apply Remark 2.3; alternatively, first do the very easy case $z = 1, b \geq 2$ and then use induction taking the pair (b', z') with $b' := b-1$ and $z' := z$. $\square$

Lemma 2.6. *Fix integers a, b and z with $b \geq a \geq 2$ and $0 \leq z \leq ab$. Let $D \subset Q$ be a general smooth curve of type $(1,1)$. Let B be a general subset of D with* $\mathrm{card}(B) = \min\{a+b-1, z\}$. *Let A be a general subset of Q with* $\mathrm{card}(A) = \max\{0, z - \mathrm{card}(B)\}$. *Set $W := A \cup B$. Then $h^0(Q, (\mathcal{I}_W)^2(2a, 2b)) = (2a+1)(2b+1) - 3(\mathrm{card}(W))$, i.e. $h^1(Q, (\mathcal{I}_W)^2(2a,2b)) = 0$.*

P r o o f. We may assume $z = ab$; not only the proof of the general case is similar, but it follows from the statement of the case $z = ab$ just taking a subset of W. Hence $\mathrm{card}(A) = (a-1)(b-1)$. If $a \geq 2$ we use induction on the integer a and assume $h^1(Q, (\mathcal{I}_A)^2(2a-2, 2b-2)) = 0$. If $a = 1$ we assume nothing. Notice that $(\mathcal{I}_W)^2(2a, 2b)_{|D}$ is a line bundle, M, of degree $2a + 2b - 2(\mathrm{card}(B)) = 2$ because $W \cap D = B$. Since $D \cong \mathbb{P}^1$ we have $H^1(D, M) = 0$. Let $2W$ (resp. $2A$) be the zero-dimensional subscheme of Q with $(\mathcal{I}_W)^2$ (resp. $(\mathcal{I}_A)^2$) as ideal sheaf and call q the bihomogeneous equation of $D \subset Q$. Take any $f \in H^0(Q, \mathcal{O}_Q(2a-1, 2b-1))$. We have $fq \in H^0(Q, (\mathcal{I}_W)^2(2a,2b))$ if and only if $f \in H^0(Q, \mathcal{I}_B \otimes (\mathcal{I}_A)^2(2a-1, 2b-1))$; in other words, the residual scheme of $2W$ with respect to the Cartier divisor D of Q is $2A \cup B$. Hence we have an exact sequence

$$0 \to \mathcal{I}_B \otimes (\mathcal{I}_A)^2(2a-1, 2b-1) \to (\mathcal{I}_W)^2(2a,2b) \to (\mathcal{I}_W)^2(2a,2b)_{|D} \to 0$$

Since $H^1(D, M) = 0$, to prove the lemma it is sufficient to prove $h^1(Q, \mathcal{I}_B \otimes$

$(\mathcal{I}_A)^2(2a-1,2b-1)) = 0$. The residual scheme of $2A \cup B$ with respect to D is $2A$, while $\mathcal{I}_B \otimes (\mathcal{I}_A)^2(2a-1,2b-1)_{|D}$ is a degree $a+b-1$ line bundle on D. Hence, just as we did before, to conclude it is sufficient to prove $h^1(Q,(\mathcal{I}_A)^2(2a-2,2b-2)) = 0$. This equality is satisfied if $a \geq 2$ by our inductive assumption. Hence from now on we assume $a = 1$ and hence $A = \emptyset$ and $\mathrm{card}(B) = b-1$. We have

$$\begin{aligned} h^0(Q,\mathcal{I}_D(1,2b-1)) &= h^0(Q,\mathcal{O}_Q(0,2b-2)) = 2b-1 \leq 4b - \mathrm{card}(B) \\ &= h^0(Q,\mathcal{O}_Q(1,2b-1)) - \mathrm{card}(B). \end{aligned}$$

Hence $h^1(Q,\mathcal{I}_B(1,2b-1)) = 0$ because B is general in D. □

Remark 2.7. We stress that results much stronger than Lemma 2.6 may be obtained using the method of [1] and [4]. We do not give them explicitly because the real obstruction to obtain stronger results using our approach is given by the bounds coming from the proof of Lemma 2.9 below.

The following result is weaker than Lemma 2.9 below, but it would be sufficient for the proof of Theorem 1.2.

Lemma 2.8. *Fix integers a, b, z with $a > 0, b > 0$ and $0 \leq z \leq a+b-1$. Let $S \subset Q$ be a general set with $\mathrm{card}(S) = z$. Then*

$$h^0(Q,\mathcal{I}_S(a,b)) = (a+1)(b+1) - z, h^0(Q,(\mathcal{I}_S)^2(2a,2b)) = (2a+1)(2b+1) - 3z$$

and the multiplication map

$$\mu_S : H^0(Q,\mathcal{I}_S(a,b)) \otimes H^0(Q,\mathcal{I}_S(a,b)) \to H^0(Q,(\mathcal{I}_S)^2(2a,2b))$$

is surjective.

