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ON THE FACTORIZATION OF THE POINCARÉ POLYNOMIAL: A SURVEY

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Communicated by P. Pragacz

Dedicated to the memory of my teachers
Prof. Dr. C. Arf and Prof. Dr. M.G. Ikeda

Abstract. Factorization is an important and very difficult problem in mathematics. Finding prime factors of a given positive integer n, or finding the roots of the polynomials in the complex plane are some of the important problems not only in algorithmic mathematics but also in cryptography. For a given smooth m-dimensional real manifold X, one has the associated Poincaré polynomial $P(X,t) = \sum_{i=0}^{m} b_i(X)t^i$ of X, where $b_i(X) = \dim_{\mathbb{R}} H^i(X;\mathbb{R})$ is the i-th Betti number of X. It is clear that the factorization of $P(X,t)$ as series over the complex numbers $\mathbb{C}$ will carry lots of information about the topological and geometric invariants of X. This is possibly why a factorization of even such a special polynomial $P(X,t)$ is expected to be hard. However we can still search for algorithms to write

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$P(X,t)$ as a product of some nontrivial power series. One notes that the factorizations

$$P(\mathbb{P}^n, t^{1/2}) = \sum_{i=0}^{n} t^i = \frac{1-t^{n+1}}{1-t},$$

$$P(GL_n/B, t^{1/2}) = \prod_{i=1}^{n} \frac{1-t^i}{1-t}$$

are examples of such kind. Here $\mathbb{P}^n$ is the n-dimensional complex projective space and GL_n/B is the complex full flag manifold associated to the upper triangular matrices B in the invertible complex matrices GL_n. The aim of this survey article is to give first a direct self-contained elementary algebraic treatment of the problem and then provide examples of nonsingular complex projective varieties X so that the $\mathbb{C}$-algebra $H^*(X;\mathbb{C})$ fits into this treatment. This will allow us to factorize $P(X,t)$ as above for such a variety X. These varieties X will include all the homogeneous spaces G/P, their smooth Schubert subvarieties and more. It is also interesting to note that in this approach, one can read off smoothness of a Schubert variety from the factorization of its Poincaré polynomial, which is discussed in Section 2 and 3.

1. Poincaré series and geometry of homogeneous regular sequences. In this section we give a self contained treatment of Poincaré series based on [8], [19] and [22] only. Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a finitely generated associative, commutative graded algebra over a field k ($R_0 = k$). Since R is finitely generated, $\dim_k(R_i) < \infty$, and therefore the formal power series

$$P(R,t) = \sum_{i=0}^{\infty} \dim_k(R_i)t^i \ \in \mathbb{Z}[[t]]$$

makes sense. This series is called the Poincaré (Hilbert) series of R. A special case of a well-known theorem of Hilbert, improved by Serre, implies that $P(R,t)$ is a rational function of t. In fact it is known that if R is generated as a k-algebra by homogeneous elements $x_1,\dots,x_n$ of degrees $k_1,\dots,k_n$ respectively (i.e. $x_i \in R_{k_i},\ i = 1,\dots,n$), then the Poincaré series $P(R,t)$ has a factorization of the form

$$P(R,t) = \frac{f(t)}{\prod_{i=1}^{n}(1-t^{k_i})} \tag{1}$$

for some polynomial $f(t) \in \mathbb{Z}[t]$, ([8, Theorem 11.1]). Note that when the polynomial ring $k[x_1, \ldots, x_n]$ is considered with the usual grading $k[x_1, \ldots, x_n] = \bigoplus_{i=0}^{\infty} R_i$, where R_i consists of all homogeneous polynomials of degree i,

$$P(k[x_1, \ldots, x_n], t) = \frac{1}{(1-t)^n}.$$

This fact can be easily generalized. In fact, let R be the polynomial ring $k[x_1, \ldots, x_n]$ which is graded by taking the degrees of x_i to be the positive integers $k_i \geq 1$, $i = 1, \ldots, n$. Then it can be checked that

$$P(R, t) = \frac{1}{\prod_{i=1}^{n}(1 - t^{k_i})},$$

namely $f(t) = 1$ in the formula (1).

Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a finitely generated graded k-algebra with $\dim R = n$. We denote by $\dim R$ the dimension of R, the maximum number of elements of R which are algebraically independent over k. By a *homogeneous system of parameters* (h.s.o.p) in R we mean a set of n homogeneous elements $\phi_1, \ldots, \phi_n$ of positive degrees such that $R_{(\phi_1, \ldots, \phi_n)} = R/(\phi_1, \ldots, \phi_n)$ is a finite dimensional vector space over k. When k is an infinite field, a basic result of commutative algebra, known as the *Noether normalization lemma*, implies that a h.s.o.p for R always exists ([8, p. 69]). For a given h.s.o.p $\phi_1, \ldots, \phi_n$ in R, it is clear that $\phi_1, \ldots, \phi_n$ are algebraically independent and R is finitely generated $k[\phi_1, \ldots, \phi_n]$-module. The following proposition shows how to compute $P(R, t)$ from such a h.s.o.p $\phi_1, \ldots, \phi_n$, when R is a free $k[\phi_1, \ldots, \phi_n]$-module.

Proposition 1.1. *Let $\phi_1, \ldots, \phi_n$ be a homogeneous system of parameters in R. If R is a free $k[\phi_1, \ldots, \phi_n]$-module with*

(2) $$R = \bigoplus_{i=1}^{m} \psi_i k[\phi_1, \ldots, \phi_n]$$

where for each $i = 1, \ldots, m$, ψ_i is a homogeneous element of R, then

$$P(R, t) = \left(\sum_{i=1}^{m} t^{\deg(\psi_i)}\right) / \prod_{i=1}^{n}(1 - t^{\deg(\phi_i)}).$$

