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OSCILLATION CRITERIA FOR FIRST ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract. This paper is concerned with the oscillatory behavior of first-order delay differential equation of the form

$$\dot{x}\,(t) + p\,(t)\,x\,(\tau\,(t)) = 0,$$

where $p,\ \tau \in C\,[[t_0, \infty)\,, R^+]$, $R^+ = [0, \infty)$, $\tau\,(t)$ is nondecreasing, $\tau\,(t) < t$ for $t \geq t_0$ and $\lim\limits_{t\to\infty} \tau\,(t) = \infty$. Let the numbers k and l be defined by

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \quad \text{and} \quad l = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds.$$

It is proved here that when $l < 1$ and $0 < k \leq \frac{1}{e}$ all solutions of this equation oscillate in several cases in which the condition

$$l > \frac{e-1}{e-2}\left(k + \frac{1}{\lambda_1}\right) - \frac{1}{e-2},$$

holds, where λ_1 is the smaller root of the equation $\lambda = e^{k\lambda}$.

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1. Introduction. The problem of establishing sufficient conditions for the oscillation of all solutions of the differential equation

(1) $$\dot{x}(t) + p(t)\,x(\tau(t)) = 0,$$

where p, $\tau \in C\,[[t_0,\infty), R^+]$, $R^+ = [0,\infty)$, $\tau(t)$ is nondecreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$, has been the subject of many investigations. See, for example, [1]–[26] and the references cited therein.

By a solution of Eq.(1) we understand a continuously differentiable function defined on $[\tau(T_0), \infty)$ for some $T_0 \geq t_0$ and such that (1) is satisfied for $t \geq T_0$. Such a solution is called oscillatory if it has arbitrarily large zeros, and otherwise it is called nonoscillatory.

The first systematic study for the oscillation of all solutions of Eq. (1) was made by Myshkis [24]. In the 1950 paper [24] he proved that every solution of (1) oscillates if

(C_1) $$\lim_{t\to\infty} \sup(t-\tau(t)) < \infty \text{ and } \lim_{t\to\infty} \inf(t-\tau(t)).\lim_{t\to\infty} \inf p(t) > \frac{1}{e}.$$

In 1972 Ladas et al [19] proved that the same conclusion holds if

(C_2) $$\lim_{t\to\infty} \sup \int_{\tau(t)}^{t} p(s)ds > 1.$$

In 1979 Ladas [18] and in 1982 Koplatadze et al [13] improved (C_1) to

(C_3) $$\lim_{t\to\infty} \inf \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}.$$

Concerning the constant $\frac{1}{e}$ in (C_3), it should be pointed out that if the inequality

$$\int_{\tau(t)}^{t} p(s)ds \leq \frac{1}{e},$$

eventually holds, then, according to a result in [13] , Eq. (1) has a nonoscillatory solution.

In 1984 Ladas et al [20] and in 1984 Fukagai et al [10] established the oscillation criteria (of the type of conditions (C_2) and (C_3)) for Eq. (1) with oscillating coefficient $p(t)$.

It is obvious that there is a gap between the conditions (C_2) and (C_3) when the limit

$$\lim_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds,$$

does not exist.How to fill this gap is an interesting problem which has been recently investigating by several authors.

In 1988 Erbe et al [9] developed new oscillation criteria by employing the upper bound of the ratio $\dfrac{x(\tau(t))}{x(t)}$ for possible nonoscillatory solutions $x(t)$ of Eq. (1). Their result, when formulated in terms of the numbers k and l defined by

$$(C)\qquad k=\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds \quad\text{and}\quad l=\limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds,$$

says that all the solutions of Eq. (1) are oscillatory, if $0<k\leq\dfrac{1}{e}$ and

$$(C_4)\qquad l>1-\frac{k^2}{4}.$$

Since then several authors tried to obtain better results by improving the upper bound for $\dfrac{x(\tau(t))}{x(t)}$. In 1991 Jian Chao [2] derived the condition

