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WEAK COMPATIBILITY AND COMMON FIXED POINT THEOREMS FOR A-CONTRACTIVE AND E-EXPANSIVE MAPS IN UNIFORM SPACES

M. Aamri, D. El Moutawakil

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Abstract. The purpose of this paper is to define the notion of A-distance and E-distance in uniform spaces and give several new common fixed point results for weakly compatible contractive or expansive selfmappings of uniform spaces.

1. Introduction. The concept of compatibility was introduced by G. Jungck [3] in 1998 which is more general than that of weak commutativity introduced by Sessa [10], as follows

Definition 1.1 [3]. *Let T and S be two selfmappings of a metric space (X, d). S and T are said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever (x_n) is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$.*

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This notion was frequently used to prove existence theorems in the theory of common fixed point for contractive or expansive selfmappings of complete metric spaces. Further, in 1998, Jungck and Rhoades [5] introduced the following concept of weakly compatible

Definition 1.2 [5]. *Two selfmapping T and S of a metric space X are said to be weakly compatible if they commute at there coicidence points, i.e. if $Tu = Su$ for some $u \in X$, then $TSu = STu$.*

In [6] O. Kada, T. Suzuki and W. Takahashi have introduced the concept of a W-distance on metric spaces and have generalized some important results in non-convex minimizations and in fixed point theory for both W-contractive or W-expansive maps. On the other hand, it has always been tempting to generalize certain existence fixed or common fixed point theorems to uniform spaces. Following ideas in [6] J. R. Montes and J. A. Charris established in [9], some results on fixed and coincidence points of maps by means of appropriate W-contractive or W-expansive assumptions in uniform spaces. In this paper, we give many common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the concept of an A-distance or an E-distance.

The paper is divided into three sections. In section 2 we introduce the concept of an A-distance and an E-distance and we give a formulation of the concept of comatibilty and weak compatibility in the setting of uniform spaces. In section 3 we prove some common fixed point theorems for weakly compatible A (resp. E)-contractive maps and weakly compatible E-expansive maps. We begin by recalling some basic concepts of the theory of uniform spaces needed in the sequal. For more information we refer the reader to the book by N. Bourbaki [1], chapter II. We call uniform space (X, ϑ) a nonempty set X endowed of an uniformity ϑ, the latter being a special kind of filter on $X \times X$, all whose elements contain the diagonal $\Delta = \{(x, x)/x \in X\}$. If $V \in \vartheta$ and $(x, y) \in V$, $(y, x) \in V$, x and y are said to be V-close, and a sequence (x_n) in X is a Cauchy sequence for ϑ if for any $V \in \vartheta$, there exists $N \geq 1$ such that x_n and x_m are V-close for $n, m \geq N$. An uniformity ϑ defines a unique topology $\tau(\vartheta)$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X/(x, y) \in V\}$ when V runs over ϑ. A uniform space (X, ϑ) is said to be Hausdorff if and only if the intersection of all the $V \in \vartheta$ reduces to the diagonal Δ of X, i.e., if $(x, y) \in V$ for all $V \in \vartheta$ implies $x = y$. This guarantees the uniqueness of limits of sequences. $V \in \vartheta$ is said to be symmetrical if $V = V^{-1} = \{(y, x)/(x, y) \in V\}$. Since each $V \in \vartheta$ contains a symmetrical $W \in \vartheta$ and if $(x, y) \in W$ then x and y are both W and V-close, then for our purpose, we assume that each $V \in \vartheta$ is symmetical. When topological concepts are mentioned in the context of a uniform space (X, ϑ), they always refer to the topological space $(X, \tau(\vartheta))$.

