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VARIATIONAL PRINCIPLES FOR MONOTONE AND MAXIMAL BIFUNCTIONS

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Abstract. We establish variational principles for monotone and maximal bifunctions of Brøndsted-Rockafellar type by using our characterization of bifunction's maximality in reflexive Banach spaces. As applications, we give an existence result of saddle point for convex-concave function and solve an approximate inclusion governed by a maximal monotone operator.

1. Introduction. Given X a real Banach space with topological dual X^*, the Brøndsted-Rockafellar's principle ([2] and [5]) states that if ϕ is an extended proper convex lower semicontinuous function defined on X, with domain $\mathrm{dom}\phi$ and subdifferential $\partial\phi$, if $x \in X, x^* \in X^*$, $\alpha, \beta > 0$, and

$$\inf_{u\in\ \mathrm{dom}\phi}\{\phi(u) - \phi(x) + \langle x^*, x - u\rangle\} \geq -\alpha\beta, \tag{1.1}$$

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then there exists (y, y^*) in the graph of $\partial\phi$ (i.e. $y^* \in \partial\phi(y)$) such that $\|x - y\| \le \alpha$ and $\|x^* - y^*\| \le \beta$.
Torralba [8] generalized, in reflexive Banach space, this principle to the family of maximal monotone operators by stating that if $T : X \to 2^{X^*}$ is a maximal monotone operator with graph G(T), if $x \in X, x^* \in X^*$, $\alpha, \beta > 0$, and

$$\inf_{(u,u^*)\in \text{ G(T)}} \{\langle u^* - x^*, u - x\rangle\} \ge -\alpha\beta, \tag{1.2}$$

then there exists $(y, y^*) \in$ G(T) (i.e. $y^* \in T(y)$) such that $\|x - y\| \le \alpha$ and $\|x^* - y^*\| \le \beta$.
Note that in general Banach space, this result was established by Revalsky and Théra [6] for maximal monotone operators of type (D). By modifying the question slightly, Simons [7] obtains his statement for maximal monotone operators of type (ED).

In this paper, we establish the following variational principle of Brøndsted-Rockafellar type for monotone and maximal bifunctions:

Theorem 1.1. *Let X be a reflexive Banach space, X^* its topological dual, K be a closed convex subset of X and $f : K \times K \to \mathbb{R}$ be a monotone and maximal bifunction such that $f(x, .)$ is convex and lower semicontinuous and $f(x, x) = 0 \ \forall x \in K$. Then f satisfies the Brøndsted-Rockafellar's property (BR in brief) on K, i.e. for any $x \in K, x^* \in X^*$ and $\alpha, \beta > 0$ the following inequality*

$$\inf_{u\in K}\{f(x, u) + \langle x^*, x - u\rangle\} \ge -\alpha\beta, \tag{1.3}$$

implies that there exists $(y, y^*) \in X \times X^*$ such that
$\inf_{u\in K}\{f(y, u) + \langle y^*, y - u\rangle\} \ge 0$, *and* $\|y - x\| \le \alpha$, $\|y^* - x^*\| \le \beta$.

As corollary, we obtain a result (Corollary 2.3) of existence for a perturbed equilibrium problem without any hypothesis of compactness. By taken then particular bifunctions, we find Brøndsted-Rockafellar's principle for convex lower semicontinuous function, we give a result of existence of saddle point for perturbed convex-concave function (Remark 2.2) and we solve an approximate inclusion governed by a maximal monotone operator (see Remark 2.3).

2. Variational principles. We will need the following definition and two lemmas.

Definition 2.1. *Let $f : K \times K \to \mathbb{R}$ be a real bifunction.*

(i) f is said to be monotone if $f(x,y) + f(y,x) \leq 0$, for each $x, y \in K$.

(ii) f is said to be maximal if $(x, \zeta) \in K \times X^*$ and $f(x,u) \leq \langle -\zeta, u - x\rangle$ $\forall\, u \in K$ imply that $f(x,u) + \langle -\zeta, u - x\rangle \geq 0 \;\forall u \in K$.

