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AUTOMORPHISMS OF THE PLANAR TREE POWER SERIES ALGEBRA AND THE NON-ASSOCIATIVE LOGARITHM

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Abstract. In this note we present the formula for the coefficients of the substitution series $f(g(x))$ of planar tree power series $g(x)$ into $f(x)$. The coefficient $c_T(f(g(x)))$ relative to a finite planar reduced rooted tree T is the sum

$$\sum c_S(f)c_{(T-\text{In}(S))}(g)$$

which is extended over all open subtrees S of T. In this expression $c_S(f)$ is the coefficient of f with respect to S and $T-\text{In}(S)$ is the complement of the inner vertices of S in T while $c_{(T-\text{In}(S))}(g)$ is the product of the coefficients of g relative to the connected components of the forest $T-\text{In}(S)$.

Also we give the formula for the compositional inverse of a planar tree power series g in terms of coefficients of g. It is shown that the generic non-associative exponential series EXP has an inverse LOG. Results about coefficients of LOG are presented.

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Key words: planar rooted tree, planar formal power series, non-associative exponential series, substitution endomorphisms and automorphisms for planar power series, Hopf algebra.

The group $G = \mathrm{Aut}(K\{\{x\}\}_\infty)$ of automorphisms of the algebra $K\{\{x\}\}_\infty$ of planar tree power series in a single variable x over a field K, see [6], is an affine group. The finite planar reduced rooted trees T are indexing a set of variables y_T of $K[G]$ and the composition in G induces a coproduct

$$\Delta\colon K[G] \to K[G] \otimes K[G]$$

for which

$$\Delta(y_T) = \sum y_S \otimes y_{(T-\mathrm{In}(S))}$$

where the summation is as in the formula above.
It is an interesting question how $(K[G], \Delta)$ which is indeed a Hopf algebra is related to the Hopf algebra of Kreimer, see [12], in which the coproduct on a variable y_T is defined by constructing admissible cuts on the set of edges of the planar rooted tree T while the above $\Delta(y_T)$ can be described by admissible cuts on the set of vertices of T.

Introduction. Let $K\{\{x\}\}_\infty$ be the algebra of planar tree power series in a variable x over a field K, see [6]. Then any $g \in K\{\{x\}\}_\infty$ of order > 0 induces a substitution homomorphism

$$\varphi_g\colon K\{\{x\}\}_\infty \to K\{\{x\}\}_\infty$$

which maps $f(x) \in K\{\{x\}\}_\infty$ into $f(g(x))$ by substituting $g(x)$ for x.
In this article we present a formula for the coefficients of $\varphi_g(f)$. It follows that φ_g is an automorphism if and only if the linear term of g is non-zero. A formula for the inverse of φ_f which is φ_g with $\varphi_g(f) = x$ in terms of the coefficients of g is given.

If $b(T)$ (resp. $a(T)$) is the coefficient of g (resp. f) relative to a planar reduced rooted tree T, then the following formula is obtained, see Theorem 4.2:

$$a(T) = \sum_{r=1}^{<\infty} (-1)^{r-1} \beta_r(T)$$

where

$$\beta_r(T) = \sum_{S \in \Omega_r(T)} \hat{b}_S$$

and $\Omega_r(T)$ is the system of flags of open subtrees of T of length r, see Definition 1.12.
In this formula

$$\hat{b}_S = b(S_0 - \mathrm{In}(S_1)) \cdot \ldots \cdot b(S_{r-1} - \mathrm{In}(S_r))$$

where

$$S = (S_0, \dots, S_r) \in \Omega_r(T),$$

$S_i - \mathrm{In}(S_{i+1})$ is the closed subforest in S_i complementary to S_{i+1} and

$$b(S_i - \mathrm{In}(S_{i+1}))$$

is the product of the coefficients of g with respect to the connected components of $S_i - \mathrm{In}(S_{i+1})$.
The Hopf algebra dual to the affine group $\mathrm{Aut}(K\{\{x\}\}_\infty)$ of automorphisms of $K\{\{x\}\}_\infty$ is explicitly constructed in Section 6. The formula for the antipode corresponds to the formula for the coefficients of the inverse of φ_g above.
There are Hopf algebras of trees by Kreimer [12], Brouder-Frabetti [1], Grossman-Larson [8], Connes-Kreimer [3], Loday [11], and others, see [5] in which the coproduct is defined by an admissible cutting procedure on the set of edges of trees. It is of interest to understand the relation between $(K[G], \Delta)$ and the other Hopf algebras of trees. The dual to the Hopf algebra of Kreimer is the Butcher group in the theory of geometric numerical integration of ordinary differential equations, see [9], Chap III. The question comes up naturally how the Butcher group is related to the group $G = \mathrm{Aut}(K\{\{x\}\}_\infty)$ of planar tree diffeomorphisms in one variable.

If $g = \mathrm{EXP}(x) - 1$ where $\mathrm{EXP}(x)$ is the generic non-associative exponential series, then the inverse of φ_g is φ_{LOG} where LOG is the generic logarithm, see Section 5. J. L.-Loday has suggested in [10] to apply LOG to the identity on the algebra $K\{\{X\}\}_\infty$ of X-labeled planar tree power series relative to the convolution product to obtain projections onto the space of primitive elements in $K\{\{x\}\}_\infty$, see [7], and idempotents in group algebras.

In Section 1 we discuss open subtrees and closed subgraphs of rooted trees. The arity $\mathrm{ar}_T(a)$ of a vertex a in a rooted tree T is defined to be the number of growing down edges of a. *A* subtree S of T is defined to be open in T, if the root of T is in S and if

$$\mathrm{ar}_S(a) = \mathrm{ar}_T(a)$$

for all inner vertices a in S. *A* subtree U of T is defined to be closed, if

$$\mathrm{ar}_U(a) = \mathrm{ar}_T(a)$$

for all vertices a of U.
A subforest of T is called closed, if every connected component is closed. It is

shown that there is a duality between open subtrees and closed subforests of any finite rooted tree.

The system $\Omega_r(T)$ of flags of open subtrees of length r is introduced. It appears as domain of summation in the formula for the inverse of φ_g in Theorem 4.2.
Some results about planar rooted trees are considered in Section 2. If S is an open subtree of T and $V_1, \ldots, V_m$ is the ordered set of components of the complement $T - \mathrm{In}(S)$ of the inner vertices of S in T, then

$$T = \cdot_S(V_1, \ldots, V_n)$$

is the grafting product of $V_1, \ldots, V_n$ relative to S, see Proposition 2.7. This result is applied in Sections 3 and 4 where we investigate substitution homomorphisms of planar tree power series algebras.