$$(C_5)\qquad l>1-\frac{k^2}{2(1-k)},$$

while in 1992 Yu et al [27] and [28] obtained the condition

$$(C_6)\qquad l>1-\frac{1-k-\sqrt{1-2k-k^2}}{2}.$$

In 1990 Elbert et al [7] and in 1991 Kwong [16], using different techniques, improved (C_4), in the case where $0<k\leq\dfrac{1}{e}$, to the conditions

$$(C_7)\qquad l>1-\left[1-\frac{1}{\sqrt{\lambda_1}}\right]^2,$$

and

$$(C_8)\qquad l>\frac{\ln\lambda_1+1}{\lambda_1},$$

respectively, where λ_1 is the smaller root of the equation

$$\lambda = e^{k\lambda}. \tag{2}$$

In 1994 Koplatadze et al [14] improved (C_6), while in 1999 Philos et al [25], in 1998 Jaros et al [11], in 2000 Kon et al [17] and in 2003 Sficas et al [26] derived new conditions.

$$l > 1 - \frac{k^2}{2(1-k)} - \frac{k^2}{2}\lambda_1, \tag{C_9}$$

$$l > \frac{1+\ln\lambda_1}{\lambda_1} - \frac{1-k-\sqrt{1-2k-k^2}}{2}, \tag{C_{10}}$$

$$l > 2k + \frac{2}{\lambda_1} - 1, \tag{C_{11}}$$

and

$$l > \frac{\ln\lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2k\lambda_1}}{\lambda_1}, \tag{C_{12}}$$

respectively.

Following this historical (and chronological) review we also mention that in the case where

$$\int_{\tau(t)}^{t} p(s)ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \frac{1}{e},$$

this problem has been studied in 1995 by Elbert et al [8] and in 1995 by Kozakiewicz [15], Li [22, 23] and by Domshlak et al [5].

The purpose of this paper is to improve the methods previously used to show that the conditions (C_2) and $(C_4) - (C_{12})$ may be weakened to

$$l > \frac{e-1}{e-2}\left(k + \frac{1}{\lambda_1}\right) - \frac{1}{e-2}. \tag{C_{13}}$$

One has to notice that as $k \to 0$, then all conditions $(C_4) - (C_{11})$ and also our condition (C_{13}) reduce to the condition (C_2). However the improvement is clear as $k \to \frac{1}{e}$. For illustrative purpose, we give the values of the lower bound on l under these conditions, when $k = \frac{1}{e}$, are

(C_2) : 1.000000000
(C_4) : 0.966166179
(C_5) : 0.892951367
(C_6) : 0.863457014
(C_7) : 0.845181878
(C_8) : 0.735758882
(C_9) : 0.709011646
(C_{10}) : 0.599215896
(C_{11}) : 0.471517764
(C_{12}) : 0.459987065
(C_{13}) : 0.367879441

We see that our condition (C_{13}) essentially improves all the known results in the literature.

2. Main results. In what follows we will denote by k and l the lower and upper limits of the average $\int_{\tau(t)}^{t} p(s)ds$ as $t \to \infty$, respectively, see (C).

Set

$$\omega(t) = \frac{x(\tau(t))}{x(t)}.$$

We begin with the preliminary analysis of asymptotic behavior of the function $\omega(t)$ for a possible nonoscillatory solution $x(t)$ of Eq.(1) in the case when $k \leq \frac{1}{e}$. For this purpose, assume that (1) has a solution $x(t)$ which is positive for all large t. Dividing first Eq. (1) by $x(t)$ and then integrating it from $\tau(t)$ to t, we get the integral equality

$$\omega(t) = \exp \int_{\tau(t)}^{t} p(s)\omega(s)\,ds, \tag{3}$$

which holds for all sufficiently large t.

Also set

$$F(t) = \frac{p(t)}{\mu(t)},$$

where the function $\mu(t)$ satisfies the following conditions:

(i) $\mu(t)$ is nonincreasing,

(ii)

$$1 \leq N := \lim_{t\to\infty} \inf \mu(t) \leq \frac{1}{e-2}.$$

For the next lemma see [11].