Proof. Let $k[\phi_1, \ldots, \phi_n] = \bigoplus_{i=0}^{\infty} S_i$ be the decomposition of the graded k-algebra $k[\phi_1, \ldots, \phi_n]$ into homogeneous parts. Since $\phi_1, \ldots, \phi_n$ are algebraically independent, $k[\phi_1, \ldots, \phi_n]$ is isomorphic as a graded k-algebra to the polynomial ring $k[y_1, \ldots, y_n]$ which is graded by $\deg y_i = \deg \phi_i$, $i = 1, \ldots, n$. Thus $P(k[\phi_1, \ldots, \phi_n], t) = \left(\prod_{i=1}^{n} \left(1 - t^{\deg(\phi_i)}\right) \right)^{-1}$. On the other hand, since $\{\psi_1, \ldots, \psi_m\}$ is a homogeneous free basis of the graded algebra $R = \bigoplus_{i=0}^{\infty} R_i$ over $k[\phi_1, \ldots, \phi_n]$, we get for each $i = 0, 1, \ldots$, $R_i = \bigoplus \psi_\ell S_j$, where the direct sum is over all $\ell = 1, 2, \ldots, m$ and $j = 0, 1, \ldots$ such that $\deg(\psi_\ell) + j = i$. The claim then follows by comparing the coefficients of t^i in both sides of the formula. □

Note that for the free $k[\phi_1, \ldots, \phi_n]$-module R, the homogeneous elements $\psi_1, \ldots, \psi_m$ of R satisfy (2) if and only if their images $\{\overline{\psi_1}, \ldots, \overline{\psi_m}\}$ in $R_{(\phi_1, \ldots, \phi_n)} = R/(\phi_1, \ldots, \phi_n)$ form a vector space basis for $R_{(\phi_1, \ldots, \phi_n)}$. This observation gives us the following:

Corollary 1.1. *Let $\phi_1, \ldots, \phi_n$ be a homogeneous system of parameters in R, and let $\psi_1, \ldots, \psi_m$ be homogeneous elements of R satisfying* (2) *above, then*

$$P(R_{(\phi_1, \ldots, \phi_n)}, t) = \sum_{i=1}^{m} t^{\deg(\psi_i)} = P(R, t) P(k[\phi_1, \ldots, \phi_n], t).$$

When R is a free $k[\phi_1, \ldots, \phi_n]$-module, this corollary gives us an algorithm to factorize the Poincaré series of $R_{(\phi_1, \ldots, \phi_n)}$. In particular, if R is the polynomial ring $k[x_1, \ldots, x_n]$ graded by deg $(x_i) = k_i \geq 1$, $i = 1, \ldots, n$, and R is a free $k[\phi_1, \ldots, \phi_n]$-module then we get

$$P(R_{(\phi_1, \ldots, \phi_n)}, t) = \prod_{i=1}^{n} \frac{1 - t^{\deg \phi_i}}{1 - t^{k_i}}. \tag{3}$$

A typical example is the polynomial ring $k[x_1, \ldots, x_n]$ with the usual grading and $\phi_i = \sigma_i(x_1, \ldots, x_n)$, the i-th elementary symmetric functions in $x_1, \ldots, x_n$, $i = 1, \ldots, n$. In this case the formula (3) becomes

$$P(R_{(\sigma_1, \ldots, \sigma_n)}, t) = \prod_{i=1}^{n} \frac{1 - t^i}{1 - t}.$$

We shall discuss later a far-reaching generalization of this example proved by Chevalley ([17, p. 73], [15]). A characterization of those homogeneous systems of parameters $\phi_1, \dots, \phi_n$ in R for which R is a free $k[\phi_1, \dots, \phi_n]$-module is well-known in commutative algebra (see [22, p. 482-483]), and they are called homogeneous regular sequences in R. By a *regular sequence in R* we mean n elements ($n = \dim R$) $\phi_1, \dots, \phi_n$ in R such that ϕ_1 is not a zero divisor and for each $i = 1, \dots, n-1$, ϕ_{i+1} is not a zero divisor in $R/(\phi_1, \dots, \phi_i)$ ([19, p. 95]). For the sake of completeness of this note we are going to give a geometric characterization of the homogeneous regular sequences in the polynomial algebra $R = k[x_1, \dots, x_n]$ where the grading is determined by $\deg(x_i) = k_i \geq 1$, $i = 1, \dots, n$. Let $\phi_1, \dots, \phi_n$ be a homogeneous system of parameters in R, and let $\phi : \mathbb{A}^n \to \mathbb{A}^n$ be the morphism given by $\phi(x) = (\phi_1(x), \dots, \phi_n(x))$, and let $R_{(\phi_1, \dots, \phi_n)} = R/(\phi_1, \dots, \phi_n)$. We note that a surjective flat morphism is called faithfully flat.

Theorem 1.1. *The following are equivalent.*
(i) $\phi_1, \dots, \phi_n$ *is a regular sequence in R,*
(ii) $\phi : \mathbb{A}^n \to \mathbb{A}^n$ *is faithfully flat,*
(iii) R is a free $k[\phi_1, \dots, \phi_n]$-module.

Proof. For (i) $\Rightarrow$ (ii): Since ϕ is a finite morphism, it is enough to prove that $\dim_k\ A(\phi^{-1}(\lambda)) = \dim_k(R/(\phi_1 - \lambda_1, \dots, \phi_n - \lambda_n)) = \dim_k A(\phi^{-1}_{(0)}) = \dim_k R_{(\phi_1, \dots, \phi_n)}$ for any $\lambda = (\lambda_1, \dots, \lambda_n) \in k^n$. Let I_λ be the ideal of R generated by $\phi_1 - \lambda_1, \dots, \phi_n - \lambda_n$, and let $\mathrm{gr}(I_\lambda)$ be the ideal generated by the leading terms $f_\star$ of f in I_λ. It is clear $\mathrm{gr}(I_\lambda)$ is a homogeneous ideal containing $I_0 = \mathrm{gr}(I_0) = (\phi_1, \dots, \phi_n)$. We claim $\mathrm{gr}(I_\lambda) = I_0$, for any $\lambda \in k^n$. Let $f = \sum_{i=1}^{n} (\phi_i - \lambda_i) f_i$ be an arbitrary element of I_λ. Since the property of being homogeneous regular sequence is independent of the order of the sequence ([19, p. 96-100]), without loss of generality we may assume $\sum_{i=1}^{k} \phi_i f_i \neq 0$, $\sum_{j=k+1}^{n} \phi_j f_j = 0$. But $\sum_{j=k+1}^{n} \phi_j f_j = 0$ implies that $f_j \in (\phi_1, \dots \phi_n)$ for each $j = k+1, \dots, n$, ([19]). Thus $f = \sum_{i=1}^{k} \phi_i f_i - \sum_{i=1}^{k} \lambda_i f_i + g$ for some $g \in (\phi_1, \dots, \phi_n)$. This implies immediately $f_\star \in (\phi_1, \dots, \phi_n)$, because $(\phi_1, \dots, \phi_n)$ is a homogeneous ideal with deg $(\phi_i) = k_i \geq 1$. The rest follows from the fact that $\dim_k A(\phi^{-1}(\lambda)) = \dim_k \mathrm{gr}(A(\phi^{-1}(\lambda)) = \dim_k\ A(\phi^{-1}(0))$.