2. A (resp. E)-distance.

Definition 2.1. *Let (X, ϑ) be a uniform space. A function $p : X \times X \longrightarrow \mathbb{R}^+$ is said to be an A-distance if for any $V \in \vartheta$ there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$.*

Definition 2.2. *Let (X, ϑ) be a uniform space. A function $p : X \times X \longrightarrow \mathbb{R}^+$ is said to be an E-distance if*

(p_1) *p is an A-distance,*

(p_2) $p(x, y) \leq p(x, z) + p(z, y), \quad \forall x, y, z \in X.$

Examples 2.0.1.

1. *Let (X, ϑ) be a uniform space and let d be a distance on X. Clearly (X, ϑ_d) is a uniform space where ϑ_d is the set of all subsets of $X \times X$ containing a "band" $B_\epsilon = \{(x, y) \in X^2 / d(x, y) < \epsilon\}$ for some $\epsilon > 0$. Moreover, if $\vartheta \subseteq \vartheta_d$, then d is an E-distance on (X, ϑ).*

2. *Recently, J. R. Montes and J. A. Charris introduced the concept of W-distance on uniform spaces. Every W-distance p is an E-distance since it satisfies (p_1), (p_2) and the following condition: for all $x \in X$, the function $p(x, .)$ is lower semi-continuous.*

3. *Let $X = [0, +\infty[$ and $d(x, y) = |x - y|$ the usual metric. Consider the function p defined as follows*

$$p(x, y) = \begin{cases} y, & y \in [0, 1[\\ 2y, & y \in [1, +\infty[\end{cases}$$

It is easy to see that the function p is an E-distance on (X, ϑ_d) but it is not an W-distance on (X, ϑ_d) since the function $p(x, .) : X \longrightarrow \mathbb{R}^+$ is not lower semi-continuous at 1.

The following lemma contains some useful properties of A-distances. It is stated in [6] for metric spaces and in [9] for uniform spaces. The proof is straightforward.

Lemma 2.0.1. *Let (X, ϑ) be a Hausdorff uniform space and p be an A-distance on X. Let (x_n), (y_n) be arbitrary sequences in X and (α_n), (β_n) be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x, y, z \in X$, the following holds*

(a) *If* $p(x_n, y) \leq \alpha_n$ *and* $p(x_n, z) \leq \beta_n$ *for all* $n \in \mathbb{N}$*, then* $y = z$*. In particular, if* $p(x, y) = 0$ *and* $p(x, z) = 0$*, then* $y = z$*.*

(b) *If* $p(x_n, y_n) \leq \alpha_n$ *and* $p(x_n, z) \leq \beta_n$ *for all* $n \in \mathbb{N}$*, then* (y_n) *converges to* z*.*

(c) *If* $p(x_n, x_m) \leq \alpha_n$ *for all* $m > n$*, then* (x_n) *is a Cauchy sequence in* (X, ϑ)*.*

Let (X, ϑ) be a uniform space with an A-distance p. A sequence in X is p-Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting

Definition 2.3. *Let* (X, ϑ) *be a uniform space and* p *be an A-distance on* X*.*

(1) *X is S-complete if for every p-Cauchy sequence* (x_n)*, there exists* x *in* X *with* $\lim_{n\to\infty} p(x_n, x) = 0$*.*

(2) *X is p-Cauchy complete if for every p-Cauchy sequence* (x_n)*, there exists* x *in* X *with* $\lim_{n\to\infty} x_n = x$ *with respect to* $\tau(\vartheta)$*.*

(3) $f : X \longrightarrow X$ *is p-continuous if* $\lim_{n\to\infty} p(x_n, x) = 0$ *implies* $\lim_{n\to\infty} p(f(x_n), f(x)) = 0$*.*

(4) $f : X \longrightarrow X$ *is* $\tau(\vartheta)$*-continuous if* $\lim_{n\to\infty} x_n = x$ *with respect to* $\tau(\vartheta)$ *implies* $\lim_{n\to\infty} f(x_n) = f(x)$ *with respect to* $\tau(\vartheta)$*.*

(5) *X is said to be p-bounded if* $\delta_p(X) = sup\{p(x, y)/x, y \in X\} < \infty$*.*

Remark 2.0.1. Let (X, ϑ) be a Hausdorff uniform space and let (x_n) be a p-Cauchy sequence. Suppose that X is S-complete, then there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$. Lemma 2.1(b) then gives $\lim_{n\to\infty} x_n = x$ with respect to the topology $\tau(\vartheta)$. Therefore S-completeness implies p-Cauchy completeness.