Lemma 1 [11]. *Suppose that $k > 0$ and Eq. (1) has an eventually positive solution $x(t)$. Then*

$$k \leq \frac{1}{e} \quad \text{and} \quad \lambda_1 \leq \lim_{t\to\infty} \inf \omega(t) \leq \lambda_2$$

where λ_1 and λ_2 are the roots of the equation $\lambda = e^{k\lambda}$.

Lemma 2. *Let $0 < k \leq \frac{1}{e}$ and $x(t)$ be an eventually positive solution of Eq. (1). Assume that there exists $\theta > 0$ such that*

$$\int_{\tau(u)}^{\tau(t)} F(s)ds \geq \theta \int_u^t F(s)ds \quad \text{for all } \tau(t) \leq u \leq t, \tag{4}$$

then

$$\lim_{t\to\infty} \sup \omega(t) \leq \frac{2}{N\left(1-k-\sqrt{(1-k)^2-4B}\right)}, \tag{5}$$

where B is given by

$$B = \frac{e^{\lambda_1 \theta k} - \lambda_1 \theta k - 1}{(\lambda_1 \theta)^2}. \tag{6}$$

and λ_1 is the smaller root of the equation $\lambda = e^{k\lambda}$.

P r o o f. Let $t > t_0 \geq 1$ be large enough so that $\tau(t) > t_0$ and $t_1 \equiv t_1(t) > t$ such that

$$\tau(t_1) = t, \; \delta = \int_t^{t_1} F(s)ds \leq \int_t^{t_1} p(s)ds,$$

where $\delta : 0 < \delta < k$ is arbitrary close to k.

Integrating (1) from t to t_1, we obtain

$$x(t) = x(t_1) + \int_t^{t_1} p(s)x\left(\tau(s)\right) ds,$$

and $F(s) = \dfrac{p(s)}{\mu(s)}$,

$$x(t) = x(t_1) + \int_t^{t_1} F(s)\mu(s)x\left(\tau(s)\right) ds,$$

Integrating (1) from $\tau(s)$ to t for $s < t_1$, we have

$$\begin{aligned} x(\tau(s)) &= x(t) + \int_{\tau(s)}^{t} p(u)x\left(\tau(u)\right) du \\ &= x(t) + \int_{\tau(s)}^{t} F(u)\mu(u)x\left(\tau(u)\right) du. \end{aligned}$$

Combining the last two equalities, we obtain

$$x(t) = x(t_1) + \int_t^{t_1} F(s)\mu(s)\left(x(t) + \int_{\tau(s)}^{t} F(u)\mu(u)x\left(\tau(u)\right) du\right) ds. \tag{7}$$

Let $0 < \lambda < \lambda_1$, then the function

$$\phi(t) = x(t)e^{\lambda \int_{t_0}^{t} F(s)ds}, \tag{8}$$

is decreasing for large $t \geq t_0$ since $x(t)$ is also decreasing. Indeed, by Lemma 1,

$$\frac{x(\tau(t))}{x(t)} > \lambda, \text{ since } \mu(t) \geq 1 \quad \text{for } t \geq t_0 \geq 1,$$

then

$$\frac{\mu(t)x\left(\tau(t)\right)}{x(t)} > \lambda,$$

for all sufficiently large t, and consequently

$$0 = x'(t) + F(t)\mu(t)x\left(\tau(t)\right) \geq x'(t) + \lambda F(t)\, x(t),$$

which implies $\phi'(t) \leq 0$ for sufficiently large t.

Substituting into (7), we get for sufficiently large t the inequality

$$x(t) \geq x(t_1) + \delta x(t)$$

$$+ \int_t^{t_1} F(s) \left(\int_{\tau(s)}^{t} \mu(u) F(u) \phi(\tau(u)) e^{-\lambda \int_{t_0}^{\tau(u)} F(\xi) d\xi} du \right) ds$$

$$\geq x(t_1) + \delta x(t)$$

$$+\mu(t) \phi(\tau(t)) \int_t^{t_1} F(s) \left(\int_{\tau(s)}^{t} F(u) e^{-\lambda \int_{t_0}^{\tau(u)} F(\xi) d\xi} du \right) ds$$

$$= x(t_1) + \delta x(t)$$

$$+\mu(t) \phi(\tau(t)) e^{-\lambda \int_{t_0}^{\tau(t)} F(s) ds} \int_t^{t_1} F(s) \left(\int_{\tau(s)}^{t} F(u) e^{\lambda \int_{\tau(u)}^{\tau(t)} F(\xi) d\xi} du \right) ds.$$