Proof. The equality $h^0(Q,\mathcal{I}_S(a,b)) = (a+1)(b+1) - z$ is obvious by Remark 2.2 and the generality of S. For the equality $h^0(Q,(\mathcal{I}_S)^2(2a,2b)) = (2a+1)(2b+1) - 3z$, see Lemma 2.6. Now we prove the surjectivity of the map μ_S. The case $a = 1$ is given by Lemma 2.5. Hence we may assume $a \geq 2$. If $z = 1$ and $a = 1$ we may apply Lemma 2.5. The case $z = 1$ and $a \geq 2$ is strictly easier and may be done as in the proof of Lemma 2.5. Now assume $z \geq 2$. Fix $P, P' \in S$ with $P \neq P'$ and set $A := S\backslash\{P\}, E := S\backslash\{P'\}$ and $B := S\backslash\{P,P'\}$. We have

$$\begin{aligned} h^0(Q,(\mathcal{I}_A)^2(2a-2,2b)) &= h^0(Q,(\mathcal{I}_E)^2(2a-2,2b)) \\ &= (2a-1)(2b+1) - 3(\mathrm{card}(E)) = h^0(Q,(\mathcal{I}_S)^2(2a,2b)) - 2(2b+1) + 3 \\ &= h^0(Q,(\mathcal{I}_B)^2(2a-4,2b)) + 2(2b+1) - 3. \end{aligned}$$

Take as D the line of type $(1,0)$ containing P instead of the line of type $(0,1)$ containing P. With this modification we may repeat the proof of Lemma 2.5 (case $b = z$). □

Lemma 2.9. *Fix integers a, b and z with $b \geq a \geq 1$ and $0 \leq z \leq ab - a$. Let $S \subset Q$ be a general set with* $\mathrm{card}(S) = z$. *Then*
$h^0(Q, \mathcal{I}_S(a, b)) = (a + 1)(b + 1) - z, h^0(Q, (\mathcal{I}_S)^2(2a, 2b)) = (2a + 1)(2b + 1) - 3z$
and the multiplication map

$$\mu_S : H^0(Q, \mathcal{I}_S(a, b)) \otimes H^0(Q, \mathcal{I}_S(a, b)) \to H^0(Q, (\mathcal{I}_S)^2(2a, 2b))$$

is surjective.

Proof. The equality $h^0(Q, \mathcal{I}_S(a, b)) = (a + 1)(b + 1) - z$ is obvious by the Remark 2.2 and the generality of S. For the equality $h^0(Q, (\mathcal{I}_S)^2(2a, 2b)) = (2a + 1)(2b + 1) - 3z$, see Lemma 2.6. Now we prove the surjectivity of the map μ_S. First we assume $z = ab - a$. Since the case $a = 1$ is covered by Lemma 2.5, we may assume $a \geq 2$ and use double induction on a and b. We will prove the surjectivity of μ_W for some particular $W \subset Q$ with $\mathrm{card}(W) = ab - a$. The set W will satisfy $h^0(Q, \mathcal{I}_W(a, a)) = (a+1)(b+1) - \mathrm{card}(W)$ and $h^0(Q, (\mathcal{I}_W)^2(2a, 2a)) = (2a + 1)(2b + 1) - 3(\mathrm{card}(W))$. The surjectivity of μ_S for a general $S \subset Q$ with $\mathrm{card}(S) = \mathrm{card}(W)$ will then follow from semicontinuity.