Before we state our main results, we give a formulation of the concept of compatibility and weak compatibility in the setting of uniform spaces as follows

Definition 2.4. *Let* (X, ϑ) *be a Hausdorff uniform space and* p *be an A-distance on* X*. Two selfmappings* f *and* g *of* X *are said to be p-compatible if, for each sequence* (x_n) *of* X *such that* $\lim_{n\to\infty} p(f(x_n), u) = \lim_{n\to\infty} p(g(x_n), u) = 0$ *for some* $u \in X$*, one has* $\lim_{n\to\infty} p(f(g(x_n)), g(f(x_n))) = 0$*.*

Definition 2.5. *Let* (X, ϑ) *be a Hausdorff uniform space and* p *be an A-distance on* X*. Two selfmappings* f *and* g *of* X *are said to be weak compatible if they commute at there coicidence points.*

3. Common fixed point results.

3.1. Common fixed point theorems for weakly compatible A (resp. E)-contractive maps. In the sequal, we involve a nondecreasing function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

(ψ_1) For each $t \in]0, +\infty[$, $0 < \psi(t)$.

(ψ_2) $\lim_{n\to\infty} \psi^n(t) = 0$, $\forall t \in]0, +\infty[$

It is easy to see that under the above properties, ψ satisfies also

$$\psi(t) < t, \quad \text{for each} \quad t > 0$$

Theorem 3.1.1. *Let (X, ϑ) be a Hausdorff uniform space and p be an A-distance on X such that X is p-bounded. Let f and g be two weakly compatible selfmappings of X such that*

(1) $f(X) \subseteq g(X)$,

(2) $p(f(x), f(y)) \le \psi(p(g(x), g(y)))$, $\quad \forall x, y \in X$.

If the range of f or g is a S-complete subspace of X, then f and g have a common fixed point.

P r o o f. Let $x_0 \in X$. Choose $x_1 \in X$ such that $f(x_0) = g(x_1)$. Choose $x_2 \in X$ such that $f(x_1) = g(x_2)$. In general, choose $x_n \in X$ such that $f(x_{n-1}) = g(x_n)$. We have

$$\begin{aligned} p(f(x_n), f(x_{n+m})) &\le \psi(p(g(x_n), g(x_{n+m}))) = \psi(p(f(x_{n-1}), f(x_{n+m-1}))) \\ &\le \psi^2(p(g(x_{n-1}), g(x_{n+m-1}))) = \psi^2(p(f(x_{n-2}), f(x_{n+m-2}))) \\ &\vdots \\ &\le \psi^n(p(f(x_0), f(x_m))) \le \psi^n(\delta_p(X)) \end{aligned}$$

where $\delta_p(X) = \sup\{p(x, y)/x, y \in X\}$. Then, by ($\psi_2$) and Lemma 2.1(c), we deduce that the sequence $(f(x_n))$ is a p-Cauchy sequence.