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1. Open and closed subtrees of rooted trees. If G is a graph, we denote the set of vertices of G by G^0 and the set of edges of G by $\overline{G}$. Throughout it is assumed that G is simple which means that there are no loops and no multiple edges in G. Thus any $k \in \overline{G}$ can be identified with the subset $\{a, b\}$ of $G^0, a \neq b$, consisting of the vertices which are incident with k.

Assume now that G is connected. For $a,\ b \in G^0$ we denote by $\mathrm{dist}_G\ (a, b)$ the minimum of the lengths of all paths in G connecting a with b. It is easy to check that dist_G is a metric on G^0 and that dist_G completely determines the graph G because $\mathrm{dist}_G(a, b) = 1$ if and only if $\{a, b\}$ is an edge in G.

Proposition 1.1. *Let T be a rooted tree with root ρ_T, U be a connected subgraph of T.*
Then U is a tree and there is a unique vertex σ of U such that the distance $\mathrm{dist}_T(\sigma, \rho_T)$ is minimal. We call σ the canonical root of U relative to T.

Remark 1.2. If S is a subtree of a rooted tree T, it is rooted by the canonical root relative to T. Thus we can and will consider S to be a rooted tree. If U is a subgraph of T, any connected component of U is a tree and canonically rooted.

For any vertex a of a graph G we denote by $\mathrm{val}_G(a)$ the valence of a with respect to G. It is the number of edges of G incident with a.

Definition 1.3. *Let T be a rooted tree and a be a vertex of T. Then*

$$\mathrm{ar}_T(a) := \begin{cases} \mathrm{val}_T(a): & a = \rho_T \\ \mathrm{val}_T(a) - 1: & a \neq \rho_T \end{cases}$$

is called the arity of the vertex a relative to T.

Let T be a finite rooted tree. Then

Definition 1.4. $L(T) := \{a \in T^0 : ar_T(a) = 0\}$ *is called the set of leaves of T and* $\mathrm{In}(T) := T^0 - L(T)$ *is called the set of inner vertices of T.*

Definition 1.5. *T is called reduced, if* $\mathrm{ar}_T(a) \neq 1$ *for all $a \in T^0$.*

Definitoin 1.6. *Let T be a rooted tree and S a subtree of T considered as rooted tree.*

(i) *S is called an open subtree of T, if $\rho_T \in S^0$ and* $\mathrm{ar}_S(a) = \mathrm{ar}_T(a)$ *for all inner vertices a of S.*

(ii) *S is called closed in T, if* $\mathrm{ar}_S(a) = \mathrm{ar}_T(a)$ *for all vertices a of S.*

The root of T and the whole T are open subtrees of T. All other open subtrees are called proper.

Definition 1.7. *Let U be a subgraph of a rooted tree. U is called closed in T if any connected component of U which is a canonically rooted tree is closed in T.*

A graph F is called a forest if all connected components of F are trees.

Definition 1.8. *A forest F together with a subset ρ_F of vertices of F is called rooted forest, if for any connected component C of F exactly one vertex of ρ_F is contained in C.*

Let F be a rooted forest and $a \in F^0$ and let

$$\mathrm{ar}_F(a) := \begin{cases} \mathrm{val}_F(a) & : a \in \rho(F) \\ \mathrm{val}_F(a) - 1 & : \text{otherwise.} \end{cases}$$

It is called the arity of a relative to F. We denote by $L(F)$ the set of vertices a of F of arity 0. It is called the set of leaves of F.

Corollary 1.9. *Let U be a subgraph of a rooted forest F. Then U is a forest and there is a canonical set ρ_U of vertices of U turning (U, ρ_U) into a rooted forest.*

Proposition 1.10. *Let S be an open subtree of a finite rooted tree T. Then*

$$T - \mathrm{In}(S) := (T^0 - \mathrm{In}(S), \overline{T} - \overline{S})$$

is a closed subgraph of T.

Proof. 1) Let k be an edge of T which is not in $S, k = \{a, b\}$. We have to show that $a, b \in T^0 - \mathrm{In}(S^0)$. If k is incident with a vertex of S, then it is incident with a leaf, say a, of S as S is open in T. Then $b \notin S^0$ because otherwise k would be in S. It follows that $a, b \notin \mathrm{In}(S)$ and thus $a, b \in T^0 - \mathrm{In}(S)$. This shows that $T - \mathrm{In}(S)$ is a subgraph of T.

2) Let now a be a vertex of $T - \mathrm{In}(S)$. If $a \in L(S)$, then the set of edges of $T - \mathrm{In}(S)$ is in $1-1$ correspondence with the edges of T which are not on the simple path between a and ρ_T.
Thus

$$\mathrm{ar}_{(T-\mathrm{In}(S))}(a) = \mathrm{ar}_T(a).$$

If a is a vertex of $T - \mathrm{In}(S)$ which is not in $L(S)$, then a is not a root of $T - \mathrm{In}(S)$ and of T and $\mathrm{val}_T(a) = \mathrm{val}_{(T-\mathrm{In}(S))}(a)$. □

Proposition 1.11. *Let F be a closed subgraph of a finite rooted tree T. Then*

$$S_F := T - (F - \rho(F)) = ((T^0 - F^0) \cup \rho(F), \overline{T} - \overline{F})$$

is an open subtree of T with $L(S_F) = \rho(F)$ and $T - \mathrm{In}(S_F) = F$.

Proof. 1) Let k be an edge of T which is not an edge of $F, k = \{a, b\}$ with $a, b \in T^0$. If k is incident with a root from F, say with a, then $b \notin F^0$ which shows that $a, b \in (T^0 - F^0) \cup \rho(F)$. If k is not incident with a root of F, then $a, b \notin F^0$ and a, b in $T^0 - F^0$. This shows that S_F is a subgraph of T and $L(S_F) = \rho(F)$.