For (ii) $\Rightarrow$ (iii): Let $\psi_\alpha, \alpha \in \wedge$, be the homogeneous elements of R such that $\{\overline{\psi_\alpha} : \alpha \in \wedge\}$ is a k-basis of $R_{(\phi_1,\ldots,\phi_n)} = A(\phi^{-1}(0))$. It is easy to see by induction on degree that $\{\psi_\alpha : \alpha \in \wedge\}$ spans R as $k[\phi_1,\ldots,\phi_n]$-module. This immediately implies that R is a free $k[\phi_1,\ldots,\phi_n]$-module, because $\phi = (\phi_1,\ldots,\phi_n)$ is a faithfully flat morphism.

(iii) $\Rightarrow$ (i): It is enough to show that whenever $f_{i+1}\phi_{i+1} + \cdots + f_1\phi_1 = 0$, $i = 0,\ldots,n-1$, then $f_{i+1} \in (\phi_1,\ldots,\phi_i)$. We prove this by using induction on i. For $i = 0$, $f_1\phi_1 = 0$ gives $f_1 = 0$ because ϕ_1 is a member of a homogeneous system of parameters in the integral domain R. Now assume the claim for $i = t-1 \leq n-1$. It is clear that R is a free $k[\phi_1,\ldots,\phi_t]$-module if and only if R is a free $k[\phi_1,\ldots,\phi_{t-1}]$ module and $R/(\phi_1,\ldots,\phi_{t-1})$ is a free $k[\phi_t]$-module. By the induction hypothesis the claim follows. □

The following corollary can also be found in [12, p. 296].

Corollary 1.2. *Let R be the polynomial ring $k[x_1,\ldots,x_n]$ graded by $\deg(x_i) = k_i \geq 1$, for $i = 1,\ldots,n$. If $\phi_1,\ldots,\phi_n$ is a homogeneous regular R-sequence, then the Poincaré polynomial $P(R_{(\phi_1,\ldots,\phi_n)},t)$ of the graded k-algebra $R_{(\phi_1,\ldots,\phi_n)} = R/(\phi_1,\ldots,\phi_n)$ has the following factorization:*

$$P(R_{(\phi_1,\ldots,\phi_n)},t) = \prod_{i=1}^{n} \frac{1-t^{\deg(\phi_i)}}{1-t^{k_i}}.$$

Poincaré polynomial of coinvariant algebra R_G of finite pseudo-reflection group G. Let $G \subset GL_n$ be a finite subgroup of the group of $n \times n$ invertible matrices GL_n over $\mathbb{C}$. G naturally acts on the polynomial ring $R = \mathbb{C}[x_1,\ldots,x_n]$. Let $R^G = \{f \in R : g \cdot f = g \text{ for every } g \text{ in } G\}$ be the ring of invariants of G, and let I^G be the ideal generated by $f \in R^G$ with $f(0) = 0$. Since G preserves the degrees of polynomials, I^G is a homogeneous ideal in the graded algebra

$$R = \mathbb{C}[x_1,\ldots,x_n], \quad \text{where} \quad \deg(x_i) = 1, \quad i = 1,\ldots,n.$$

The following theorem was proved by Shepard and Todd, Chevalley and Serre, see [22, p. 486] for the historical development.

Theorem 1.2. *There exists homogeneous regular sequence $\phi_1,\ldots,\phi_n$ in R such that $I^G = (\phi_1,\ldots,\phi_n)$ if and only if G is generated by pseudo-reflections.*

Recall that $g \in GL_n$ is called a pseudo-reflection if precisely one eigenvalue of g is not equal to one.

Corollary 1.3. *Let G be a finite subgroup of GL_n generated by pseudo-reflections, and let $\phi_1, \ldots, \phi_n$ be homogeneous elements of R such that $I^G = (\phi_1, \ldots, \phi_n)$. Then the Poincaré series $P(R_G, t)$ of the coinvariant algebra $R_G = R/I^G$ has the following factorization*

$$P(R_G, t) = \prod_{i=1}^{n} \frac{1 - t^{\deg(\phi_i)}}{1 - t}.$$

In particular if $G = \{\sigma(\mathrm{Id}) : \sigma \in S_n\}$ is the group of $n \times n$ permutation matrices in GL_n, then $I^G = (\sigma_1, \ldots, \sigma_n)$, where σ_i is the i-th elementary symmetric function in $x_1, \ldots, x_n$. Thus

$$P(R_G, t) = \prod_{i=1}^{n} \frac{1 - t^i}{1 - t}$$

as mentioned above.

2. Cohomology of (G_a, G_m)-varieties. Let X be a smooth n-dimensional complex projective variety having algebraic G_a- and G_m-actions.

$$\begin{aligned} \varphi &: G_a \times X \to X, \ \ ((z, x) \to \varphi(z) \cdot x) \\ \lambda &: G_m \times X \to X, \ \ ((t, x) \to \lambda(t) \cdot x) \end{aligned}$$

satisfying

(i) G_a-action φ has only one fixed point, say s_0.

(ii) there is a positive integer $p \geq 1$ such that $\lambda(t)\varphi(z)\lambda(t^{-1}) = \varphi(t^p z)$ for all t in G_m and z in G_a.