Suppose that $g(X)$ is S-complete, then $\lim_{n\to\infty} d(g(u), f(x_n)) = 0$, for some $u \in X$, and therefore $\lim_{n\to\infty} p(g(u), g(x_n)) = 0$. We show that $f(u) = g(u)$. Indeed:

$$d(f(u), f(x_n)) \le \psi(d(g(u), g(x_n)))$$

therefore $\lim_{n\to\infty} d(f(u), f(x_n)) = \lim_{n\to\infty} d(g(u), f(x_n)) = 0$ and Lemma 2.1(a) then gives $f(u) = g(u)$. The assumption that f and g are weakly compatibile implies

$fg(u) = gf(u)$. Also $f(f(u)) = f(g(u)) = g(f(u)) = g(g(u))$. Suppose that $p(f(u), f(f(u))) \neq 0$. From (2), it follows

$$p(f(u), f(f(u))) \leq \psi(p(g(u), g(f(u)))) = \psi(p(f(u), f(f(u)))) < p(f(u), f(f(u)))$$

which is a contradiction. Thus $p(f(u), f(f(u))) = 0$. Suppose that $p(f(u), f(u)) \neq 0$. Also from (2), we have

$$p(f(u), f(u)) \leq \psi(p(g(u), g(u))) = \psi(p(f(u), f(u))) < p(f(u), f(u))$$

a contradiction. Thus $p(f(u), f(u)) = 0$. Since $p(f(u), f(u)) = 0$ and $p(f(u), f(f(u))) = 0$, lemma 2.1(a) then gives $f(f(u)) = f(u)$. Hence $g(f(u)) = f(f(u)) = f(u)$, and therefore $f(u)$ is a common fixed point of f and g.

Now, if the range of f is a S-complete subspace of X, then there exists $x \in X$ such that $\lim_{n\to\infty} d(f(x), f(x_n)) = 0$. Since $f(X) \subseteq g(X)$, there exists $u \in X$ such that $f(x) = g(u)$ and the proof that $g(u)$ is a common fixed point of f and g is the same as that given when $g(X)$ is S-complete. $\square$

Clearly, one would ask whether the common fixed point is unique. This will be happen if we assume that the function p is an E-distance.

Theorem 3.1.2. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X such that X is p-bounded. Let f and g be two weakly compatible selfmappings of X such that*

(1) $f(X) \subseteq g(X)$,

(2) $p(f(x), f(y)) \leq \psi(p(g(x), g(y))), \quad \forall x, y \in X.$

If the range of f or g is a S-complete subspace of X, then f and g have a unique common fixed point.

Proof. Since an E-distance function p is an A-distance, f and g have a common fixed point. Suppose that there exists $u, v \in X$ such that $f(u) = g(u) = u$ and $f(v) = g(v) = v$. If $p(u, v) \neq 0$, then

$$p(u, v) = p(f(u), f(v)) \leq \psi(p(g(u), g(v))) = \psi(p(u, v)) < p(u, v)$$

which is a contradiction. Thus $p(u, v) = 0$. Similarly, we show that $p(v, u) = 0$. Consequently, by (p_2), we have $p(u, u) \leq p(u, v) + p(v, u)$ and therefore $p(u, u) = 0$. Now we have $p(u, u) = 0$ and $p(u, v) = 0$, which implies $u = v$. Hence we have the theorem. $\square$

Example 3.1.1. *Let $X = [0,1]$ and $d(x,y) = |x-y|$ the usual metric. Let f and g defined by*

$$fx = \begin{cases} x^2, & x \in [0, \frac{1}{2}[\\ 0, & x \in [\frac{1}{2}, 1] \end{cases} \qquad gx = \begin{cases} x, & x \in [0, \frac{1}{2}[\\ 1, & x \in [\frac{1}{2}, 1] \end{cases}$$

Consider the functions p and ψ defined as follows

$$\psi(x) = \begin{cases} x^2, & x \in [0, \frac{1}{2}[\\ \frac{1}{2}x, & x \in [\frac{1}{2}, 1] \end{cases}$$

and

$$p(x,y) = \begin{cases} y, & y \in [0, \frac{1}{2}[\\ 1, & y \in [\frac{1}{2}, 1] \end{cases}$$