We have to mention here that by taking $f(x,u) = sup_{\xi \in A(x)}\langle \xi, u - x\rangle$, Oettli-Riahi in [4] have established the relation between monotonicity and maximality of an operator A and those of the corresponding bifunction f.

Lemma 2.1 (Extended Ky Fan's Minimax inequality, see [3]**).** *Let X be a topological vector space, K a closed convex subset of X and $\varphi, \psi : K \times K \to \mathbb{R}$. Suppose that*

(a) for each $x, u \in K$ if $\psi(x,u) \leq 0$ then $\varphi(x,u) \leq 0$;

(b) for each $x \in K$ $\varphi(x, .)$ is lower semicontinuous on any compact subset of K;

(c) for every finite subset A of K and every $u \in conv\ A$ one has $\min_{x\in A} \psi(x,u) \leq 0$;

(d) (coercivity hypothesis) there exist a convex compact $C \subset K$ and $x_0 \in C$ such that $\forall u \in K\backslash C, \psi(x_0, u) > 0$.
Then, there exists $\overline{u} \in C$ such that $\varphi(x, \overline{u}) \leq 0$ for all $x \in K$.

In the sequel, without restriction, we suppose that the reflexive Banach space X with its dual are strictly convex. This implies that the duality mapping from X into X^* which is defined by

$$H(x) := \left\{x^* \in X^* / \ \|x^*\| = \|x\| \text{ and } \langle x, x^*\rangle = \|x\|^2\right\}$$

is one to one and strictly monotone, see Zeidler [9].

Lemma 2.2. *Suppose that K is closed convex and $f : K \times K \to \mathbb{R}$ is monotone and convex lower semicontinuous with respect to the second argument and $f(x,x) = 0\ \forall x \in K$. Then the following assertions are equivalent:*

(i) f is maximal;

(ii) $\forall x \in X, \forall \lambda > 0$, *there exists a unique solution* $z = J_\lambda^f(x) \in K$ *to the problem* $P(x, \lambda)$*:*

$$\lambda f(z,u) + \langle H(x-z), z-u\rangle \geq 0, \ \forall u \in K.$$

P r o o f. (ii) $\Rightarrow$ (i) Let $(x,\zeta) \in K \times X^*$ be such that $\forall u \in K \ f(u,x) \leq \langle -\zeta, u-x\rangle$. Setting $u = J_1^f(x+x_0)$, with $x_0 = H^{-1}(\zeta)$, in the equation above and $u = x$ in (ii), we have

$$f(J_1^f(x+x_0), x) \leq \left\langle -\zeta, J_1^f(x+x_0) - x \right\rangle \tag{2.1}$$

and

$$f(J_1^f(x+x_0), x) + \left\langle H\left(x + x_0 - J_1^f(x+x_0)\right), J_1^f(x+x_0) - x\right\rangle \geq 0. \tag{2.2}$$

Adding (2.1) to (2.2), it follows that

$$\left\langle H(x - J_1^f(x+x_0) + x_0) - H(x_0), \left(x - J_1^f(x+x_0) + x_0\right) - x_0 \right\rangle \leq 0.$$

From the strict monotonicity of H we deduce that $x - J_1^f(x+x_0) + x_0 = x_0$, and thus $x = J_1^f(x+x_0)$. Using (ii) we deduce that $f(x,u) + \langle -\zeta, u-x\rangle \geq 0 \ \ \forall u \in K$, which means that f is maximal.