Let $D \subset Q$ be a general smooth curve of type $(1, 1)$ and call $2D$ the double of D inside Q. Hence $2D \subset Q$ is the unique curve of type $(2, 2)$ with $(2D)_{\mathrm{red}} = D$. Let q be the bihomogeneous equation of $D \subset Q$; hence q^2 is the bihomogeneous equation of $2D$. Take a general $A \subset Q$ with $\mathrm{card}(A) = (a - 1)(b - 2)$ and a general $B \subset D$ with $\mathrm{card}(B) = ab - a - \mathrm{card}(A) = a + b - 2$. Set $W := A \cup B$. By the inductive assumption, the multiplication map
$\mu_A : (H^0(Q, \mathcal{I}_A(a - 1, b - 1)))^{\otimes 2} \to H^0(Q, (\mathcal{I}_A)^2(2a - 2, 2b - 2))$ is surjective. Since $\mathrm{card}(A) = (a - 1)(b - 2) \leq h^0(Q, \mathcal{O}_Q(a - 1, b - 1))$ and A is general, we have $h^1(Q, \mathcal{I}_A(a - 1, b - 1)) = 0$ (Remark 2.2). Hence the restriction map $\alpha : H^0(Q, \mathcal{I}_W(a, b)) \to H^0(D, \mathcal{I}_B \otimes \mathcal{O}_D(a, b))$ is surjective; here we use that $W \cap D = B$. The surjectivity of the maps α and μ_A implies that $\mathrm{Im}(\mu_W)$ contains all equations of type uq^2 with $u \in H^0(Q, (\mathcal{I}_A)^2(2a-2, 2b-2))$. Obviously $(\mathcal{I}_W)^2(2a, 2b)_{|2D} = (\mathcal{I}_B)^2(2a, 2b)_{|2D}$. We will write $\mathcal{I}_{B,2D}$ for the ideal sheaf of B in $2D$. Since $H^1(Q, (\mathcal{I}_A)^2(2a - 2, 2b - 2)) = 0$ (Lemma 2.6), the restriction map $\rho : H^0(Q, (\mathcal{I}_W)^2(2a, 2b)) \to H^0(2D, (\mathcal{I}_W)^2(2a, 2b)_{|2D})$ is surjective. Hence to prove the lemma it is sufficient to prove that the multiplication map

$$\mu_{B,2D} : H^0(2D, \mathcal{I}_B(a, b)_{|2D}) \otimes H^0(2D, \mathcal{I}_B(a, b)_{|2D}) \to H^0(2D, (\mathcal{I}_B)^2(2a, 2b)_{|2D})$$

is surjective. The trouble comes from the fact that neither B nor $2B$ is a Cartier divisor of $2D$. Since $\mathrm{card}(B) = a + b - 2$ and $h^1(Q, \mathcal{O}_Q(a - 2, b - 2)) = 0$, there is a curve $E \subset Q$ of type $(a - 1, b - 1)$ with $B = E \cap D$ (scheme-theoretically). Set $F := E \cap 2D$. Hence F is a Cartier divisor of $2D$ associated to a global section of $\mathcal{O}_{2D}(a - 1, b - 1)$. Hence $\mathcal{I}_{F,2D} \otimes \mathcal{O}_{2D}(a, b) \cong \mathcal{O}_{2D}(1, 1)$. Thus the projective normality of $Q \subset \mathbb{P}^3$ and the vanishing of $H^1(Q, \mathcal{O}_Q(-1,-1))$ implies that the line bundle $\mathcal{I}_{F,2D} \otimes \mathcal{O}_{2D}(a,b)$ is quadratically normal, i.e. the multiplication map $\mu_{F,2D} : (H^0(2D, \mathcal{I}_{F,2D} \otimes \mathcal{O}_{2D}(a,b)))^{\otimes 2} \to H^0(2D, (\mathcal{I}_{F,2D})^2 \otimes \mathcal{O}_{2D}(2a,2b))$ is surjective. Obviously $\mathrm{Im}(\mu_{B,2D})$ contains $\mathrm{Im}(\mu_{F,2D})$. Hence by the surjectivity of $\mu_{F,2D}$ the surjectivity of $\mu_{B,2D}$ is equivalent to

$$\dim(\mathrm{Im}(\mu_{B,2D})/\,\mathrm{Im}(\mu_{F,2D})) = \mathrm{length}(2F) - \mathrm{length}(2B)) = a+b-2.$$