2) If $a \in \mathrm{In}(S_F)$ then $\mathrm{val}_T(a) = \mathrm{val}_{S_F}(a)$ because $\overline{S}_F \cup \overline{F} = \overline{T}$. Thus $\mathrm{ar}_T(a) = \mathrm{ar}_{S_F}(a)$ if $a \in \rho(F) = L(S_F)$. Then $\mathrm{ar}_F(a) = \mathrm{val}_F(a)$ and $\mathrm{ar}_{S_F}(a) = 0$. This shows that S_F is an open subtree of T. □

Definition 1.12. *A subgraph U of a rooted forest F is open (resp. closed), if the intersection of U with any connected component T of F which is always a rooted tree is an open (resp. closed) subtree of T.*

If F is a finite rooted forest and $F_1, \dots, F_n$ are the connected components of F, then the set of open subforests of F corresponds bijectively to

$$\{(S_1, \dots, S_n) : S_i \text{ open subtree of } \quad F_i\}.$$

If S is an open forest of F, then $S_i = S \cap F_i$ is an open subtree of F_i for all i and

$$S = \bigcup_{i=1}^{n} S_i.$$

Corollary 1.13. *Let T be a rooted tree, S an open subtree of T and $F = T - \mathrm{In}(S)$ the complementary closed subforest.*
If S' is an open subforest of F, then $S \cup S'$ is an open subtree of T and

$$F - \mathrm{In}(S') = T - \mathrm{In}(S \cup S').$$

Proof. Apply Proposition 1.10 to the connected components of F. □

Let T be a finite rooted tree and $C \subseteq T^0$.

Definition 1.14. *C is called an admissible cut of T if for any leaf b of T the line $[b, \rho_T]$ intersects C in exactly one vertex.*

Proposition 1.15. *Let $C \subseteq T^0$. Then C is an admissible cut of T if and only if there is an open subtree S of T such that $C = L(S)$.*

Proof. 1) Let C be an admissible cut of T. Then

$$S := \bigcup_{c \in C} [c, \rho_T]$$

is an open subtree of T with $L(S) = C$. Here $[c, \rho_T]$ is the line between c and ρ_T which is the smallest connected subgraph of T containing c and ρ_T.

2) Let S be an open subtree of T. Obviously $[b, \rho_T]^0 \cap L(S)$ consists of exactly one vertex of $L(S)$ for any leaf b of T because $[b, \rho_T] \cap S$ is a line. Thus $L(S)$ is an admissible cut of T. □

Remark 1.16. The cuts of the definition above consist of vertices while the cut systems in [12, 5] consist of edges. It is not obvious how these two concepts are related.

Let T be a finite reduced rooted tree and $S = (S_0, S_1, \ldots, S_r)$ be a system of open subtrees S_i of T.

Definition 1.17. *S is called an open flag of T of length r if*

$$\begin{aligned} S_i &\supseteq S_{i+1} \text{ for all } 0 \le i < r \\ S_i &\neq S_{i+1} \text{ for all } 0 \le i < r \\ S_0 &= T \\ S_r &= \text{root of } T. \end{aligned}$$

The set of open flags of T of length r will be denoted by $\Omega_r(T)$.

Obviously $\Omega_1(T) = \{(T, \rho_T)\}$ and

$$\Omega_2(T) = \{(T, S, \rho_T) : S \text{ proper open subtree of } T\}.$$

Proposition 1.18. *Let T be a finite reduced rooted tree, $r \geq 2$. Then the set $\Omega_r(T)$ of open flags of T corresponds bijectively to $\Omega'_r(T) := \{(U, S): U$ is proper open subtree of T, $S \in \Omega_{r-1}(U)\}$.*

Proof. If $(U, S) \in \Omega'_r(T)$ and $S = (S_0, \ldots, S_{r-1})$ then $(T, S_0, S_1, \ldots, S_{r-1}) \in \Omega_r(T)$ and the map $(U, S) \to (U, S_0, \ldots, S_{r-1})$ is a bijective map $\Omega'_r(T) \to \Omega_r(T)$. $\square$

2. Planar rooted forests. Let F be a finite rooted forest and $\leq_{L(F)}$be a total order on

$$L(F) := \{a \in F^0 : \mathrm{ar}_F(a) = 0\}$$

where $\mathrm{ar}_F(a) = \mathrm{ar}_T(a)$, if T is the connected component of T containing a.

Definition 2.1. $(F, \leq_{L(F)})$ *is called planar rooted forest, if $L(U)$ is an interval of $L(F)$ relative to $\leq_{L(F)}$ for any closed connected subtree U of F.*

Let F, F' be finite planar rooted forests and φ an isomorphism of graph from F to F'.

Definition 2.2. *φ is called an isomorphism of planar rooted forests, if φ maps the roots of F bijectively to the roots of F' and if the restriction of φ onto the set $L(F)$ of leaves of F is a monotonic map from $L(F)$ into $L(F')$ relative to the orders $\leq_{L(F)}, \leq_{L(F')}$.*

Remark 2.3. Denote by **PRT** the set of isomorphism classes of finite reduced planar rooted trees, see [6]. The set of finite planar reduced rooted forests with n connected components corresponds bijectively with

$$\mathbf{PRT}^n = \{(T_1, \ldots, T_n) : T_i \in \mathbf{PRT}\}.$$

We associate to $(T_1, \ldots, T_n) \in \mathbf{PRT}^n$ the disjoint union $F = T_1 \cup \cdots \cup T_n$ and define $\rho_F = \{\rho_{T_i} : 1 \leq i \leq n\}$. Then (F, ρ_F) is a rooted forest with $L(F) = L(T_1) \cup \cdots \cup L(T_n)$.

Define the planarity order on $L(F)$ by putting $b \leq_{L(F)} b'$, if $b \in L(T_i), b' \in L(T_j)$ and $i \leq j$ or $i = j$ and $b \leq_{L(T_i)} b'$.

If $(F, \leq_{L(F)})$ is a planar rooted tree, we call $\leq_{L(F)}$ the planarity order of $(F, \leq_{L(F)})$. One can canonically extend $\leq_{L(F)}$ to an order $\leq_F$ on the set F^0 of vertices of F in the following way:

Let $a_1, a_2 \in F^0$ and T_1, T_2 be the connected components of F containing a_1, a_2.

Then $a_1 \leq_F a_2$, if the line $[a_1, \rho_1]_{T1}$ is contained in the line $[a_2, \rho_2]_{T_2}$ in which case $T_1 = T_2, \rho_1 = \rho_2$.
Assume now that neither $[a_1, \rho_1]_{T_1}$ is contained in $[a_2, \rho_2]_{T_2}$ nor vice versa.
Choose leaves b_i in $L(T_i)$ such that $a_i \in [b_i, \rho_i]_{T_i}$. Then $a_1 \leq_F a_2$ if $b_1 \leq_{L(F)} b_2$.
It is easy to check that this definition is independent of the choice of b_1, b_2 and that $b_1 \neq b_2$ because otherwise a_1, a_2 would be vertices on the line $[b_1, \rho_1]_{T_1}$ from which one can conclude that $[a_1, \rho_1]_{T_1} \subseteq [a_2, \rho_2]_{T_2}$ or vice versa.