We call such a X a (G_a, G_m)-variety. If X is a (G_a, G_m)-variety then it is known that the fixed points X^{G_m} of the G_m-action λ form a finite set and $s_0 \in X^{G_m}$ ([7]). Let $X^{G_m} = \{s_0, s_1, \ldots, s_r\}$, we now recall the Bialynicki-Birula decomposition of X induced from the G_m-action λ. We set

$$X_i^- = \{x \in X : \lim_{t \to \infty} \lambda(t) \cdot x = s_i\}, \quad i = 0, 1, \ldots, r.$$

The X_i^- are called minus cells and the decomposition $X = \bigcup_{i=0}^{r} X_i^-$ is called the minus BB-decomposition ([9]). The G_m-action λ on X induces, via tangent action $d\lambda$, an action of G_m on the tangent space $T_{s_i}X$ of X at the fixed point s_i, $i = 0, 1, \ldots, r$. Since $\dim X^{G_m} = 0$, it follows from [9] that all the weights of $d\lambda$ on $T_{s_i}(X)$ are nonzero, and thus we get a G_m-invariant decomposition

$$T_{s_i}(X) = T_{s_i}(X)^- \oplus T_{s_i}(X)^+$$

of $T_{s_i}(X)$, where $T_{s_i}(X)^-$ (resp. $T_{s_i}(X)^+$) is a direct sum of negative (resp. positive) weight spaces (v is a negative (resp. positive) weight vector, if $d\lambda(t)\cdot v = t^k v$ for every $t \in G_m$ and for some $k < 0$ (resp. $k > 0$)). It follows from ([9]) that s_0 is the sink of the G_m-action λ, namely $T_{s_0}(X) = T_{s_0}(X)^-$, and each minus cell X_i^- is G_m-equivariantly isomorphic to the affine space $T_{s_i}(X)^-$. Thus, $X = \bigcup_{i=0}^{r} X_i^-$ is a G_m-invariant decomposition of X into complex affine spaces X_i^- with dim $X_i^- = \dim_{\mathbb{C}} T_{s_i}(X)^-$ = the number of negative weights of $d\lambda$ in $T_{s_i}(X)$, $i = 0, 1, \ldots, r$. It follows from this observation that odd Betti numbers are all zero and each even Betti number $b_{2k}(X)$ equals the number of fixed points s_i of the G_m-action $d\lambda$ at which exactly k weights are negative. Thus the Poincaré polynomial of X is given by

$$P(X, t^{1/2}) = \sum_{k=0}^{n} b_{2k}(X)t^k = \sum_{i=0}^{r} t^{v_i},$$

where $v_i = \dim(X_i^-) = \dim_{\mathbb{C}} T_{s_i}(X)^-$.

So far we have discussed the contribution of the G_m-action λ to the topology of X, now it is time to look at the G_a-action φ on X. We keep the notations as above and let $V = \left.\dfrac{d\varphi}{dz}\right|_{z=0}$ be the holomorphic vector field associated to φ, and let Z be the zero scheme of V. It follows from the property $\lambda(t)\varphi(z)\lambda(t^{-1}) = \varphi(t^p z)$ that the fixed point scheme X^{G_a} of φ is a G_m-invariant closed subscheme of X. Since X^{G_a} equals to Z as a scheme ([5]) and the support of Z is equal to $\{s_0\}$, Z is a G_m-invariant subscheme of $U = X_0^- \cong T_{s_0}(X) = T_{s_0}(X)^-$. The G_m-action λ on U induces G_m-action on the coordinate ring $A(U)$ of U in the usual manner: $(\lambda(t)\cdot f)(x) = f(\lambda(t^{-1})\cdot x)$. This G_m-action induces a graded algebra structure on $A(U) = \bigoplus_{k=0}^{\infty} A(U)_k$, where

$$A(U)_k = \{f \in A(U) : \lambda(t)\cdot f = t^k f \quad \text{for all} \quad t \in G_m\}.$$

Since Z is a G_m-invariant closed subscheme of U, the ideal $I(Z)$ of Z is a homogeneous ideal in $A(U)$, and therefore the coordinate ring $A(Z) = A(U)/I(Z)$

has a natural induced graded algebra structure. In fact, if $e_1, \dots, e_n$ is a basis of $T_{s_0}(X)$ of weight vectors of weights $a_1, \dots, a_n$, respectively, and $x_1, \dots, x_n$ is the dual basis, then $\mathrm{Sym}(T_{s_0}(X)^\star) = \mathbb{C}[x_1, \dots, x_n]$ and the grading is given by the fact that x_i is homogeneous of degree $\deg\ x_i = -a_i$. The following proposition gives the graded algebra structures of $A(U)$ and $A(Z)$ in terms of the weights of the G_m-action $d\lambda$ on $T_{s_0}(X)$ and the vector field V as follows:

Proposition 2.1. *Let $a_1, \dots, a_n$ be all the weights of the G_m-action $d\lambda$ on $T_{s_0}(X)$, and let R be the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ with homogeneous generators $x_1, \dots, x_n$ where $\deg x_i = -a_i$, $i = 1, \dots, n$. Then*

(i) *All the weights a_i are negative, and thus R is positively graded polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$ with $\deg(x_i) = -a_i \geq 1$, $i = 1, \dots, n$.*

(ii) *The algebra $A(U)$ is isomorphic to R as a graded algebra.*

(iii) *Viewing V as a derivation on $\mathbb{C}[x_1, \dots, x_n]$, $V(x_i) = \phi_i(x_1, \dots, x_n)$ is a homogeneous element of R having $\deg(\phi_i) = p - a_i$, $i = 1, \dots, n$. Moreover $\phi_1, \dots, \phi_n$ form a homogeneous regular sequence in R.*

(iv) *$A(Z)$ is isomorphic as a graded algebra to $R_{(\phi_1,\dots,\phi_n)} = R/(\phi_1, \dots, \phi_n)$.*

Proof. Since $\left.\dfrac{\partial}{\partial x_i}\right|_{s_0} = e_i$, $(d\lambda(t)\cdot V)_u = d\lambda(t)(V_{\lambda(t)^{-1}\cdot u}) = \sum_{i=1}^{n} \phi_i(\lambda(t)^{-1}\cdot u)$ $d\lambda(t) \cdot e_i$ and e_i has weight a_i, it follows that $\lambda(t) \cdot \phi_i = t^{p-a_i}\phi_i$ by condition (ii) above. This shows that $\deg(\phi_i) = p - a_i$; $i = 1, \dots, n$. Using this we can combine Proposition 3.1 and Lemma 3.3 of [22] to deduce that $\phi_1, \dots, \phi_n$ is a regular sequence, since $R/(\phi_1, \dots, \phi_n)$ has finite dimension (see the next theorem). The rest basically follows from the discussions above, for more details we refer the reader to [6], [7]. □

Corollary 2.1. *The Poincaré series $P(A(Z), t)$ of $A(Z)$ is given by*

$$P(A(Z), t) = P(R_{(\phi_1,\dots,\phi_n)}, t) = \prod_{i=1}^{n} \frac{1 - t^{p-a_i}}{1 - t^{-a_i}}.$$

In the following, we recall the calculation of $H^\star(X; \mathbb{C})$ associated to the vector field V from the references [3], [6] and [7], [14]. Let V be a holomorphic vector field on a nonsingular complex projective variety X with finitely many zeros and let $i(V) : \Omega_X^p \to \Omega_X^{p-1}$ be the contraction operator associated to V.