From (8), we have

$$(9) \quad x(t) \geq x(t_1) + \delta x(t) + \mu(t) x(\tau(t)) \int_t^{t_1} F(s) \left[\int_{\tau(s)}^{t} F(u) e^{\lambda \int_{\tau(u)}^{\tau(t)} F(\xi) d\xi} du \right] ds.$$

In view of (4) we obtain

$$\begin{aligned} \int_{\tau(s)}^{t} F(u) e^{\lambda \int_{\tau(u)}^{\tau(t)} F(\xi) d\xi} du &\geq \int_{\tau(s)}^{t} F(u) e^{\lambda\theta \int_{u}^{t} F(\xi) d\xi} du \\ &= \frac{1}{\lambda\theta} \left(e^{\lambda\theta \int_{\tau(s)}^{t} F(\xi) d\xi} - 1 \right). \end{aligned}$$

Thus

$$\int_t^{t_1} F(s)\left(\int_{\tau(s)}^{t} F(u)e^{\lambda\int\limits_{\tau(u)}^{\tau(t)}F(\xi)d\xi}du\right)ds \geq \frac{-\delta}{\lambda\theta}+\frac{1}{\lambda\theta}\int_t^{t_1}F\left(s\right)e^{\lambda\theta\int\limits_{\tau(s)}^{t}F(\xi)d\xi}ds$$

$$= \frac{-\delta}{\lambda\theta}+\frac{1}{\lambda\theta}\int_{t_1}^{t}F(s)e^{\lambda\theta\int\limits_{\tau(s)}^{s}F(\xi)d\xi-\lambda\theta\int\limits_{t}^{s}F(\xi)d\xi}ds$$

$$\geq \frac{-\delta}{\lambda\theta}+\frac{1}{\lambda\theta}e^{\lambda\theta\delta}\int_t^{t_1}F\left(s\right)e^{-\lambda\theta\int\limits_{t}^{s}F(\xi)d\xi}ds$$

$$= \frac{-\delta}{\lambda\theta}+\frac{e^{\lambda\theta\delta}}{\left(\lambda\theta\right)^2}\left(1-e^{-\lambda\theta\int\limits_{t}^{t_1}F(\xi)d\xi}\right)$$

$$= \frac{-\delta}{\lambda\theta}+\frac{e^{\lambda\theta\delta}}{\left(\lambda\theta\right)^2}\left(1-e^{-\lambda\theta\delta}\right)$$

$$= \frac{-\delta}{\lambda\theta}+\frac{1}{\left(\lambda\theta\right)^2}\left(e^{\lambda\theta\delta}-1\right).$$

and (9) yields

$$x(t)\geq x\left(t_1\right)+\delta x\left(t\right)+B^*\mu\left(t\right)x\left(\tau\left(t\right)\right), \tag{10}$$

where

$$B^*=\frac{e^{\lambda\theta\delta}-\lambda\theta\delta-1}{\left(\lambda\theta\right)^2}.$$

From (10), we have

$$x(t)\geq d_1\mu\left(t\right)x\left(\tau(t)\right),$$

where

$$d_1=\frac{B^*}{1-\delta}.$$

Observe that

$$x(t_1)\geq d_1\mu\left(t_1\right)x\left(\tau(t_1)\right)\geq d_1x(t),$$

since $\mu(t) \geq 1$ for $t \geq t_0 \geq 1$ and therefore (10) yields

$$x(t) \geq d_2 \mu(t)\, x(\tau(t))$$

where

$$d_2 = \frac{B^*}{1 - d_1 - \delta},$$

Following this iterative procedure (cf. [26, 27]), we obtain

$$x(t) \geq d_{n+1} \mu(t)\, x(\tau(t)),$$

where

$$d_{n+1} = \frac{B^*}{1 - d_n - \delta}, \quad n = 1, 2, 3, \ldots.$$

It is easy to see that the sequence $\{d_n\}$ is strictly increasing and bounded.