Since $\mathrm{Im}(\mu_{F,2D})$ spans $(\mathcal{I}_{F,2D})^2(2a,2b)$, we have

$$\dim(\mathrm{Im}(\mu_{B,2D})/\,\mathrm{Im}(_{F,2D})) = a+b-2$$

if and only if $\mathrm{Im}(\mu_{B,2D})$ spans $(\mathcal{I}_{B,2D})^2 \otimes \mathcal{O}_{2D}(2a,2b)$ at each point of B. Assume that this is not the case. Recall that B is general in D. Since D is irreducible the symmetric product of $a+b-2$ copies of D is irreducible. Hence our assumption implies $\mathrm{Supp}(\mathrm{Im}(\mu_{B,2D})/(\mathcal{I}_{F,2D})^2 \otimes \mathcal{O}_{2D}(2a,2b)) = B$. Hence $(\mathcal{I}_{F,2D})^2 \otimes \mathcal{O}_{2D}(2a,2b)$ is the subsheaf of $(\mathcal{I}_{B,2D})^2 \otimes \mathcal{O}_{2D}(2a,2b)$ spanned by $\mathrm{Im}(\mu_{B,2D})$ and we have $\mathrm{Im}(\mu_{F,2D}) = \mathrm{Im}(\mu_{B,2D})$. Now we move E among the curves of type $(a-1,b-1)$ containing B. Moving E the divisor $F := E \cap 2D$ moves and hence $(\mathcal{I}_{F,2D})^2 \otimes \mathcal{O}_{2D}(2a,2b)$ moves, while $(\mathcal{I}_{B,2D})^2 \otimes \mathcal{O}_{2D}(2a,2b)$ is fixed. Hence we obtain a contradiction and conclude the proof of the case $z = ab-a$. If $z < ab-a$ we just take $W = A \cup B$ with $\mathrm{card}(A) = \min\{(a-1)(b-2), z\}$ and $\mathrm{card}(B) = z - \mathrm{card}(A) = \max\{0, z-(a-1)(b-2)\}$ and then repeat the proof just given. $\square$

3. Nodal curves in a quadric surface.

Remark 3.1. (i) Fix integers g and k with $g \geq 2k-1 \geq 5$. Let C be a general k-gonal curve of genus g and $L \in \mathrm{Pic}^k(C)$ the degree k pencil. By [2] we have $h^0(C, L^{\otimes t}) = t+1$ if $0 \leq t \leq [g/(k-1)]$ and $h^0(C, L^{\otimes t}) = tk+1-g$ if $t > [g/(k-1)]$.
(ii) Indeed, the statement (i) above can be proved by showing a pair (X,R) where X is a smooth curve of genus g and $R \in \mathrm{Pic}^k(X)$ with $h^0(X,R) \geq 2$, $h^0(X, R^{\otimes t}) = t+1$ if $0 \leq t \leq [g/(k-1)]$ and $h^0(X, R^{\otimes t}) = tk+1-g$ if $t > [g/(k-1)]$. Such a pair (X,R) comes from a certain construction in the smooth quadric surface Q. We stress that it is essential to check that for such pair (X,R) one has $h^0(X, R^{\otimes t}) = t+1$ if $0 \leq t \leq [g/(k-1)]$ and $h^0(X, R^{\otimes t}) = tk+1-g$ if $t > [g/(k-1)]$.
(iii) To be more precise, let x be the minimal integer such that $g \leq kx-x-k+1$. Set $y := kx-x-k+1-g$. Hence $0 \leq y \leq k-2$. Take a general $S \subset Q$ with $\mathrm{card}(S) = y$ and call $A(S)$ the set of all integral curves $Y \subset Q$ with type (k,x) and $\mathrm{Sing}(Y) = S$. It was proved in [2] that the set $A(S)$ is a non-empty open

subset of a projective space of dimension $(k+1)(x+1)-1-3y$. Fix any $Y \in A(S)$ and let $\pi : X \to Y$ be the normalization map of Y. By the adjunction formula we have $p_a(Y) = kx - k - x + 1$ and hence X has genus g. The pencil of lines of type $(0,1)$ on Q induces $R \in \mathrm{Pic}^k(X)$ with $h^0(X,R) \geq 2$ and R is base point free. Indeed, it was proved in [2] (or see [5, Theorem 1.1] that $h^0(X, R^{\otimes t}) = t+1$ if $0 \leq t \leq [g/(k-1)]$ and $h^0(X, R^{\otimes t}) = tk + 1 - g$ if $t > [g/(k-1)]$.

In the next remark we will use the following well-known result (see e.g. [11, Lemma 2.1].

Lemma 3.2. *Let X be a smooth curve of genus g and $L \in \mathrm{Pic}(X)$ with L very ample and $\deg(L) \geq g+1$. The line bundle L is projectively normal if and only if it is quadratically normal.*

Proof. The only if part is tautological. Now assume that L is quadratically normal, i.e. assume the surjectivity of the multiplication map $H^0(X,L) \otimes H^0(X,L) \to H^0(X, L^{\otimes 2})$. By induction on the integer k it is sufficient to prove that for all integers $k \geq 2$ the multiplication map $\mu_k : H^0(X,L) \otimes H^0(X, L^{\otimes k}) \to H^0(X, L^{\otimes (k+1)})$ is surjective. By M. Green's H^0-Lemma ([9, Theorem 4.e.1]) the map μ_k is surjective if $h^1(X, L^{\otimes (k-1)}) \leq h^0(X,L) - 2$. Since L is very ample, $h^0(X,L) \geq 3$. Since $\deg(L) \geq g+1$, we have $h^1(X, L^{\otimes (k-1)}) = 0$ if $k \geq 3$. Now assume $k = 2$. By Riemann - Roch we have $h^0(X,L) - h^1(X,L) = \deg(L) + 1 - g \geq 2$, proving the lemma. $\square$