Here Ω_X^p (resp. $\mathcal{O}_X$) denotes the sheaf of germs of holomorphic p-forms (resp. functions) on X. It is clear that the structure sheaf $\mathcal{O}_Z$ of the zero scheme Z of V is $\mathcal{O}_X/i(V)\Omega_X^1$. That is, Z is the scheme (possibly unreduced) defined by the sheaf of ideals $J(Z) = i(V)\Omega_X^1$ in $\mathcal{O}_X$. We have the fundamental Koszul complex of sheaves:

$$0 \to \Omega_X^n \to \Omega_X^{n-1} \to \cdots \to \Omega_X^1 \to \mathcal{O}_X \to 0$$

in which the differential is $i(V)$, $n = \dim X$. It follows from general facts on hypercohomology that there are two spectral sequences $\{'E_r\}$ and $\{''E_r\}$ abutting to $\mathrm{Ext}^\star(X; \mathcal{O}_Z, \Omega_X^n)$ where $'E_1^{p,q} = H^q(X; \Omega_X^{n-p})$ and $''E_2^{p,q} = H^p(X; \mathrm{Ext}^q(\mathcal{O}_Z; \Omega_X^n))$. The key fact proved in [14] is that the first spectral sequence degenerates at $'E_1$. Thus, as a consequence of the finiteness of Z and $H^\circ(X; \mathcal{O}_Z) \cong \mathrm{Ext}^n(X; \mathcal{O}_Z, \Omega_X^n)$ we find

(i) $H^q(X; \Omega_X^p) = 0$ if $p \neq q$ (consequently $H^{2p+1}(X; \mathbb{C}) = 0$ and $H^{2p}(X; \mathbb{C}) = H^p(X; \Omega_X^p)$),

(ii) $A(Z) = H^\circ(X; \mathcal{O}_Z)$ has a filtration $A(Z) = F_n \supset \cdots \supset F_0$ such that $F_p/F_{p-1} \cong H^p(X; \Omega_X^p)$ and $F_p \cdot F_q \subseteq F_{p+q}$,

(iii) a graded algebra isomorphism

$$\Phi_V : \mathrm{Gr}(A(Z)) = \oplus F_p/F_{p-1} \to H^\star(X; \mathbb{C}).$$

The main difficulty in realizing the cohomology ring of X on Z lies in computing the mysterious filtration F_p. When X is a (G_a, G_m)-variety, the following theorem ([6]) says that the filtration F_p of $A(Z)$ is nothing but the filtration induced from the graded algebra structure on $A(Z)$ discussed in Proposition 2.1. Namely $\mathrm{Gr}(A(Z)) \cong A(Z) \cong R_{(\phi_1,\ldots,\phi_n)}$.

Theorem 2.1. *There exists an algebra isomorphism* $\Phi : A(Z) \to H^\star(X; \mathbb{C})$ *which carries* $A(Z)_{ip}$ *onto* $H^{2i}(X; \mathbb{C})$. *In particular* $A(Z)_k$ *is trivial unless* $k = ip$ *for some* i, $0 \leq i \leq n$.

Remark. $A(Z)$ together with $\Phi : A(Z) \xrightarrow{\sim} H^\star(X; \mathbb{C})$ is called the nilpotent description of $H^\star(X; \mathbb{C})$ obtained from the holomorphic field induced from the G_a-action φ. In view of [14], there is also another description of $H^\star(X; \mathbb{C})$ obtained from the holomorphic vector field induced from the G_m-action λ. This description is called semi-simple description of the cohomology algebra $H^\star(X; \mathbb{C})$, see [3] for details.

Corollary 2.2. *The Poincaré polynomial $P(X, t^{p/2})$ of X has the following factorization:*

$$P(X, t^{p/2}) = \prod_{i=1}^{n} \frac{1 - t^{p-a_i}}{1 - t^{-a_i}}$$

and moreover we have also the following identity:

$$\sum_{i=0}^{r} t^{v_i} = \prod_{i=1}^{n} \frac{1 - t^{p-a_i}}{1 - t^{-a_i}},$$

where $v_i = \dim X_i^- = \dim_{\mathbb{C}} T_{s_i}(X)^-$, $i = 0, 1, \dots, r$.

Lemma 2.1. *Let Y be a G_a-invariant non-empty closed subvariety of the (G_a, G_m)-variety X. Then Y is smooth if and only if Y is smooth at s_0.*

P r o o f. Since Y is closed and G_a-invariant, G_a has a fixed point in Y. Since the support of X^{G_a} is $\{s_0\}$, we get $s_0 \in Y$. Let Z be the singular locus of Y. Since Z is a G_a-invariant closed subvariety of Y, Z is non-empty if and only if $s_0 \in Z$. This finishes the proof. □

Let Y be a non-empty G_a-and G_m-invariant closed subvariety of the (G_a, G_m)-variety X, and let $\Omega(Y)$ be the set of all G_m weights that occur in the Zariski tangent space $T_{s_0}(Y)$ of Y at s_0. The following result is proved in [13].