Proposition 2.4. *Let U be a subgraph of a finite planar rooted forest. Then the restriction of the order $\leq_F$ on the set of leaves of U induces a planarity order on U and turn U into a planar rooted forest.*

P r o o f. By standard methods of graph theory. □

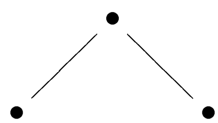
Let $T \in \mathbf{PRT}$.

Definition 2.5. $\deg(T) := \#L(T)$ *is called the degree of T.*

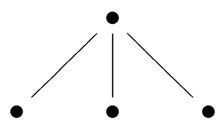
Example 2.6. *There is a unique $T \in \mathbf{PRT}$ with $\deg(T) = 1$. It will be denoted by x in the sequel.*

Let $m \in \mathbb{N}_{\geq 2}$. There is a unique planar reduced rooted tree x^m of degree m with the following properties:
The set of vertices of x^m is the disjoint union of the root of x^m and $L(x^m)$. It is called the m-corona.

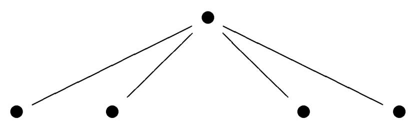
$x^2 =$



$x^3 =$



$x^4 =$



Proposition 2.7. *Let S be a finite planar rooted tree of degree m and $V_1, \ldots, V_m \in \mathbf{PRT}$. Then there is a unique $T \in \mathbf{PRT}$, such that*

(*i*) *S is an open subtree of T.*

(*ii*) *$T - \mathrm{In}(S)$ is a planar rooted forest isomorphic with*

$$V = V_1 \cup \cdots \cup V_m$$

where the order $\leq_{L(V)}$ is such that $b \leq_{L(V)} b'$, if $b \in L(V_i), b' \in L(V_{i+1})$ for $1 \leq i < m$.

T is denoted by $\cdot_S(V_1, \ldots, V_m)$ and is called the grafting product of $(V_1, \ldots, V_m)$ over S.
Moreover S is an open subtree of T and $T - \mathrm{In}(S) = V_1 \cup \cdots \cup V_m$.

Proof. Let T^0 be the set of equivalence classes of the disjoint union $S^0 \cup V_1^0 \cup \cdots \cup V_n^0$ modulo the identification of the i-th leaf b_i of S with the root of V_i.
Let $\overline{T}$ be the disjoint union

$$\overline{S} \cup \overline{V}_1 \cup \cdots \cup \overline{V}_m.$$

It is easy to see that $T := (T^0, \overline{T})$ is a rooted tree with the root ρ_S. There is a unique order on $L(T) = L(V_1) \cup \cdots \cup L(V_m)$ extending the planarity orders on $L(V_i)$ such that $b \leq_{L(T)} b'$, if $b \in L(V_i), b' \in L(V_j)$ with $i < j$. Thus T is a tree in **PRT**, S is an open subtree of T and obviously $T - \mathrm{In}(S) = V_1 \cup \cdots \cup V_m$.

Remark 2.8. If $S = x^m, m \in \mathbb{N}_{\geq 2}$, then $\cdot_{x^m}$ is also denoted by $\cdot_m$ and $\cdot_m(V_1, \ldots, V_m)$ is also denoted by $V_1 \cdot V_2 \cdot \ldots \cdot V_m$. It is the m-array grafting product, see [6], § 1.

Proposition 2.9. *Let $S, T \in$ **PRT**.*

(i) *If S is an open subtree of $T, S \neq \rho_T$, then* $\mathrm{ar}(S) = \mathrm{ar}(T)$.

(ii) *If* $\mathrm{ar}(S) = \mathrm{ar}(T)$, *$S = S_1 \cdot S_2 \cdot \ldots \cdot S_m$, $T = T_1 \cdot T_2 \cdot \ldots \cdot T_m$, then S is an open subtree of T if and only if S_i is an open subtree of T_i for all i.*

Proof. 1) (i) is obviously true.
2)$S - \rho_S, T - \rho_T$ are planar forests with m connected components. The i-th connected component of $S - \rho_S$ is S_i and the i-th connected component of $T - \rho_T$ is T_i. From this observation the statements of (ii) easily follow. □

Corollary 2.10 *Let $S, T \in$ **PRT**. There is at most one open subtree U of T which is isomorphic with S.*

P r o o f. 1) Assume that $S = x$. Then ρ_T is the only subtree of T isomorphic to S.
2)Let now $\mathrm{ar}(S) = m > 1$. If $m \neq \mathrm{ar}(T)$, then there is no open subtree U of T isomorphic to S. If $m = \mathrm{ar}(T), S = S_1 \cdot \ldots \cdot S_m$, $T = T_1 \cdot \ldots \cdot T_m$, then there is an open subtree U of T isomorphic to S if and only if T_i has an open subtree isomorphic to S_i by Proposition 2.9. □

Example 2.11. *List of proper open subtrees for trees in* **PRT** *of degree* ≤ 4 :

Tree	*Proper open subtrees*
x	
x^2	
x^3	
$x \cdot x^2$	x^2
$x^2 \cdot x$	x^2
x^4	
$x \cdot x^3$	x^2
$x^3 \cdot x$	x^2
$x(x \cdot x^2)$	$x^2, x \cdot x^2$
$x(x^2 \cdot x)$	$x^2, x^2 \cdot x$
$(x \cdot x^2) \cdot x$	x^2, xx^2
$(x^2 \cdot x) \cdot x$	x^2, x^2x
$x^2 \cdot x^2$	x^2, xx^2, x^2x
$x \cdot x \cdot x^2$	x^3
$x \cdot x^2 \cdot x$	x^3
$x^2 \cdot x \cdot x$	x^3

3. Substitution endomorphisms. Let K be a field. Denote by **A** or $K\{\{x\}\}_\infty$ the unital K-algebra of planar tree power series in a variable x over

K, see [6]. For any $f \in \mathbf{A}$, $T \in \mathbf{PRT}' := \mathbf{PRT} \cup \{1\}$, 1 is the empty tree, we denote by $c_T(f)$ the coefficient of f relative to $T \in \mathbf{PRT}'$. Let

$$\operatorname{ord}(f) := \begin{cases} \infty : f = 0 \\ \min\{\sharp L(T) : T \in PRT,\ c_T(f) \neq 0\} : f \neq 0. \end{cases}$$

Then ord defines a topology on $\mathbf{A}$ for which a system of neighbourhoods of 0 is given by $(U_n)_{n\geq 0}$ with $U_n = \{f \in \mathbf{A} : \operatorname{ord}(f) \geq n\}$.