Remark 3.3. (i) To fix our notation we will check the well-known fact that $H^0(X, \omega_X) \cong H^0(Q, \mathcal{I}_S(k-2, x-2))$. Let $f : M \to Q$ be the blowing - up of Q at S. Call $E_i, 1 \leq i \leq y$, the exceptional divisors of f. We have

$$\omega_M \cong f^*(\omega_Q) + \sum_{1 \leq i \leq y} E_i = f^*(\mathcal{O}_Q(-2,-2)) + \sum_{1 \leq i \leq y} E_i;$$

here we mix the additive and the multiplicative notation for line bundles on M. Since Y is nodal, X is isomorphic to the strict transform of Y in M and, identifying X with this curve in M, we have $X \in |f^*(\mathcal{O}_Q(k,x)) - 2\sum_{1 \leq i \leq y} E_i|$. By the adjunction formula we have

$$\omega_X \cong \omega_M(X)_{|X} \cong (f^*(\mathcal{O}_Q(k-2, x-2)) - \sum_{1 \leq i \leq y} E_i)_{|X}$$

and hence $H^0(X, \omega_X) \cong H^0(Q, \mathcal{I}_S(k-2, x-2))$, as wanted.
(ii) Since $h^0(X, R^{\otimes t}) = t+1$ if $0 \leq t \leq [g/(k-1)]$ and $h^0(X, R^{\otimes t}) = tk + 1 - g$ if $t > [g/(k-1)]$, we obtain $h^0(X, \omega_X \otimes (R^*)^{\otimes t}) = h^0(Q, \mathcal{I}_S(k-2, x-2-t))$ for every integer $t \geq 0$. Now we fix an integer t with $0 \leq t \leq [g/(k-1)]$ and a general finite subset B of X. Set $E := f(B)$ and $F := S \cup E$. Since B is general, $\mathrm{card}(E) = \mathrm{card}(B)$ and F may be regarded as a general subset of Y.

Note that $h^0(X, R^{\otimes t}(B)) = \max\{t+1, \deg(B)+tk+1-g\}$ since B is general. Set $A := \omega_X \otimes (R^{\otimes t})^*(-B)$. Hence

$$h^1(X, A) = \max\{t+1, deg(B)+tk+1-g\}.$$

(iii) In Remark 3.4 we will connect the surjectivity of the multiplication map $\tau(t, B) : H^0(X, A) \otimes H^0(X, A) \to H^0(X, A^{\otimes 2})$ with the surjectivity of the multiplication map

$$\mu(t, E) : (H^0(Q, \mathcal{I}_F(k-2, x-2-t)))^{\otimes 2} \to H^0(Q, (\mathcal{I}_F)^2(2k-4, 2x-4-2t)).$$

A priori E is not general in Q but only in Y. If $3y + \mathrm{card}(B) < (k+1)(x+1)$ and we take Y general in $A(S)$, then F is general in Q. Hence we may apply the results of section 2 to obtain the surjectivity of $\mu(t, E)$. By Lemma 3.2 the line bundle A is normally generated if and only if it is quadratically generated. Hence if $\tau(t, B)$ is surjective, then A is normally generated, i.e. the image curve $\phi_A(X) \subset \mathbb{P}(H^0(X, A))$ is projectively normal. We need to check that A is very ample; however, $\deg(A) \geq g+1$ for the pairs of integers $(t, \deg(B))$ in the range of our consideration. Hence $\deg(A^{\otimes 2}) \geq 2g+2$ and $A^{\otimes 2}$ is very ample. Therefore if $\tau(t, B)$ is surjective, then A is base point free and separates distinct points and non-zero tangent vectors, i.e. A is very ample.

Remark 3.4. We continue the discussion of Remark 3.3. We have $H^0(X, A) \cong H^0(Q, \mathcal{I}_{S \cup E}(k-2, x-2-t))$, while there is a restriction map

$$\delta(t, B) : H^0(Q, (\mathcal{I}_{S \cup E})^2(2k-4, 2x-4-2t)) \to H^0(X, A^{\otimes 2})$$

such that $\tau(t, B) = \delta(t, B) \circ \mu(t, E)$. Hence if $\tau(t, B)$ is surjective, then $\delta(t, B)$ must be surjective, while if $\delta(t, B)$ and $\mu(t, E)$ are surjective, then $\tau(t, B)$ is surjective. Concerning the surjectivity of $\delta(t, B)$ we will see that the case $B = \emptyset$ is easier and we will do this first. This case is sufficient to prove Theorems 3.6 and 1.2.