(i) $\Rightarrow$ (ii) Fix $\lambda > 0$ and $x \in K$. We shall verify the assumptions of Lemma 2.1 for $\varphi(z,u) = \lambda f(z,u) - \langle H(u-x), z-u\rangle$ and $\psi(z,u) = -\lambda f(u,z) - \langle H(u-x), z-u\rangle$, when X is endowed with the weak topology.
Assumptions (a) and (b) are immediate, and (c) comes from the convexity of the set $\{x \in K : \psi(x,u) > 0\}$, which follows from the convexity of $f(u,.)$.
For (d), let us consider $B = \{v \in K : \|v - x\| \leq R_1\}$ where R_1 is a sufficiently large positive real number for which B is nonempty. As $f(x,.)$ is convex lower semicontinuous and B is weakly compact, there exists $\alpha_0 \in \mathbb{R}$ such that $f(x,u) \geq \alpha_0$ for all $u \in B$.
Let $u \in K\backslash B$, since $f(x,x) = 0$ and $f(x,.)$ is convex, it follows that

$$\alpha_0 \leq f\left(x, \frac{R_1}{\|x-u\|}u + (1 - \frac{R_1}{\|x-u\|})x\right) \leq \frac{R_1}{\|x-u\|} f(x,u).$$

Using f monotone we conclude that $f(u,x) \le -(\alpha_0/R_1)\,\|x-u\|$, and thus

$$\psi(x,u) \ge \lambda \frac{\alpha_0}{R_1}\,\|x-u\| + \|x-u\|^2 .$$

Then for some $R_2 > R_1$, the assumption (d) is satisfied by taking $C = \{u \in K : \|x-u\| \le R_2\}$.
According to Lemma 2.1, there exists $x_\lambda := J_\lambda^f(x)$ such that $\varphi(u,x_\lambda) \le 0 \ \forall u \in K$. By maximality of f, $J_\lambda^f x$ becomes a solution of $(EP)_\lambda$. The uniqueness of $J_\lambda^f x$ comes from the strict monotonicity of H. □

Let us now prove Theorem 1.1.

P r o o f. For $(x,x^*) \in K \times X^*$ satisfying relation (1.3), we set $g(x,u) = f(x,u) + \langle x^*, x-u\rangle$ for $u \in K$. According to Lemma 2.2 applied to g for $\lambda = \alpha/\beta$, there exists $y \in K$ such that $\forall u \in K$

$$\lambda g(y,u) + \langle H(x-y), y-u\rangle \ge 0.$$

Taking $u = x$ we have

$$f(y,x) + \left\langle x^* - \frac{1}{\lambda} H(y-x), y-x \right\rangle \ge 0. \tag{2.3}$$

On the other hand, according to (1.3), one has

$$f(x,y) + \langle x^*, x-y\rangle \ge -\alpha\beta. \tag{2.4}$$

Summing (2.3) and (2.4) and using monotonicity of f, it follows

$$-\frac{1}{\lambda}\|y-x\|^2 = \left\langle -\frac{1}{\lambda} H(y-x), y-x \right\rangle \ge -\alpha\beta$$

which implies that $\|y-x\| \le \alpha$. Setting $y^* = x^* - \frac{1}{\lambda}H(y-x)$, we conclude $\|y^*-x^*\| = \frac{1}{\lambda}\|H(y-x)\| = \frac{\beta}{\alpha}\|y-x\| \le \beta$, and thus (y,y^*) is the desired pair in $K \times X^*$. □

Corollary 2.3. *Under the hypotheses of Theorem* 1.1*, for each* $\varepsilon > 0$ *and* $x \in K$ *such that* $f(x,u) \ge -\varepsilon \ \forall u \in K$*, there exists* $y \in K$ *such that* $\|y-x\| \le \sqrt{\varepsilon}$ *and* $f(y,u) + \sqrt{\varepsilon}\|y-u\| \ge 0 \ \forall u \in K$.