Proposition 3.1. *Let* $g \in \mathbf{A}$, $\operatorname{ord}(g) \geq 1$. *Then there is a unique continuous unital* K*-algebra homomorphism*

$$\varphi_g : \mathbf{A} \to \mathbf{A}$$

with

$$\varphi_g(x) = g.$$

It is called the substitution endomorphism induced by g. *For* $f \in \mathbf{A}$ *the power series* $\varphi_g(f)$ *is also denoted by* $f(g(x))$ *or* $f(g)$*; it is obtained by substituting* g *for* x *in* f.

Proof. 1) Let $(h_n)_{n\geq 0}$ be a sequence in $\mathbf{A}$ and $h \in \mathbf{A}$. Then $\lim h_n = h$ is defined to mean that

$$\lim \operatorname{ord}(h_n - h) = \infty \quad \text{for} \quad n \to \infty.$$

Now it is easy to see that a sequence $(h_n)_{n\geq 0}$ in $\mathbf{A}$ has a unique limit $h \in \mathbf{A}$ if and only if

$$\lim \operatorname{ord}(h_n - h_m) \to \infty \text{ for } m, n \to \infty.$$

It follows that the partial sums

$$p_n = \sum_{i=0}^{n} h_i$$

of a sequence $(h_n)_{n\geq 0}$ in $\mathbf{A}$ are convergent in $\mathbf{A}$ if and only if

$$\lim_{n\to\infty} \operatorname{ord}(h_n) = \infty.$$

2) Let $V \in \mathbf{PRT}'$.
Define $\varphi_g(V) = 1$ if $V = 1$ and $\varphi_g(V) = g$ if $V = x$.
If $\operatorname{ar}(V) = m \geq 2$, $V = V_1 \cdot \ldots \cdot V_m$, then $\varphi_g(V) := \varphi_g(V_1) \cdot \ldots \cdot \varphi_g(V_m)$. By

induction on $\deg(V)$, this defines $\varphi_g(V)$ for all $V \in \mathbf{PRT}'$.
3) If $f \in \mathbf{A}$, then

$$h_n := \sum_{\deg(V)=n} c_V(f) \cdot \varphi_g(V)$$

is a power series of order $\geq n$, as $\operatorname{ord}(\varphi_g(V)) \geq n$, if $\operatorname{ord}(V) \geq n$. Thus

$$\sum_{n=0}^{\infty} h_n$$

is a series in $\mathbf{A}$ which is defined to be $\varphi_g(f)$.
4) It is easy to check that φ_g is K-algebra homomorphism. $\square$

Let $\operatorname{End}(\mathbf{A})$ denote the monoid of K-algebra endomorphisms $\varphi : \mathbf{A} \to \mathbf{A}$. Then $\operatorname{End}(\mathbf{A}) = \{\varphi_g : g \in \mathbf{A},\ \operatorname{ord}(g) \geq 1\}$ by the above proposition. Also $\varphi_x = \mathrm{Id}$ and

$$\varphi_g \circ \varphi_f = \varphi_g(f) = \varphi_{f(g(x))}.$$

Proposition 3.2. *Let $a(T) = c_T(f)$, $b(T) = c_T(g)$ for any $T \in \mathbf{PRT}$. For any finite planar rooted forest F let*

$$b(F) := \prod_{i=1}^{r} b(T_i)$$

where $T_1, \ldots, T_r$ is the system of connected components of F.

Then

$$c_T(f(g(x))) = \sum a(S) \cdot b(T - \operatorname{In}(S))$$

where the summation is extended over the system of all open subtrees S of T.

Proof. 1) As

$$\varphi_g(f) = \sum_{V \in \mathbf{PRT}'} a(V)\varphi_g(V)$$

$$\text{and} \quad c_T(\varphi_g(f)) = \sum_{V \in \mathbf{PRT}'} a(V) \cdot c_T(\varphi_g(V)).$$

It is enough to show that

$$c_T(\varphi_g(V)) = b(T - \operatorname{In}(V))$$

for all $V \in \mathbf{PRT}$ which are isomorphic with open subtrees of T.

2) If $\deg(V) = 1$, then $V = x$ and there is only one open subtree of T isomorphic with x; it consists only of the root of T. Then $T - \mathrm{In}(V) = T$ and we have to show that

$$c_T(\varphi_g(x)) = c_T(g)$$

is equal to $b(T)$ which is trivial by definition.

3) Let now $\deg(V) > 1$, V of arity $m \geq 2$ and $V = V_1 \cdot V_2 \cdot \ldots \cdot V_m$ the unique factorization of $V, V_i \in \mathbf{PRT}$. If V is isomorphic with an open subtree of $T \in \mathbf{PRT}$, then

$$\mathrm{ar}(T) = m, \quad T = T_1 \cdot T_2 \cdot \ldots \cdot T_m, \quad T_i \in \mathbf{PRT},$$

and V_i is isomorphic with an open subtree of T_i. Moreover

$$T - \mathrm{In}(V) = F_1 \cup \cdots \cup F_m$$

where

$$F_i = T_i - \mathrm{In}(V_i)$$

thus

$$\begin{aligned} b(T - \mathrm{In}(V)) &= b(F_1) \cdots b(F_m) \\ &= \prod_{i=1}^{m} b(T_i - \mathrm{In}(V_i)). \end{aligned}$$

By induction on the degree we can assume that

$$c_{T_i}(\varphi_g(V_i)) = b(T_i - \mathrm{In}(V_i)).$$

Thus it is here that the multiplication comes into play

$$\begin{aligned} c_T(\varphi_g(V)) &= \prod_{i=1}^{n} b(T_i - \mathrm{In}(V_i)) \\ &= b(T - \mathrm{In}(V)). \end{aligned}$$

□

Example 3.3. *Apparently*

$$c_x(f(g(x))) = a(x) \cdot b(x).$$

If $T \in \mathbf{PRT}, \mathrm{ar}(T) = m \geq 2, T = T_1 \cdot \ldots \cdot T_m$ *then*

$$\begin{aligned} c_T(f(g(x))) = a(x)b(T) + a(T) \cdot b(x)^m \\ + \sum_{S \in \Omega(T)} a(S)b(T - \mathrm{In}(S)), \end{aligned}$$

where $\Omega(T)$ denotes the set of proper open subtrees of T because $T - \mathrm{In}(\rho_T) = T$, $T - \mathrm{In}(T) = L(T)$ and $b(L(T)) = b(x)^m$.
If $T = x^m$, then

$$c_T(f(g(x))) = a(x)b(x^m) + a(x^m)b(x)^m$$

because $\Omega(x^m)$ is empty.