(i) We first consider the case $B = \emptyset$, i.e. $E = \emptyset$. Note that there is a natural isomorphism

$$H^0(Q, (\mathcal{I}_S)^2(2k-4, 2x-4-2t)) \cong H^0(M, f^*(\mathcal{O}_Q(2k-4, 2x-4-2t)) - \sum_{1 \leq i \leq y} 2E_i).$$

Using this isomorphism we may regard $\tau(t, \emptyset)$ as the restriction map

$$\lambda(t) : H^0(M, f^*(\mathcal{O}_Q(2k-4, 2x-4-2t)) - \sum_{1 \leq i \leq y} 2E_i) \to H^0(X, A^{\otimes 2}).$$

Since $X \in |f^*(\mathcal{O}_Q(k, x)) - 2\sum_{1 \leq i \leq y} E_i|$,

$$\mathrm{Coker}(\lambda(t)) \subseteq H^1(M, f^*(\mathcal{O}_Q(k-4, x-4-2t))) \cong H^1(Q, \mathcal{O}_Q(k-4, x-4-2t)).$$

If $k \geq 4$ the latter cohomology group vanishes if and only if $x \geq 2t+3$. If $H^1(Q, (\mathcal{I}_S)^2(2k-4, 2x-4-2t)) = 0$, then $\mathrm{Coker}(\lambda(t)) = H^1(Q, \mathcal{O}_Q(k-4, x-$

$4-2t)$) and hence we have both a criterion for the surjectivity of $\delta(t,\emptyset)$ (i.e. of the quadratic normality of A) and a measure of the failure of the quadratic normality of A when the criterion is not satisfied.
(ii) Here we assume $B \neq \emptyset$. Notice that F is the residual scheme of $(\mathcal{I}_F)^2$ with respect to C. Hence as in (i) above, we obtain $\mathrm{Coker}(\delta(t,B)) \subseteq H^1(Q,\mathcal{I}_F(k-4,x-4-2t))$ and that $\mathrm{Coker}(\delta(t,B)) = H^1(Q,\mathcal{I}_F(k-4,x-4-2t))$ if $H^1(Q,(\mathcal{I}_E)^2(2k-4,2x-4-2t)) = 0$.

Remark 3.5. In the set-up of Remark 3.4 we always have

$\dim(\mathrm{Coker}(\lambda(t))) \geq h^1(Q,\mathcal{I}_F(k-4,x-4-2t)) - h^1(Q,(\mathcal{I}_E)^2(2k-4,2x-4-2t))$

and a similar statement holds true for the normalization, X, of a nodal curve in any smooth rational surface, say Π, just with respect to any special line bundle all whose global sections is induced by a line bundle, D, on Π for the following reason. For each $P \in \Pi$ one expects (and quite often it is true) that $(\mathcal{I}_{\{P\}})^2$ imposes 3 independent conditions on the corresponding linear system on Π, while certainly $(\mathcal{I}_{\{P\}},X)^2$ imposes at most two independent conditions on the corresponding linear system on X. Since $h^1(\Pi,\mathcal{O}_\Pi) = h^1(\Pi,\omega_\Pi) = 0$ for every smooth rational surface Π, such $D \in \mathrm{Pic}(\Pi)$ are quite common. In the case $E = \emptyset$, i.e. in the set-up of Remark 3.4.(i), if $\mathrm{Sing}(Y) \neq \emptyset$ we see as in Theorem 3.6 below a different obstruction for the quadratic normality of certain line bundles on the normalization of Y; however, this new obstruction works in a more restricted range of pairs of integers (genus,degree).

Theorem 3.6. *Fix integers g,k and t with $g \geq 2k \geq 8$. Let x be the minimal integer such that $g \leq xk - x - k + 1$ and assume $0 \leq t \leq x-3$. Set $y := xk - x - k + 1 - g$. Take a general $S \subset Q$ with $\mathrm{card}(S) = y$. Let $A(S)$ be the set of all integral nodal curves of type (k,x) on Q with S as singular locus. Then $A(S) \neq \emptyset$ and $A(S)$ is an irreducible variety of dimension $xk + x + k - 3y$. Fix any $Y \in A(S)$. Let $\pi : X \to Y$ be its normalization and $R \in \mathrm{Pic}^k(X)$ the degree k pencil on X induced by the pencil of lines of type $(0,1)$ on Q. We have $h^0(X,\omega_X \otimes (R^*)^{\otimes t}) = g - tk + t$. If $2t \leq x-3$, then $A := \omega_X \otimes (R^*)^{\otimes t}$ is very ample and normally generated, i.e. the complete linear system associated to A is an embedding and the image curve is projectively normal. If $x-2 \leq 2t \leq 2x-6$, then A is not quadratically normal and the cokernel of the multiplication map has dimension $(k-3)(2t-x+3)$.*