Therefore

$$\lim_{t \to \infty} d_n = d,$$

exists and satisfies

$$d^2 - (1 - \delta)\, d + B^* = 0,$$

since $\{d_n\}$ is strictly increasing it follows that

$$d = \frac{1 - \delta - \sqrt{(1 - \delta)^2 - 4B^*}}{2}.$$

Observe that for large t one has

$$\frac{x(t)}{\mu(t) x(\tau(t))} \geq \frac{1 - \delta - \sqrt{(1 - \delta)^2 - 4B^*}}{2},$$

and since $0 < \delta < k$ is arbitrarily close to k, by letting $\lambda \to \lambda_1$ the last inequality leads to (5).

The proof is complete. $\square$

Remark 1. Assume that $\tau(t)$ is continuously differentiable and that there exists $\theta > 0$ such that

$$F(\tau(t))\, \tau'(t) \geq \theta F(t) \tag{11}$$

eventually for all t. Then it is easy to see that (11) implies (4). Indeed, the function

$$v\left(u\right)=\int_{\tau(u)}^{\tau(t)}F(s)ds-\theta\int_{u}^{t}F(s)ds,\quad \tau\left(t\right)\leq u\leq t,$$

satisfies the conditions

$$v\left(t\right)=0,$$

and

$$v'\left(u\right)=-F\left(\tau\left(u\right)\right)\tau'\left(u\right)+\theta F\left(u\right)\leq 0.$$

If $F\left(t\right)>0$ eventually for all t and

$$\liminf_{t\rightarrow\infty}\frac{F\left(\tau\left(t\right)\right)\tau^{'}\left(t\right)}{F\left(t\right)}=\theta_0>0,$$

then θ can be any number satisfying $0<\theta<\theta_0$.

Our main result can now be stated as follows:

Theorem 1. *Consider the differential equation* (1) , *Assume that* $l<1$, $0<k\leq\dfrac{1}{e}$ *and there exist* $\theta>0$ *such that* (4) *is satisfied. Assume that*

$$l>\frac{\ln\lambda_1+1}{\lambda_1}-\frac{N(1-k-\sqrt{(1-k)^2-4B})}{2} \tag{12}$$

where λ_1 *the smaller root of the equation* $\lambda=e^{k\lambda}$ *and* B *is given by* (6). *Then every solution of* (1) *oscillates.*

P r o o f. Assume, for the sake of contradiction, that $x(t)$ is an eventually positive solution of equation (1) .

Let σ be any number in $\left(\dfrac{1}{\lambda_1},1\right)$. From Lemma 1, there is a $T_1>t_0$ such that

$$\frac{x\left(\tau\left(t\right)\right)}{x\left(t\right)}>\sigma\lambda_1,\quad t\geq T_1, \tag{13}$$

and

$$\frac{x\left(t\right)}{x\left(\tau\left(t\right)\right)}>\sigma M,\quad t\geq T_1, \tag{14}$$

where

$$M = \lim_{t \to \infty} \inf \frac{x(t)}{x(\tau(t))}.$$

Now let $t \geq T_1$. Since the function $g(s) = \dfrac{x(\tau(t))}{x(s)}$ is continuous, $g(\tau(t)) = 1 < \sigma\lambda_1$ and $g(t) > \sigma\lambda_1$, there is a $t^*(t) \in (\tau(t), t)$ such that

$$\frac{x(\tau(t))}{x(t^*(t))} = \sigma\lambda_1.$$

Dividing (1) by $x(t)$, integrating from $\tau(t)$ to $t^*(t)$, and taking into account (13) yields

$$\int_{\tau(t)}^{t^*(t)} p(s)\,ds \leq -\frac{1}{\sigma\lambda_1}\int_{\tau(t)}^{t^*(t)} \frac{x'(s)}{x(s)}ds = \frac{\ln(\sigma\lambda_1)}{\sigma\lambda_1}. \tag{15}$$