4. Substitution automorphism. Let $G = \mathrm{Aut}(K\{\{x\}\}_\infty)$ be the group of all K-algebra automorphisms of $K\{\{x\}\}_\infty$. It is also called the group of planar diffeomorphisms in one variable.

Proposition 4.1. $G = \{\varphi_g : g \in K\{\{x\}\}_\infty, \mathrm{ord}(g) = 1\}$.

Proof. 1) Let $\mathrm{ord}(g) = 1$. We will show that there is $f \in K\{\{x\}\}_\infty$, $\mathrm{ord}(f) > 0$, such that

$$\varphi_g(f) = x.$$

We consider the coefficients $a(T) = c_T(f)$ as unknown which have to satisfy the system of linear equations of Proposition 3.2. This means

$$a(x) = (b(x))^{-1} \text{ if } b(T) = c_T(g) \text{ and}$$
$$a(T) \cdot (b(x))^{\deg(T)} = -\sum a(S)b(T - \mathrm{In}(S))$$

for any $T \in \mathbf{PRT}$ of degree $n > 1$, where the summation is extended over all open subtrees S of T different from T. This system of equations has a unique solution as $b(x) \neq 0$ and $\deg(S) < \deg(T)$ for any open subtree S of T which is different from T.

2) $(\varphi_g \circ \varphi_f)(x) = \phi_{g(f)}(x) = x$ and thus $\varphi_g \circ \varphi_f = \varphi_x$ and φ_x is the identity in G.
As $\mathrm{ord}(f) = 1$ there is a power series h in $K\{\{x\}\}_\infty$ of order 1 such that

$$\varphi_f \circ \varphi_h = \varphi_x.$$

Therefore $\varphi_h = \varphi_g$ and φ_f is the inverse of φ_g in G. □

Let $g \in K\{\{x\}\}_\infty$, $\mathrm{ord}(g) = 1$, $b(T) = c_T(g)$ the coefficient of g relative to $T \in \mathbf{PRT}$ and assume that $b(x) = 1$. Let $f \in K\{\{x\}\}_\infty$ such that

$$\varphi_g(f) = x$$

and $a(T) = c_T(f)$ the coefficient of f relative to $T \in \mathbf{PRT}$.
Let $\Omega_r(T)$ denote the set of open flags of T of length r, see Definition 1.17.

For any $S \in \Omega_r(T), S = (S_0, \ldots, S_r)$,
let

$$\hat{b}_S := \prod_{i=0}^{r-1} b(S_i - \mathrm{In}(S_{i+1}))$$

where

$$b(S_i - \mathrm{In}(S_{i+1})) = b(F_1) \cdot \ldots \cdot b(F_n)$$

$$S_i - \mathrm{In}(S_{i+1}) = F_1 \cup \ldots \cup F_n$$

and all F_i are connected graphs.

Theorem 4.2. *Let*

$$\beta_r(T) = \sum_{S \in \Omega_r(T)} \hat{b}_S.$$

Then

$$a(T) = \sum_{r=1}^{<\infty} (-1)^{r-1} \beta_r(T).$$

Proof. 1) From Proposition 3.2 it follows that

$$c_T(f(g(x))) = \sum a(S) \cdot b(T - \mathrm{In}(S)) = \begin{cases} 1: & T = x \\ 0: & \text{otherwise} \end{cases}$$

where the summation is over all open subtrees S of T. If $\deg(T) > 1$ and $S = T$, then we get $T - \mathrm{In}(T)$ is a disjoint system of isolated vertices. Thus $b(T - \mathrm{In}(T)) = 1$, because $b(x) = 1$. If $S = \rho(T)$, then $T - \mathrm{In}(\rho_T) = T, a(\rho(T)) = 1$ and thus

$$a(T) = -b(T) - \sum a(S) \cdot b(T - \mathrm{In}(S))$$

where the summation is extended over all proper open subtrees S of T .

2) In order to prove the formula we proceed by induction on $n = \deg(T)$ noting that it is trivial if $n = 1$. Let now $n > 1$. We start with the formula in 1) and we may assume by induction hypothesis that for any proper open subtree S of T we have

$$a(S) = \sum_{r=1}^{<\infty} (-1)^{r-1} \beta_r(S).$$

Substituting this expression into the formula in 1) gives

$$a(T) = -\sum_{r=1}^{<\infty} (-1)^{r-1} \sum_{\substack{S \text{ open in } T \\ S \neq T}} \beta_r(S) \cdot b(T - \mathrm{In}(S)).$$

After applying Proposition 1.18 we get

$$a(T) = \sum_{r=0}^{<\infty} (-1)^r \beta_{r+1}(T). \qquad \square$$

Example 4.3. *List of $\beta_r(T, g)$ for $T \in \mathbf{PRT}$ of degree ≤ 4.*

The coefficient of $g \in K\{\{x\}\}_\infty$ relative to $T \in \mathbf{PRT}'$ is denoted by $b(T)$. Always $\beta_r(T) = 0$ for $r \geq 4$ and $\beta_1(T) = b(T)$ while

$$\beta_2(T) = \sum_{S \in \Omega(T)} b(T - \mathrm{In}(S)) \cdot b(S)$$

and

$$\beta_3(T) = \sum_{\substack{(S_1, S_2) \in \Omega(T) \\ S_1 \supset S_2, S_1 \neq S_2}} b(T - \mathrm{In}(S_1)) \cdot b(S_1 - \mathrm{In}(S_2)) \cdot b(S_2)$$

where $\Omega(T)$ denotes the set of proper open subtrees of T. The computations have been carried out by using the list of proper open subtrees in Example 2.11.