Proof. The non-emptiness, irreducibility and dimension of $A(S)$ was proved in [2]. The value of $h^0(X,A)$ (i.e. by Serre duality and Riemann-Roch, the equality $h^0(X,R^{\otimes t}) = t+1$) was also proved in [2]. We use the notation introduced in Remark 3.3. In particular $\omega_M(X) \cong f^*(\mathcal{O}_Q(k-2,x-2)) - \sum_{1\leq i\leq y} E_i$.

Thus $H^0(X, A) \cong H^0(Q, \mathcal{I}_S(k-2, x-2-t))$ and $H^0(X, A^{\otimes 2}) \cong H^0(Q, (\mathcal{I}_S)^2(2k-4, 2x-4-2t))$. We have $h^0(Q, (\mathcal{I}_S)^2(2k-4, 2x-4-2t)) = (2k-3)(2x-3-2t)-3y$ because $h^1(Q, (\mathcal{I}_S)^2(2k-4, 2x-4-2t)) = 0$ by Lemma 2.6 applied to the integers $z := y$, $a := k-2$ and $b := x-2-t$. By the assumption we have $t \leq x-3$. If $2t \leq x-3$ and A is very ample, the quadratic normality of A follows from Remark 3.4.(i) and Lemmas 2.8 and 2.9, while the projective normality of A follows from its quadratic normality; Lemma 3.2.

Now we will check the very ampleness of A. Since $h^0(X, A) \geq 2$, the line bundle $\omega_X \otimes (R^*)^{\otimes t}$ is very ample if and only if for every length 2 subscheme Z of X we have $h^0(X, A(-Z)) = h^0(X, A) - 2$. Fix any length 2 subscheme Z of X. We distinguish 3 cases. First assume either Z reduced, say $Z = \{P, P'\}$ and $\pi(P) \neq \pi(P')$, or Z not reduced but $\pi(Z_{\mathrm{red}}) \notin S$. Then the result follows from the inequalities $y \leq k-2$, $x-2-t \geq 1$ and the fact that the linear system $\mathbb{P}(H^0(Q, \mathcal{I}_S(k-2, x-2-t)))$ separates distinct points of Q and non-zero tangent vectors of $Q \backslash S$. Now we assume Z reduced, say $P \neq P'$, and $\pi(P) = \pi(P')$. The result follows by using that $\mathbb{P}(H^0(Q, \mathcal{I}_S(k-2, x-2-t)))$ contains a curve smooth at $\pi(P)$ and tangent to one of the two branches of Y at $\pi(P)$ and hence not tangent to the other branch of Y at $\pi(P)$. Now assume that Z is not reduced and $\pi(Z_{\mathrm{red}}) \in S$. For the failure of the quadratic normality of A when $x-2 \leq 2t \leq 2x-6$, use the second part of Remark 3.4.(i). □

Proof of Theorem 1.2. The conditions "very ample" and "normally generated are open conditions in a family of line bundles with constant cohomology. Hence the result follows from Theorem 3.6 because the set of all k-gonal curves is irreducible. □

Remark 3.7. If $y = 0$ (i.e. if $k-1$ divides g) we have $S = \emptyset$ and hence we may do also the case $t = [g/(k-1)] - 1$ in the statement of Theorem 3.6 and hence in the statement of Theorem 1.2.

Theorem 3.8. *Take integers g, k, t, y, a set $S \subset Q$ and curves Y, X as in the statement of Theorem 3.6. Assume $2t \leq x-3$ and Y general in $A(S)$. Take an integer z with $0 \leq z \leq (k-3)(x-3-2t)$. Let B be a general subset of X with $\mathrm{card}(B) = z$. Set $L := \omega_X \otimes (R^{\otimes t})^*(-B)$. Then L is quadratically normal.*

Proof. Set $F := \pi(B)$. Hence $\mathrm{card}(F) = z$ and F is just the union of z general points of Y. We may consider it as general in Q because we took a general $Y \in A(S)$ and $z \leq kx+k+x-3y = \dim(A(S))$. Since $z \leq (k-3)(x-3-2t)$, we have $H^1(Q, \mathcal{I}_F(k-4, x-4-2t)) = 0$. Hence the result follows from 3.4.(ii). □