Proposition 2.2. *Y is smooth if and only if the Poincaré polynomial of Y has the following factorization:*

$$P(Y, t^{p/2}) = \prod_{a_i \in \Omega(Y)} \frac{1 - t^{p-a_i}}{1 - t^{-a_i}}.$$

P r o o f. If Y is smooth, the factorization follows from Corollary 2.2 above. Now if we have the above factorization of $P(Y, t^{p/2})$, then it is easy to see that the Zariski tangent space of Y at s_0 has dimension $\dim Y$, and therefore Y is nonsingular at s_0. This finishes the proof in view of Lemma 2.1. □

3. Homogeneous spaces. For the rest of the note we fix the notation as follows:

$G \supset B \supset H$	:	a semi-simple linear algebraic group over $\mathbb{C}$, a Borel subgroup and a maximal torus
$B^- \supset U^-$	:	The Borel subgroup of G such that $B^- \cap B = H$ and its maximal unipotent subgroup
$\boldsymbol{g}$, $\boldsymbol{b}$, $\boldsymbol{h}$	:	Lie algebras of G, B and H, respectively
$\Delta \subset h^\star$	:	the root system of $(\boldsymbol{g}, \boldsymbol{h})$
$\boldsymbol{g}_\alpha = \mathbb{C}e_\alpha(\alpha \in \Delta)$	:	the α weight space in $\boldsymbol{g}$
$\sum = \{\alpha_i : i = 1, \ldots, n\}$	:	the basis of Δ corresponding to $\boldsymbol{b}$
Δ_+	:	the positive roots: $\alpha \in \Delta$, $\alpha > 0$
$\mathrm{ht}(\alpha)$	:	the height of $\alpha \in \Delta_+$, given by $\quad \mathrm{ht}(\sum_{i=1}^n k_i\alpha_i) = k_i$
W, n_w	:	the Weyl group $N_G(H)/H$ of $\boldsymbol{g}$, $\boldsymbol{h}$ and a representative of $w \in W$
r_α	:	the reflection corresponding to $\alpha \in \Delta$
$\ell(w), w_0$	:	the length function on W with respect to $\sum$ and the longest element of W
$A(\boldsymbol{h}) = \mathrm{Sym}\ (\boldsymbol{h}^\star)$	:	the coordinate ring of $\boldsymbol{h}$.

A. Borel-Chevalley description of $H^\star(G/B;\mathbb{C})$ and factorization of $P(G/B, t^{1/2})$. The Weyl group W acts on H as : $w \cdot s = n_w s n_w^{-1}$, $w \in W$, $s \in H$, and thus W acts on $\boldsymbol{h}$ via the adjoint action $w \cdot h = \mathrm{Ad}(w)(h)$, $w \in W$, $h \in \boldsymbol{h}$. It is known that W is a finite subgroup of $GL(\boldsymbol{h})$ and is generated by the reflections on $\boldsymbol{h}$. Thus the induced action of W on $A(\boldsymbol{h})$ produces the coinvariant algebra

$$R_W = A(\boldsymbol{h})/I^W \cong \mathbb{C}[x_1, \ldots, x_n]/(\phi_1, \ldots, \phi_n)$$

of W having the Poincaré series

$$P(R_W, t) = \prod_{i=1}^n \frac{1 - t^{\deg(\phi_i)}}{1 - t}.$$

In fact in this case it is known that the positive integers $\{\deg \phi_i : \quad i = 1, \ldots, n\}$ are independent of the choice of the generators of I^W ([17, p. 58]). These integers deg ϕ_i, $i = 1, \ldots, n$, are called degrees of W.

Let $\chi : H \to G_m$ be a character of H and L_χ be the associated line bundle on G/B:

$$L_\chi = G \times \mathbb{C}/B, \quad \text{where the action of} \quad B$$

on $G\times\mathbb{C}$ is given by $(g,z)\cdot b=(gb,\alpha(b^{-1})z)$. Here χ is extended on $B=U\rtimes H$ as usual: $\chi(u)=1$, $u\in U$, where $U=w_0U^-w_0$. Now let $\beta:A(\boldsymbol{h})\to H^\star(G/B;\mathbb{C})$ be the degree doubling graded algebra homomorphism determined by $\beta(d\chi)=c_1(L_\chi)$, where $d\chi\in\mathbf{h}^\star$ is the differential of χ at the identity and $c_1(L_\chi)$ is the first Chern class of L_χ.

Theorem 3.1 (Borel-Chevalley). *The algebra homomorphism $\beta:A(\boldsymbol{h})\to H^\star(G/B;\mathbb{C})$ is surjective with the kernel I^W, and therefore β induces an algebra isomorphism*

$$\overline{\beta}:R_W\xrightarrow{\sim}H^\star(G/B;\mathbb{C})$$

such that $(R_W)_i\cong H^{2i}(G/B;\mathbb{C})$, $i=1,2,\ldots$.

Remark. This theorem was originally proved in [11]. An alternative proof can be found in [2]. In [2] R_W together with $\overline{\beta}:R_W\xrightarrow{\sim}H^\star(G/B;\mathbb{C})$ has been viewed as a semi-simple description of $H^\star(G/B;\mathbb{C})$ associated to the holomorphic vector field induced from the G_m-action $\lambda(t)=\exp(th)$, where h is a regular semi-simple element of $\boldsymbol{h}$; for example, h can be taken as the unique element of $\boldsymbol{h}$ such that $\alpha_i(h)=1$, $i=1,\ldots,n$, as will be considered later.

Corollary 3.1. *The Poincaré polynomial $P(G/B,t^{1/2})$ of G/B has the following factorization:*

$$P(G/B,t^{1/2})=P(R_W,t)=\prod_{i=1}^{n}\frac{1-t^{m_i}}{1-t}$$

where $m_1,\ldots,m_n$ are the exponents of G.

When $G=GL_n$, $B=$ the group of upper triangular matrices, $H=$ the group of diagonal matrices, we get $W\cong S_n$, the symmetric group on the set $\{1,2,\ldots,n\}$; the action of W on $A(\boldsymbol{h})=\mathbb{C}[x_1,\ldots,x_n]$ is nothing but $\sigma\cdot f(x_1,\ldots,x_n)=f(x_{\sigma 1},\ldots,x_{\sigma n})$, $f\in A(\boldsymbol{h})$, $\sigma\in S_n$. Therefore $P(GL_n/B,t^{1/2})=\prod_{i=1}^{n}\frac{1-t^i}{1-t}$, as discussed in Section 1.

B. Nilpotent description of $H^\star(G/B;\mathbb{C})$ and Kostant-Macdonald identity. Let e be the principal nilpotent element $\sum_{i=1}^{n}e_{\alpha_i}$ in $\boldsymbol{b}$ and let h be the unique element in $\boldsymbol{h}$ such that $\alpha_i(h)=1$ for $i=1,\ldots,n$. By means of the exponential function exp, the element e and h induce one parameter subgroups G_a and G_m of B and H respectively. Now let φ and λ be the G_a- and G_m-action on G/B induced from these one parameter subgroups via the left multiplication. Then the following can be found in ([2], [13], [7]): G/B is a (G_a, G_m)-variety and

(i) $s_0 = B \in G/B$ is the unique fixed point of the G_a-action φ on G/B.