Tree T	$\beta_2(T)$	$\beta_3(T)$
x	0	0
x^2	0	0
x^3	0	0
$x \cdot x^2$	$b(x^2)^2$	0
$x^2 \cdot x$	$b(x^2)^2$	0
$x^2 \cdot x$	$b(x^2)^2$	0
x^4	0	0
$x \cdot x^3$	$b(x^2)b(x^3)$	0
$x^3 \cdot x$	$b(x^2)b(x^3)$	0
$x(x \cdot x^2)$	$2b(xx^2) \cdot b(x^2)$	$b(x^2)^3$
$x(x^2 \cdot x)$	$2b(x^2x)b(x^2)$	$b(x^2)^3$
$(xx^2) \cdot x$	$2b(xx^2)b(x^2)$	$b(x^2)^3$
$(x^2x) \cdot x$	$2b(xx^2)b(x^2)$	$b(x^2)^3$
$x^2 \cdot x^2$	$b(x^2)^3 + 2^b(x^2x)b(x^2)$	$2b(x^2)^3$
$x \cdot x \cdot x^2$	$b(x^2)b(x^3)$	0
$x \cdot x^2 \cdot$	$b(x^2)b(x^3)$	0
$x^2 \cdot x \cdot x$	$b(x^2)b(x^3)$	0

5. Logarithm. Let $K = \mathbb{Q}(q)$ be the field of rational functions in a variable q over $\mathbb{Q}$. Following a suggestion of Loday I denote the generic exponential series by EXP. It is a series in $K\{\{x\}\}_\infty$ and all coefficients of EXP are in the subring R of K of all rational functions f which have poles only in roots of unity different from 1, see [6].

For any $k \in \mathbb{Q} - \{-1\}$ there is a canonical algebra homomorphism

$$\pi_k: R\{\{x\}\}_\infty \to \mathbb{Q}\{\{x\}\}_\infty$$

mapping $f \in R$ onto $f(k) :=$ value of f at the place k.
Then $\pi_k(\mathrm{EXP}) = \exp_k(x)$ is the k-ary exponential series for any $k \in \mathbb{N}_{\geq 2}$ which means that

$$(\exp_k(x))^k = \exp_k(kx).$$

Proposition 5.1. *There is a unique* $\mathrm{LOG} \in R\{\{x\}\}_\infty$ *such that*

$$\mathrm{LOG}(\mathrm{EXP}(x)) = x$$

Moreover for any $k \in \mathbb{N}_{\geq 2}$

$$\pi_k(\mathrm{LOG})(\exp_k(x)) = x.$$

LOG *is called the generic logarithm and* $\log_k(x) := \pi_k(\mathrm{LOG})$ *is the* k*-ary logarithm.*

Proof. Immediate by Proposition 4.1 and the fact that π_k is compatible with compositions. □

Example 5.2. *We give the homogeneous terms of* LOG *of degree* ≤ 4. *Recall that*

$$[n] = \frac{q^n - 1}{q - 1} = \sum_{i=0}^{n-1} q^i$$

and that

$$[n]! = \prod_{k=1}^{n} [k].$$

So especially

$$[2] = q + 1, [3] = q^2 + q + 1, [3]! = (q + 1)(q^2 + q + 1) = q^3 + 2q^2 + 2q + 1.$$

Degree $1: \quad x$

Degree $2: \quad -\frac{1}{2}x^2$

Degree $3: \quad \frac{1}{4}\frac{q}{[2]}(x \cdot x^2 + x^2 \cdot x) - \frac{1}{3!}\frac{q-2}{[2]} \cdot x^3$

$$\textit{Degree } 4: \quad \left(\frac{q-3}{3![2]} - \frac{1}{4![3]!}(q+1)(q-2)(q-3)\right) \cdot x^4 +$$

$$+\left(\frac{1}{2}\frac{1}{3![2]} - \frac{2}{4![3]!}(q-2)\right)(x \cdot x^3 + x^3 \cdot x)$$

$$+\left(\frac{1}{3![2]!}\cdot\frac{3}{2} - \frac{1}{8} - \frac{3}{4![3]!}\right)(x\cdot(x\cdot x^2)+$$

$$+x\cdot(x^2\cdot x) + (x\cdot x^2)\cdot x + (x^2\cdot x)\cdot x)$$

$$+\left(\frac{1}{3![2]!}\cdot\frac{3}{2} - \frac{1}{8} - \frac{3(q+1)}{4![3]!}\right)\cdot x^2\cdot x^2$$

$$+\left(\frac{1}{2}\frac{q-2}{3![2]!} - \frac{2(q+1)(q-2)}{4![3]!}\right)(x\cdot x\cdot x^2 + x\cdot x^2\cdot x+$$

$$x^2\cdot x\cdot x)$$

In [4] $\log_2(x)$ *was computed to be*

$$\log_2(x) = x - \frac{1}{2}x^2 + \frac{1}{3}\left(\frac{1}{2}x\cdot x^2 + \frac{1}{2}x^2\cdot x\right) -$$

$$-\frac{1}{4}\left(\frac{4}{21}x\cdot(x\cdot x^2) + \frac{4}{21}x\cdot(x^2\cdot x) + \frac{5}{21}x^2\cdot x^2 + \frac{4}{21}(x\cdot x^2)\cdot x + \frac{4}{21}(x^2\cdot x)\cdot x\right) +$$

higher terms.
It is obtained by substituting 2 *for* q *in* LOG.

6. Hopf algebra of planar reduced rooted trees. In this section we describe the Hopf algebra which is dual to the affine group $\mathrm{Aut}(K\{\{x\}\}_\infty)$.

Let

$$H := K[y_T : T \in \mathbf{PRT}]$$

be the commutative, associative K-algebra of polynomials in the infinite set $\{y_T : T \in \mathbf{PRT}\}$ of variables where $y_T \neq y_S$ for any $T, S \in \mathbf{PRT}, T \neq S$.
There is a coproduct

$$\Delta : H \to H \otimes H$$

given by

$$\Delta(y_T) = \sum y_S \otimes y_{(T-\mathrm{In}(S))}, \text{ for any } T \in \mathbf{PRT},$$

where the summation is extended over all subtrees S of T and where

$$y_{(T-\mathrm{In}(S))} = y_{F_1} \cdot \ldots \cdot y_{F_n}$$

if the connected components of $T - \mathrm{In}(S)$ are $F_1, \ldots, F_n$.

Remark 6.1. If $T = x$, then the only open subtree of T is T and $\mathrm{In}(T)$ is empty from which follows that $T - \mathrm{In}(T) = T$ and

$$\Delta(y_x) = y_x \otimes y_x.$$

Also if $n = \deg(T) > 1$, then T is an open subtree of T with $T - \mathrm{In}(T) = L(T)$. Thus for $S = T$ we get

$$y_S \otimes y_{(T-\mathrm{In}(S))} = y_T \otimes y_x^n.$$

If also $S = x$, then $T - \mathrm{In}(S) = T$ and in $\Delta(y_T)$ we get the term $y_x \otimes y_T$. These computations show that ε is indeed the counit in what follows.