On the one hand, the function p is an E-distance but not a W-distance and X is S-complete. Moreover f, g are weakly compatible since they commute. On the other hand, we have

$$d\left(f\left(\frac{1}{3}\right), f\left(\frac{1}{4}\right)\right) = \frac{7}{144} > \psi\left(d\left(g\left(\frac{1}{3}\right), g\left(\frac{1}{4}\right)\right)\right) = \psi\left(\frac{1}{12}\right) = \frac{1}{144}$$

which implies that $d(f(x), f(y)) \leq \psi(d(g(x), g(y)))$ does not hold for all $x, y \in X$. However, we have

$$p(f(x), f(y)) \leq \psi(p(g(x), g(y))), \quad \forall x, y \in X$$

and 0 is the unique common fixed point of f and g.

Example 3.1.2. *Let $X = [0,1]$ and $d(x,y) = |x-y|$ the usual metric. Let f and g defined by*

$$fx = \begin{cases} x^2, & x \in [0, \frac{1}{2}[\\ 0, & x \in [\frac{1}{2}, 1] \end{cases} \qquad gx = \begin{cases} x, & x \in [0, \frac{1}{2}[\\ \frac{1}{4}, & x \in [\frac{1}{2}, 1] \end{cases}$$

Then f and g are weakly compatible but not commuting. Consider the functions p and ψ defined as follows

$$\psi(x) = \begin{cases} x^2, & x \in [0, \frac{1}{2}[\\ \frac{1}{2}x, & x \in [\frac{1}{2}, 1] \end{cases}$$

and

$$p(x,y) = \begin{cases} y, & y \in [0, \frac{1}{2}[\\ 1, & y \in [\frac{1}{2}, 1] \end{cases}$$

It is easy to see that p is an E-distance, fX is S-complete and:

$$p(f(x), f(y)) \leq \psi(p(g(x), g(y))), \quad \forall x, y \in X.$$

Morever, 0 *is the unique common fixed point of f and g.*

Letting $g = Id_X$, the identity, gives a generalization of ψ-contraction in metric spaces, which is given in [9, page 39] as Problem 1.4.

Corollary 3.1.1. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X Such that X is p-bounded. Let f be a selfmapping of X such that*

$$p(f(x), f(y)) \leq \psi(p(x,y)), \quad \forall x, y \in X.$$

If the range of f is a S-complete subspace of X, then f has a unique fixed point.

Also for $f = Id_X$, we get the following result

Corollary 3.1.2. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Suppose that X is p-bounded and S-complete. Let g be a surjective selfmapping of X such that*

$$p(x,y) \leq \psi(p(g(x), g(y))), \quad \forall x, y \in X.$$

Then g has a unique fixed point.

3.2. Common fixed point theorems for E-expansive weakly compatible maps. In this section, we involve a nondecreasing function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying the following conditions

(ϕ_1) For each $t > 0$, $t < \phi(t)$,

(ϕ_2) For any decreasing sequence (t_n) in $\mathbb{R}^+$, if

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \phi(t_n) = t, \text{ for some } t \in \mathbb{R}^+$$

then $t = 0$.

Theorem 3.2.1. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Let f and g be two weakly compatible selfmappings of X such that*

(1) $g(X) \subseteq f(X)$,

(2) $\phi(p(g(x), g(y))) \leq p(f(x), f(y)), \quad \forall x, y \in X.$

If the range of f is a S-complete subspace of X, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. Choose $x_1 \in X$ such that $g(x_0) = f(x_1)$. Choose $x_2 \in X$ such that $g(x_1) = f(x_2)$. In general, choose $x_n \in X$ such that $g(x_{n-1}) = f(x_n)$. Consider the sequences $y_n = p(g(x_n), g(x_{n+1}))$ and $z_n = p(g(x_{n+1}), g(x_n))$, $n = 0, 1, \ldots$. Then we have $\lim\limits_{n\to\infty} y_n = 0$ and $\lim\limits_{n\to\infty} z_n = 0$. Indeed, we have