Proof of Theorem 1.3. The result follows from Theorem 3.8. □

Theorem 3.9. *Take integers g, k, t, y, a set $S \subset Q$ and curves Y, X as* *in the statement of Theorem 3.6. Assume Y general in $A(S)$. Take an integer z satisfying the inequality* $0 \leq z \leq (k-3)(x-3-2t)$. *Let B be a general subset of X with* $\mathrm{card}(B) = z$. *Set* $L := \omega_X \otimes (R^{\otimes t})^*(-B)$. *We assume* $t \leq x-4, z+y \leq (k-2)(x-2-t)$ *and* $z > (k-3)(x-3-2t)$. *Let* $\tau : H^0(X,L) \otimes H^0(X,L) \to H^0(X, L^{\otimes 2})$ *be the multiplication map. Then* $\dim(\mathrm{Coker}(\tau)) = z-(k-3)(x-3-2t)$ *and in particular L is not quadratically normal.*

Proof. Set $E := \pi(B)$ and use 3.4.(ii). As in the proof of 3.3.(ii) by the generality of Y in $A(S)$ and the generality of S we may take as $F := S \cup E$ a general subset of Q with $\mathrm{card}(F) = y+z$. We have $H^1(Q, (\mathcal{I}_E)^2(2k-4, 2x-4-2t)) = 0$ by Lemma 2.6 because we assumed $z+y \leq (k-2)(x-2-t)$. □

4. Proof of Theorem 1.4. Finally we prove Theorem 1.4.

Proof of Theorem 1.4. The very ampleness of L follows from [8, Theorem 2.3]. The very ampleness of L follows also from the surjectivity of $\sigma(S,t)$ for every $t \geq 2$. We will prove only the surjectivity of $\sigma(S,2)$, the general case being easier. Hence it is sufficient to check the surjectivity of the multiplication map $\mu_S : H^0(\mathbb{P}^2, \mathcal{I}_S(a)) \otimes H^0(\mathbb{P}^2, \mathcal{I}_S(a)) \to H^0(\mathbb{P}^2, (\mathcal{I}_S)^2(2a))$. We copy the proof of Lemma 2.9 with the following modifications. We do only the case $z = a(a-1)/2$. Here D is a line. We specialize S to the union, W, of a general $A \subset \mathbb{P}^2$ with $\mathrm{card}(A) = (a-1)(a-2)/2$ and a general $B \subset D$ with $\mathrm{card}(B) = a-1$. We take as E a general plane curve of degree $a-1$ containing B. We have $h^0(\mathbb{P}^2, (\mathcal{I}_W)^2(2a)) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}^2(a)) - 3(\mathrm{card}(W))$ by the proofs in [10]. Equivalently, we may copy the proof of Lemma 2.6. □

REFERENCES

[1] J. Alexander, A. Hirschowitz. An asymptotic vanishing theorem for generic unions of multiple points. *Invent. Math.* **140(2)** (2000), 303–325.

[2] E. Ballico. A remark on linear series on general k-gonal curves. *Boll. Un. Mat. Ital. A (7)* **3** (1989), 195–197.

[3] E. Ballico. On projectively normal embeddings of general k-gonal curves.*Ann. Univ. Ferrara, sez VII, Sci. Mat.* **45** (1999), 123–125.

[4] E. Ballico, L. Chiantini. Nodal curves and postulation of generic fat points on $\mathbb{P}^2$. *Arch. Math. (Bazel)* **71**(1998), 501–504.

[5] M. Coppens, C. Keem, G. Martens. The primitive length of a general k-gonal curve. *Indag. Math., N. S.* **5(2)**(1994), 145–159.

[6] M. Coppens, G. Martens. Linear series on 4-gonal curves. *Math. Nachr.* **213** (2000), 35–55.

[7] M. Coppens, G. Martens. Linear series on a general k-gonal curve. *Abh. Math. Sem. Univ. Hamburg* **69** (1999), 347–371.

[8] J. D'Almeida, A. Hirschowitz. Quelques plongements projectifs non speciaux de surfaces rationnelles. *Math. Z.* **211** (1992), 479–483.

[9] M. Green. Koszul cohomology and the geometry of projecitve varieties. *J. Differential Geom.* **19** (1984), 125–171.

[10] A. Hirschowitz. La methode d' Horace pour l' interpolation ' plusieurs varables. *Manuscripta Math.* **50** (1985), 337–388.

[11] S. Kim. Projective normality of an algebraic curve. Preprint.

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