Proposition 6.2. Δ *is co-associative.*

P r o o f. Let $T \in \mathbf{PRT}$, $\deg(T) > 1$. Then

$$(\Delta \otimes \mathrm{Id})(y_T) = \sum_S \Delta(y_S) \otimes y_{(T-\mathrm{In}(S))} =$$

$$= \sum_{S',S} \Delta(y_{S'}) \otimes y_{S-\mathrm{In}(S')} \otimes y_{(T-\mathrm{In}(S))}$$

where the summation is over all pairs (S', S) of open subtrees of T with $S' \subseteq S$, $S' \neq S$.
Using Proposition 1.18 we can derive that

$$(\Delta \otimes \mathrm{Id})(T) = (\mathrm{Id} \otimes \Delta)(T).$$

Let $\mathrm{Sp}(H)$ denote the space of K-valued points of H which is the set of all unital K-algebra homomorphisms

$$\alpha : H \to K.$$

For any $\alpha \in \mathrm{Sp}(H)$ let

$$g_\alpha := \sum_{T \in \mathbf{PRT}} \alpha(y_T) \cdot T \ \in K\{\{x\}\}_\infty.$$

Then $\mathrm{ord}(g_\alpha) \geq 1$ and

$$\varphi_\alpha := \varphi_{(g_\alpha)}$$

is an endomorphism of $K\{\{x\}\}_\infty$.
If $\alpha, \beta \in \mathrm{Sp}(H)$, then

$$(\alpha \otimes \beta) \circ \Delta$$

is again a K-valued point of H. $\square$

Proposition 6.3.

(i) *The map* $\alpha \to \varphi_\alpha$ *is a bijection between* $\mathrm{Sp}(H)$ *and End* $(K\{\{x\}\}_\infty)$.

(ii) $\varphi_{\alpha\otimes\beta} = \varphi_\alpha \circ \varphi_\beta$ *for all* $\alpha, \beta \in \mathrm{Sp}(H)$.

Proof. 1) (i) is immediate because any endomorphism of $K\{\{x\}\}_\infty$ has the form φ_g with $\mathrm{ord}(g) \geq 1$ and $\mathrm{Sp}(H) \cong \{\text{set of maps}\}$ $\mathbf{PRT} \to K$ as $\{y_T : T \in \mathbf{PRT}\}$ are algebraically independent.
2) (ii) follows immediately from Proposition 4.1 and the definition of Δ above. $\square$

Let $H' = H[y_x^{-1}]$ be the ring of fractions with denominators in $\{y_x^m : m \geq 1\}$. Then the coproduct Δ on H extends uniquely to a coassociative coproduct

$$\Delta : H' \to H' \otimes H'$$

because $\Delta(y_x)$ has the inverse $y_x^{-1} \otimes y_x^{-1} \in H' \otimes H'$. Thus $\mathrm{Sp}(H') = \{\alpha \in \mathrm{Sp}(H) : \alpha(y_x) \neq 0\}$.

Let $\varepsilon \in \mathrm{Sp}(K)$ with $\varphi_\varepsilon = \mathrm{Id}$.
Then $\varepsilon(y_x) = 1$ and $\varepsilon(y_T) = 0$ for $\deg(T) > 1$. $\square$

Proposition 6.4. $(H', \Delta, \varepsilon)$ *is a Hopf algebra with counit* ε. *The antipode* σ *of it is the algebra automorphism given by*

$$\sigma(y_x) = y_x^{-1}$$

$$\sigma(y_T) = \sum_{r=1}^{<\infty} (-1)^{r-1} \eta_r(T)$$

where

$$\eta_r(T) = \sum_{S \in \Omega_r(T)} \hat{y}_S$$

$$\hat{y}_S = \prod_{i=1}^{r} y_{(S_i - \mathrm{In}(S_{i+1}))}$$

$$y_{(S_i - \mathrm{In}(S_{i+1})} = y_{F_1} y_{F_2} \cdot \ldots \cdot y_{F_n}$$

if

$$F_1, \ldots, F_n$$

are the connected components of the forest $S_i - \mathrm{In}(S_{i+1})$.

Proof. By Theorem 4.2 it is immediate that $y_{(\alpha\circ\sigma)} = y_\alpha^{-1}$ for any $\alpha \in \mathrm{Sp}(H')$. By general results from Hopf algebra theory it follows that σ is the antipode of $(H', \Delta, \varepsilon)$. □

Remark 6.5. In the literature one can find many constructions of Hopf algebras of trees, see [12, 5, 3, 1, 8, 11].
In most cases the coproduct is defined by separating a rooted tree T by an admissible cutting set of edges of T. This is in contrast to the above coproduct in which cutting sets of vertices of T are used. It is an interesting, but open question how these two types of coproducts are related.

REFERENCES

[1] Ch. Brouder, A. Frabetti. Noncommutative renormalization for massless QED, hep-th/oo 11161, 2000.

[2] R. Cori. Words and Trees. In: Combinatorics on Words (Ed. M. Lothaire), Cambridge 1981, Chap 11, 213–227.

[3] A. Connes, D. Kreimer. Hopf algebras, renormalization and noncommutative geometry. *Comm. Math. Phys* **199**, *1* (1998), 203–242, hep-th/98 08042.

[4] V. Drensky, L. Gerritzen. Nonassociative exponential and logarithm. *J. Algebra* **272**, *1* (2004), 311–320.

[5] L. Foissy. Les algèbres de Hopf des arbres enracinès decorès. These 2002, Universite de Reims.

[6] L. Gerritzen. Planar rooted trees and non-associative exponential series. *Adv. Appl. Math.* (to appear) 2004.

[7] L. Gerritzen, R. Holtkamp. Hopf co-addition for tree magma algebras and the non-associative Hausdorff series. *J. Algebra* **265** (2003), 264–284

[8] R. Grossman, R. G. Larson. Hopf-algebraic structure of families of trees. *J. Algebra* **126**, *1* (1989), 184–210.

[9] E. Hairer, C. Lubich, G. Wanner. Geometric Numerical Integration. Springer-Verlag, Berlin, 2002.

[10] J.-L. Loday. Generic logarithm. Manuscript, June 2003.

[11] J.-L. Loday, M. O. Ronco. Hopf algebra of the planar binary trees. *Adv. Math.* **139**, *2* (1998), 293–309.

[12] D. Kreimer. On the Hopf algebra structure of pertubative quantum field theories. *Adv. Theor. Math. Phys.* **2**, *2* (1998), 303–334.

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