P r o o f. Since the pair $(x, 0) \in K \times X^*$ is assumed to verify (1.3) with $\alpha = \beta = \sqrt{\varepsilon}$, Theorem 1.1 asserts the existence of $(y, y^*) \in K \times X^*$ such that $\|y - x\| \leq \sqrt{\varepsilon}$, $\|y^*\| \leq \sqrt{\varepsilon}$ and $f(y, u) + \langle y^*, y - u\rangle \geq 0$ $\forall u \in K$, which means that

$$f(y, u) + \sqrt{\varepsilon}\|y - u\| \geq 0 \qquad \forall u \in K. \qquad \square$$

Remark 2.1. Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous function which domain contains K and let $\alpha, \beta > 0$, $x \in K$ and $x^* \in X^*$. If we suppose that

$\varphi(u) - \varphi(x) + \langle x^*, x - u\rangle \geq -\alpha\beta$ $\forall u \in K$ (in other words $x^* \in \partial_{\alpha\beta} (\varphi + \delta_K) (x)$),

there exist $y \in K$, $y^* \in X^*$ such that $\|y - x\| \leq \alpha$, $\|y^* - x^*\| \leq \beta$ and $y^* \in \partial (\varphi + \delta_K) (y)$.

To prove this assertion it suffices to apply Theorem 1.1 to $f(x, u) = \varphi(u) - \varphi(x)$. Note that this result is precisely the variational principle of Brøndsted-Rockafellar for convex lower semicontinuous functions, see [2] and [5].

Remark 2.2. Let X_1, X_2 be reflexive Banach spaces, K_i a closed convex subset of X_i for $i = 1, 2$ and $\psi : K_1 \times K_2 \to \mathbb{R}$ be such that $\psi(x_1, .)$ is concave upper semicontinuous for each fixed $x_1 \in K_1$ and $\psi(., x_2)$ is convex lower semicontinuous for each fixed $x_2 \in K_2$. Setting $X = X_1 \times X_2$, endowed with the norm $\|(x_1, x_2)\| = \|x_1\| + \|x_2\|$, and $K = K_1 \times K_2$, and consider $\varepsilon > 0$ and $(x_1, x_2) \in K$ such that $\psi(u_1, x_2) - \psi(x_1, u_2) \geq -\varepsilon$ for all $(u_1, u_2) \in K$. Then there exists $(y_1, y_2) \in K$ such that $\|y_1 - x_1\| + \|y_2 - x_2\| \leq \sqrt{\varepsilon}$ and (y_1, y_2) is a saddle point of the function $\psi_\varepsilon(u_1, u_2) = \psi(u_1, u_2) + \sqrt{\varepsilon}\,\|y_1 - u_1\| - \sqrt{\varepsilon}\,\|y_2 - u_2\|$. It suffices to apply Corollary 2.3 to $f((x_1, x_2), (u_1, u_2)) := \psi(u_1, x_2) - \psi(x_1, u_2)$. One then obtain that

$$\psi_\varepsilon(u_1, y_2) \geq \psi_\varepsilon(y_1, y_2) = \psi(y_1, y_2) \geq \psi_\varepsilon(y_1, u_2) \qquad \forall (u_1, u_2) \in K.$$

Remark 2.3. Let $T : X \to X^*$ be a maximal monotone operator and $K \subset dom\, T$ be a closed convex subset of X. If we suppose that, for some $\varepsilon > 0$ and $x \in K$, we have $\langle Tx, u - x\rangle \geq -\varepsilon$ $\quad \forall u \in K$, then there exists $y \in K$ such that $\|y - x\| \leq \sqrt{\varepsilon}$ and

$$0 \in Ty + \sqrt{\varepsilon}B^* + N_K(y),$$

where B^* is the unit ball of X^* and $N_K(y) := \{y^* \in X^* : \langle y^*, u - y\rangle \leq 0\, \forall u \in K\}$ is the normal cone to K.

Indeed, if we apply Corollary 2.3 to $f(x, u) = \langle Tx, u - x\rangle$, we obtain the existence of $y \in K$ such that

$$\langle Ty, u - y\rangle + \sqrt{\varepsilon}\, \|y - u\| \geq 0 \ \ \forall u \in K$$

which is equivalent to

$$-Ty \in \partial \left(\sqrt{\varepsilon}\, \|y - \cdot\| + \delta_K\right)(y).$$

The result follows by remarking that $\partial \left(\sqrt{\varepsilon}\, \|y - \cdot\| + \delta_K\right)(y) = \sqrt{\varepsilon}B^* + N_K(y)$.

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