$$\begin{aligned} y_{n+1} = p(g(x_{n+1}), g(x_{n+2})) &< \phi(p(g(x_{n+1}), g(x_{n+2})) \\ &\leq p(f(x_{n+1}), f(x_{n+2})) = p(g(x_n), g(x_{n+1}))) \\ &< y_n \end{aligned}$$

and $y_n < \phi(y_n) \leq y_{n-1} < \phi(y_{n-1})$, which implies that (y_n) and $(\phi(y_n))$ are decreasings and then $\lim\limits_{n\to\infty} y_n$ and $\lim\limits_{n\to\infty} \phi(y_n)$ exist. therefore, on letting $n \longrightarrow +\infty$, we obtain $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} \phi(y_n) = t$, for some $t \in \mathbb{R}^+$. Condition (ϕ_2) then gives $t = 0$. Hence $\lim\limits_{n\to\infty} p(g(x_n), g(x_{n+1})) = 0$. The proof is similar to show that $\lim\limits_{n\to\infty} z_n = 0$.

Now we wish to show that the sequence $(g(x_{2n}))$ is a p-Cauchy sequence. Suppose that $(g(x_{2n}))$ is not a p-Cauchy sequence. Then there exists a positive number ϵ such that, for each positive integer $2k$, there exist integers $2n(k)$ and $2m(k)$ such that $2k \leq 2n(k) < 2m(k)$ and $p(g(x_{2n(k)}), g(x_{2m(k)})) \geq \epsilon$.

For each integer $2k$, let $2m(k)$ denotes the smallest integer satisfing the last two inequalities. Then $p(g(x_{2n(k)}), g(x_{2m(k)-2})) < \epsilon$. From (p_1), we get

$$\begin{aligned} \epsilon &\leq p(g(x_{2n(k)}), g(x_{2m(k)})) \\ &\leq p(g(x_{2n(k)}), g(x_{2m(k)-2})) + p(g(x_{2m(k)-2}), g(x_{2m(k)-1})) \\ &\quad + p(g(x_{2m(k)-1}), g(x_{2m(k)-2})) \end{aligned}$$

which gives $\lim\limits_{k\to\infty} p(g(x_{2n(k)}), g(x_{2m(k)})) = \epsilon$, since $\lim\limits_{n\to\infty} p(g(x_n), g(x_{n+1})) = 0$. On the other hand, we have

$$\begin{aligned} p(g(x_{2n(k)-1}), g(x_{2m(k)-1})) &= p(f(x_{2n(k)}), f(x_{2m(k)})) \\ &\geq \phi(p(g(x_{2n(k)}), g(x_{2m(k)}))) > p(g(x_{2n(k)}), g(x_{2m(k)})) \end{aligned}$$

and

$$p(g(x_{2n(k)-1}),g(x_{2m(k)-1})) \leq p(g(x_{2n(k)-1}),g(x_{2n(k)}))+p(g(x_{2n(k)}),g(x_{2m(k)-2})) +p(g(x_{2m(k)-2}),g(x_{2m(k)-1}))$$

on letting $k \longrightarrow \infty$, we obtain $\lim_{k\to\infty} \phi(p(g(x_{2n(k)}),g(x_{2m(k)}))) = \epsilon$.

Now we have

$$\lim_{k\to\infty} p(g(x_{2n(k)}),g(x_{2m(k)})) = \lim_{k\to\infty} \phi(p(g(x_{2n(k)}),g(x_{2m(k)}))) = \epsilon$$