(ii) $\{ws_0 = n_w s_0 : \ w \in W\}$ is the fixed point set of the G_m-action λ on G/B.

(iii) $p = 1$ and $A(Z) \cong \mathbb{C}[x_\alpha : \alpha \in \Delta_+]/I(Z)$, where the grading is determined by $\deg(x_\alpha) = \mathrm{ht}(\alpha)$, $\alpha \in \Delta_+$.

It follows from Section 2 and (iii) above that

$$P(A(Z), t) = P(G/B, t^{1/2}) = \prod_{\alpha \in \Delta_+} \frac{1 - t^{\mathrm{ht}(\alpha)+1}}{1 - t^{\mathrm{ht}(\alpha)}}.$$

On the other hand we know from ([1]) that the minus BB-decomposition of G/B obtained from the G_m-action λ is nothing but

$$G/B = \bigcup_{w \in W} B^- ws_0, \quad \text{namely} \quad X^-_{ws_0} = B^- ws_0, \ \ w \in W.$$

Thus $\dim X^-_{ws_0} = \dim B^- ws_0 = \dim Bw_0ws_0 = \ell(w_0w)$ for any $w \in W$. It follows from Corollary 2.2 that

$$P(G/B, t^{1/2}) = \sum_{w \in W} t^{\ell(w_0w)} = \sum_{\sigma \in W} t^{\ell(\sigma)} = \prod_{i=1}^{n} \frac{1 - t^{m_i}}{1 - t} = \prod_{\alpha \in \Delta_+} \frac{1 - t^{\mathrm{ht}(\alpha)+1}}{1 - t^{\mathrm{ht}(\alpha)}}$$

which is known as the Kostant-Macdonald Identity ([7]). When $G = GL_n$, this identity becomes

$$P(GL_n/B, t^{1/2}) = \sum_{\sigma \in S_n} t^{\ell(\sigma)} = \prod_{i=1}^{n} \frac{1 - t^i}{1 - t} = \prod_{1 \le i < j \le n} \frac{1 - t^{j-i+1}}{1 - t^{j-i}},$$

where $\ell(\sigma) =$ the number of (i, j) with $1 \le i < j \le n$ such that $\sigma i > \sigma j$.

The typical G_a- and G_m-invariant closed subvarieties of $X = G/B$ are the so-called Schubert varieties: $X_w = \overline{Bws_0}$, the Zariski closure of the B-orbit of ws_0, $w \in W$. We recall that the Bruhat order $\tau \le w$ on W corresponds exactly to the inclusion of Schubert varieties $X_\tau \subseteq X_w$. Since $B = w_0 B^- w_0$, the orbit spaces $B\tau s_0$ and $B^- w_0 \tau s_0$ are isomorphic. This gives us an affine cellular decomposition of $X_w = \bigcup_{\tau \le w} B\tau s_0$. Thus the Poincaré polynomial of X_w

is given by $P(X_w, t^{1/2}) = \sum_{\tau \leq w} t^{\ell(\tau)}$. Now if X_w is smooth, then it follows from Proposition 2.2 that

$$P(X_w, t^{1/2}) = \prod_{\alpha \in \Omega_w} \frac{1 - t^{\mathrm{ht}(\alpha)+1}}{1 - t^{\mathrm{ht}(\alpha)}},$$

where Ω_w is the set of all G_m-weights that occur in the Zariski tangent space $T_{s_0}(X_w)$ of X_w at $s_0 = B$. For a smooth Schubert variety X_w, the following fact is due to Lakshmibai and Seshadri, D. Peterson, for more details see [13, p. 44]:

$$\Omega_w = \{\alpha \in \Delta_+ : r_\alpha \leq w\}.$$

Corollary 3.2. *Let X_w be a smooth Schubert subvariety of G/B, then we have*

$$P(X_w, t^{1/2}) = \sum_{\tau \leq w} t^{\ell(\tau)} = \prod_{\substack{\alpha \in \Delta_+ \\ r_\alpha \leq w}} \frac{1 - t^{\mathrm{ht}(\alpha)+1}}{1 - t^{\mathrm{ht}(\alpha)}}.$$

For any parabolic subgroup $P \supseteq B$ of G, it is clear φ and λ induce respective G_a- and G_m-actions on G/P making G/P a (G_a, G_m)-variety. One can easily modify the formulas above for the algebraic homogeneous space G/P and their non-singular Schubert subvarieties. In fact let $P = P_\theta$ be the parabolic subgroup associated to the subset θ of $\sum$, (if θ is empty, $P_\theta = B$), and let Δ_θ denote the span of θ in Δ_+. Then it can be checked that

$$A(Z) \cong \mathbb{C}[x_\alpha : \alpha \in \Delta_+ \setminus \Delta_\theta]/I(Z) \quad \text{and} \quad A(Z) \cong H^\star(G/P; \mathbb{C}).$$

where $\deg(x_\alpha) = \mathrm{ht}(\alpha)$, $\alpha \in \Delta_+$. Thus we get

$$P(G/P, t^{1/2}) = \prod_{\alpha \in \Delta_+ \setminus \Delta_\theta} \frac{1 - t^{\mathrm{ht}(\alpha)+1}}{1 - t^{\mathrm{ht}(\alpha)}}.$$

Example. The Poincaré polynomial of the Grassmann manifold $\mathbf{Gr}_{k,n}$. Let $G = GL_n$, let P_k be the parabolic subgroup of all matrices in G of the form $\begin{pmatrix} A & \star \\ O & B \end{pmatrix}$, where $1 \leq k < n$, and O is the $(n-k) \times k$ zero matrix. Let $h = \mathrm{diag}\,(n-1, n-2, \ldots, 1, 0)$ and let e be the $n \times n$ upper triangular matrix having 1 just above the diagonal and zero everywhere else. Then $G/P_k \cong \mathrm{Gr}_{k,n}$

is the Grassmann manifold of k-planes in $\mathbb{C}^n$, the G_m-action λ and G_a-action φ are given by

$$\lambda(t)gP_k = \text{diag}\,(t^{n-1}, t^{n-2}, \ldots, t, 1)gP_k, \quad t \in \mathbb{C}^\star$$

$$\varphi(z)gP_k = \exp(ze)gP_k \qquad , z \in \mathbb{C}.$$

The following can be found in [4]:

Let R be the polynomial ring $\mathbb{C}[z_{k+i,j} : 1 \leq i \leq n-k,\ 1 \leq j \leq k]$ with the grading determined by deg $z_{k+i,j} = k + i - j$, $i = 1, \ldots, n-k$ and $j = 1, \ldots, k$. Then $A(Z)$ is isomorphic to the graded algebra $R/I(Z)$, where the homogeneous ideal $I(Z)$ is generated by

$$a_{i,j}(z) = z_{k+1+i,j} - z_{k+i,j-1} - z_{k+i,k}z_{k+1,j},$$

$1 \leq i \leq n-k$, $1 \leq j \leq k$. Since deg $(a_{ij}) = k + 1 + i - j$ for $i = 1, \ldots, n-k$ and $j = 1, \ldots, k$, we get

$$P(\text{Gr}_{k,n}, t^{1/2}) = \prod_{\substack{1 \leq i \leq n-k \\ 1 \leq j \leq k}} \frac{1 - t^{k+1+i-j}}{1 - t^{k+i-j}} = \prod_{j=1}^{k} \left(\prod_{i=1}^{n-k} \frac{1 - t^{k+1+i-j}}{1 - t^{k+i-j}} \right)$$

$$= \prod_{j=1}^{k} \frac{1 - t^{n+1-j}}{1 - t^{k+1-j}} = \frac{(1-t^n)(1-t^{n-1}) \cdots (1-t^{n-k+1})}{(1-t^k)(1-t^{k-1}) \cdots (1-t)}.$$

This is nothing but the Gaussian polynomial

$$\frac{(1-t)(1-t^2)\cdots(1-t^n)}{(1-t)(1-t^2)\ldots(1-t^k)(1-t)(1-t^2)\cdots(1-t^{n-k})}.$$

Remarks. We have already given a smoothness criterion for the Schubert variety Y in G/P in terms of the factorization of the Poincaré polynomial of Y. On the other hand, we would like to note that

(a) When G is of type A and $P = B$, Lascoux proved the necessity part ([18]) whereas Gasharov proved the sufficiency part ([16]) of the following result: The Schubert variety X_w is smooth if and only if the Poincaré polynomial $P(X_w, t^{1/2})$ factors into polynomials of the form $1 + t + \cdots + t^r$. This result was extended later by Billey to type B in [10].

(b) The Poincaré polynomials for G/P, where G is the symplectic group or orthogonal group and P is maximal parabolic, were computed by different methods in [20] and [21].

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REFERENCES

[1] E. Akyildiz. Bruhat decomposition via G_m-action. *Bull. Acad. Pol. Sci. Math.* **28** (1982), 541–547.

[2] E. Akyildiz. Vector fields and cohomology of G/P. Springer-Verlag, Lecture Notes in Math., vol. **956**, 1982, 1–9.

[3] E. Akyildiz. SL_2-actions and cohomology of Schubert varieties. In: Topics in Algebra, Banach Centre Publications, vol. **26**, part. II, 1990, 13–26.

[4] E. Akyildiz, Y. Akyildiz. The relations of Plücker coordinates to Schubert Calculus. *J. Differential Geom.* **29** (1989), 139–142.

[5] E. Akyildiz, B. Aubertin. Zero scheme of a vector field is equal to fixed point scheme. In: Proc. of the IV. Symp. of Turkish Math. Soc., Antakya, 1991, 13–17.

[6] E. Akyildiz, J. B. Carrell. Cohomology of projective varieties with regular SL_2-actions. *Manuscripta Math.* **58** (1987), 473–486.

[7] E. Akyildiz, J. B. Carrell. A generalization of Kostant-Macdonald identity. *Proc. Nat. Acad. Sci. USA* **86** (1989), 3934–3937.

[8] M. F. Atiyah, I. G. Macdonald. Introduction to Commutative Algebra. Addison-Wesley Publ. Com., Menlo Park, Calif.- London-Don Mills, 1969.

[9] A. Bialynicki-Birula. Some theorems on actions of algebraic groups. *Ann of Math.* **98** (1973), 480–497.

[10] S. C. Billey. Pattern avoidence and rational smoothness of Schubert varieties. *Adv. Math.* **139**, *1* (1998), 141–156.

[11] A. Borel. Sur la cohomologies des espaces fibrés principaux et les espaces homogènes des groups de Lie compacts. *Ann. of Math.* **57** (1953), 115–207.

[12] R. Bott, L. W. Tu. Differential Forms in Algebraic Topology. Graduate Texts in Mathematics, vol. **82**, New York-Heidelberg-Berlin, Springer-Verlag, 1982.

[13] J. B. Carrell. Vector fields, flag varieties and Schubert calculus. In: Proc. of the Hyderabad Conf. on Algebraic Groups, Hyderabad, 1989, 23–57.

[14] J. B. Carrell, D. I. Lieberman. Holomorphic vector fields and compact Kaehler manifolds. *Invent. Math.* **21** (1973), 303–309.

[15] C. Chevalley. Invariants of finite groups generated by reflections. *Amer. J. Math.* **77** (1955), 778–782.

[16] V. Gasharov. Factoring the Poincaré polynomials for the Bruhat order on S_n. *J. Combin. Theory Ser. A* **83**, *1* (1998), 159–164.

[17] J. E. Humphreys. Reflection Groups and Coxeter Groups. Cambridge Univ. Press, Cambridge, 1990.

[18] A. Lascoux. Ordonner le groupe symétrique: pourquoi utiliser I'algébre de Iwahori-Hecke? Doc. Math., J. DMV, Extra Vol. ICM Berlin, vol. **III**, 1998, 355–364.

[19] H. Matsumura. Commutative Algebra. 2nd edition, the Benjamin Cummings Publ. Co., London, 1980.

[20] P. Pragacz, J. Ratajski. A Pieri-type theorem for Lagrangian and odd orthogonal Grassmannians. *J. Reine Angew. Math.* **476** (1996), 143–189.

[21] P. Pragacz, J. Ratajski. A Pieri-type formula for even orthogonal Grassmannian. *Fund. Math.* **178** (2003), 49–96.

[22] P. R. Stanley. Invariants of finite groups and their applications to combinatorics. *Bull. Amer. Math. Soc.* **1**, *3* (1979), 475–551.

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