Integrating (1) over $[t^*(t), t]$ and using (14) and the fact that $x(\tau(s)) \geq x(\tau(t))$ if $s \leq t$ yields

$$\begin{aligned} \int_{t^*(t)}^{t} p(s)\,ds &\leq \frac{x(t^*(t))}{x(\tau(t))} - \frac{x(t)}{x(\tau(t))} \\ &= \frac{1}{\sigma\lambda_1} - \frac{x(t)}{x(\tau(t))} \\ &\leq \frac{1}{\sigma\lambda_1} - \sigma M. \qquad (16) \end{aligned}$$

Adding (15) and (16) yields

$$\int_{\tau(t)}^{t} p(s)\,ds \leq \frac{\ln(\sigma\lambda_1) + 1}{\sigma\lambda_1} - \sigma M.$$

Letting $t \to \infty$ yields

$$l \leq \frac{\ln(\sigma\lambda_1) + 1}{\sigma\lambda_1} - \sigma M.$$

Letting $\sigma \to 1$ we obtain

$$l \leq \frac{\ln\lambda_1 + 1}{\lambda_1} - M.$$

The last inequality, in view of Lemma 2 contradicts (12). $\square$

Corollary 1. *Assume that the hypotheses of Theorem* 1 *hold with* $\theta = 1$ *and* $N = \dfrac{1}{e-2}$. *Then its conclusion holds with* (C_{13}).

P r o o f. When $\theta = 1$ and $N = \dfrac{1}{e-2}$, we get from (6) and (12) that

$$B = \frac{\lambda_1 - \lambda_1 k - 1}{\lambda_1^2},$$

and

$$\begin{aligned} l &> \frac{k\lambda_1 + 1}{\lambda_1} - \frac{\frac{1}{e-2}(1 - k - \sqrt{(1-k)^2 - \frac{4}{\lambda_1^2}(\lambda_1 - \lambda_1 k - 1)})}{2} \\ &= \frac{e-1}{e-2}\left(k + \frac{1}{\lambda_1}\right) - \frac{1}{e-2}, \end{aligned}$$

respectively. $\square$

Remark 2. Provided that $k = \dfrac{1}{e}$ one has $\lambda_1 = e$ and (C_{13}) leads to $l > 0.367879441 = \dfrac{1}{e}$.

Example 1. *Consider the delay differential equation*

$$\dot{x}(t) + px(t - a\sin^2\sqrt{t} - \frac{b}{p}) = 0, \tag{17}$$

where $p > 0$, $a > 0$, $b = \dfrac{1}{e}$, $pa = \dfrac{2}{5} - b$ *and by taking* $\mu(t) = \dfrac{1}{e-2}$, *Hence* $N = \dfrac{1}{e-2}$. *Then*

$$k = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \liminf_{t\to\infty} \int_{\tau(t)}^{t} pds = \liminf_{t\to\infty} p\left(a\sin^2\sqrt{t} + \frac{b}{p}\right) = \frac{1}{e},$$

and

$$\begin{aligned} l &= \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \limsup_{t\to\infty} \int_{\tau(t)}^{t} pds = \limsup_{t\to\infty} p\left(a\sin^2\sqrt{t} + \frac{b}{p}\right) \\ &= pa + b = \frac{2}{5} < 1. \end{aligned}$$

In the case that $k = \frac{1}{e}$, *then* $\lambda_1 = e$ *and we find*

$$\frac{e-1}{e-2}\left(k + \frac{1}{\lambda_1}\right) - \frac{1}{e-2} = \frac{1}{e} < \frac{2}{5} = l.$$

Then

$$l > \frac{e-1}{e-2}\left(k + \frac{1}{\lambda_1}\right) - \frac{1}{e-2}.$$

Hence, the conditions of Theorem 1 *and using the Corollary* 1 *are satisfied and therefore every solution of equation* (17) *oscillates. Observe that none of the results mentioned in the introduction can be applied to this equation.*

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