By (ϕ_2), we get $\epsilon = 0$, which is a contradiction. Thus the sequence $(g(x_{2n}))$ is a p-Cauchy sequence and therefore $(g(x_n))$ is a p-Cauchy sequence. Since $f(X)$ is S-complete, then $\lim_{n\to\infty} d(g(x_n),f(u)) = 0$, for some $u \in X$, and therefore $\lim_{n\to\infty} p(f(x_n),f(u)) = 0$. we show that $f(u) = g(u)$. Indeed:

$$p(g(x_n),g(u)) < \phi(p(g(x_n),g(u))) \leq p(f(x_n),f(u)))$$

therefore $\lim_{n\to\infty} p(g(x_n),g(u)) = \lim_{n\to\infty} p(g(x_n),f(u)) = 0$ and Lemma 2.1(a) then gives $f(u) = g(u)$. The assumption that f and g are weakly compatibile implies $fg(u) = gf(u)$. Also $f(f(u)) = f(g(u)) = g(f(u)) = g(g(u))$. Suppose that $p(f(u),f(f(u))) \neq 0$. From (2), it follows

$$p(f(u),f(f(u))) = p(g(u),g(g(u))) < \phi(p(g(u),g(g(u)))) \leq p(f(u),f(f(u)))$$

which is a contradiction. Thus $p(f(u),f(f(u))) = 0$. Suppose that $p(f(u),f(u)) \neq 0$. Also from (2), we have

$$p(f(u),f(u)) = p(g(u),g(u)) < \phi(p(g(u),g(u))) \leq p(f(u),f(u))$$

a contradiction. Thus $p(f(u),f(u)) = 0$. Now we have $p(f(u),f(u)) = 0$ and $p(f(u),f(f(u))) = 0$, and lemma 2.1(a) then gives $f(f(u)) = f(u)$. Hence $g(f(u)) = f(f(u)) = f(u)$, and therefore $f(u)$ is a common fixed point of f and g. Suppose that there exists $u,v \in X$ such that $f(u) = g(u) = u$ and $f(v) = g(v) = v$. If $p(u,v) \neq 0$, then

$$p(u,v) = p(g(u),g(v)) < \phi(p(g(u),g(v))) \leq p(f(u),f(v)) = p(u,v)$$

which is a contradiction. Thus $p(u,v) = 0$. Similarly, we show that $p(v,u) = 0$. Consequently, by (p_2), we have $p(u,u) \leq p(u,v)+p(v,u)$ and therefore $p(u,u) = 0$. Now we have $p(u,u) = 0$ and $p(u,v) = 0$, which implies $u = v$. $\square$

Example 3.2.1. *Let $X = [0, +\infty[$ and $d(x, y) = |x - y|$ the usual metric. Let f and g defined by*

$$g(x) = \begin{cases} \frac{1}{3}x, & x \in [0, 1[\\ 0, & x \geq 1 \end{cases} \qquad f(x) = \begin{cases} \frac{1}{2}x, & x \in [0, 1[\\ 1, & x \geq 1 \end{cases}$$

Consider the functions p and ψ defined as follows

$$\phi(x) = \begin{cases} \frac{7}{6}x, & x \in [0, 1[\\ x + 1, & x \geq 1 \end{cases} \quad \text{and } p(x, y) = y$$

It is easy to see that p is an E-distance and $f(X)$ is S-complete. Moreover f, g are weakly compatible and

$$\phi(p(g(x), g(y))) \leq p(f(x), f(y))), \quad \forall x, y \in X$$

and 0 is the unique common fixed point of f and g.

Letting $f = Id_X$ (resp. $g = Id_X$), we get the following results

Corollary 3.2.1. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Suppose X is S-complete. Let g be a selfmapping of X such that*

$$\phi(p(g(x), g(y))) \leq p(x, y), \quad \forall x, y \in X$$

Then g has a unique fixed point.

Corollary 3.2.2. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X. Let f be a surjective selfmapping of X such that*

$$\phi(p(x, y)) \leq p(f(x), f(y)), \quad \forall x, y \in X$$

If the range of f is a S-complete subspace of X, then f has a unique fixed point.

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Department of Mathematics and Informatics
Faculty of Sciences Ben M'sik
University Hassan II - Mohammedia
Casablanca-Morocco
e-mail: d.elmoutawakil@math.net

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