Serdica

Mathematical Journal

Сердика

Математическо списание

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **29** (2003), 45-54

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ACCELERATION OF CONVERGENCE IN DONTCHEV'S ITERATIVE METHOD FOR SOLVING VARIATIONAL INCLUSIONS

M. Geoffroy, S. Hilout, A. Pietrus

Communicated by V. Drensky

ABSTRACT. In this paper we investigate the existence of a sequence (x_k) satisfying $0 \in f(x_k)+\nabla f(x_k)(x_{k+1}-x_k)+\frac{1}{2}\nabla^2 f(x_k)(x_{k+1}-x_k)^2+G(x_{k+1})$ and converging to a solution x^* of the generalized equation $0 \in f(x)+G(x)$; where f is a function and G is a set-valued map acting in Banach spaces. We show that the previous sequence is locally cubic convergent to x^* whenever the set-valued map $[f(x^*)+\nabla f(x^*)(\cdot - x^*)+\frac{1}{2}\nabla^2 f(x^*)(\cdot - x^*)^2+G(\cdot)]^{-1}$ is M-pseudo-Lipschitz around $(0, x^*)$.

1. Introduction.

Throughout this paper X and Y are two real or complex Banach spaces and we consider a generalized equation of the form

$$0 \in f(x) + G(x) \tag{1}$$

2000 *Mathematics Subject Classification:* 47H04, 65K10.
Key words: Multiapplication, Aubin Continuity, Cubic Convergence.

where f is a function from X into Y an G is a set-valued map from X to the subsets of Y.
When $G = \partial\psi_C$ is the subdifferential of the function

$$\psi_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise,} \end{cases}$$

(1) has been studied by Robinson [10]. The key of his idea is to associate to (1) a linearized equation. His study concerns especially the stability of solutions of some minimization problems.

When ∇f is locally Lipschitz Dontchev [4] associates to (1) a Newton-type method based on a partial linearization which provides a local quadratic convergence. Following his work, Pietrus [9] obtains a Newton-type sequence which converges whenever ∇f satisfies a Hölder-type condition.

In this paper we associate to (1) the relation

$$0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + \frac{1}{2}\nabla^2 f(x_k)(x_{k+1} - x_k)^2 + G(x_{k+1}), \tag{2}$$

where $\nabla f(x)$ and $\nabla^2 f(x)$ denote respectivly the first and the second Fréchet derivative of f at x. One can note that if $x_k \longrightarrow x^*$, then x^* is a solution of (1). Let us mention that relation (2) derives from a second-degree Taylor polynomial expansion of f at x_k and that such an approximation is an extension of Dontchev's original work [3].

The paper is organized as follows: in section 2 we recall a few preliminary results and make some fundamental assumptions on f. Then, in section 3 we prove the existence of a sequence (x_k) satisfying (2) and we show that it is locally cubic convergent.

2. Preliminaries and fundamental assumptions.

Definition 2.1. *A set-valued map $\Gamma : X \longrightarrow Y$ is said to be M-pseudo-lipschitz around $(x_0, y_0) \in \operatorname{graph}\Gamma := \{(x, y) \in X \times Y \mid y \in \Gamma(x)\}$ if there exist neighbourhoods V of x_0 and U of y_0 such that*

$$\sup_{y \in \Gamma(x_1) \cap U} dist(y, \Gamma(x_2)) \leq M \parallel x_1 - x_2 \parallel, \forall x_1, x_2 \in V. \tag{3}$$

When a multiapplication Γ is M-pseudo-Lipschitz, the constant M is called the modulus of Aubin continuity.

The Aubin continuity of Γ is equivalent to the openess with linear rate of Γ^{-1} (the covering property) and to the metric regularity of Γ^{-1} (a basic well-posedness property in optimization).

Finally, when f is a function which is strictly differentiable at some x_0, then the Aubin continuity of f^{-1} around $(f(x_0), x_0)$ is equivalent to the surjectivity of $\nabla f(x_0)$. For more details, the reader can refer to [1, 2, 8, 11, 12].

Let A and C be two subsets of X, we recall that the excess e from the set A to the set C is given by $e(C, A) = \sup_{x \in C} dist(x, A)$.

Then, we have an equivalent definition of M-pseudo-Lipschitzness in terms of excess by replacing (3) by

$$e(\Gamma(x_1) \cap U, \Gamma(x_2)) \leq M \parallel x_1 - x_2 \parallel, \forall x_1, x_2 \in V, \tag{4}$$

in the previous definition. In [6] the above property is called Aubin property and in [5] it has been used to study the problem of the inverse for set-valued maps. In the sequel, we will need the following fixed point statement which has been proved in [5].

Lemma 2.1. *Let (X, ρ) be a complete metric space, let ϕ a map from X into the closed subsets of X, let $\eta_0 \in X$ and let r and λ be such that $0 \leq \lambda < 1$ and*
a) dist $(\eta_0, \phi(\eta_0)) \leq r(1 - \lambda)$,
b) $e(\phi(x_1) \cap B_r(\eta_0), \phi(x_2)) \leq \lambda\ \rho(x_1, x_2)\ \forall x_1, x_2 \in B_r(\eta_0)$,
then ϕ has a fixed point in $B_r(\eta_0)$. That is, there exists $x \in B_r(\eta_0)$ such that $x \in \phi(x)$. If ϕ is single-valued, then x is the unique fixed point of ϕ in $B_r(\eta_0)$.

The previous lemma is a generalization of a fixed-point theorem in [7], where in (b) the excess e is replaced by the Haussdorff distance.
We suppose that $x^* \in X$ is a solution of equation (1). Before studying our problem, we make the following assumptions:

(H0) *G has closed graph;*

(H1) *f is Fréchet differentiable on some neighborhood V of x^*;*

(H2) *$\nabla^2 f$ is Lipschitz on V with constant L;*

(H3) *For all $y \in V$, the application*

$$[f(x^*) + \nabla f(x^*)(\cdot - x^*) + \frac{1}{2}\nabla^2 f(x^*)(\cdot - x^*)^2 + G(\cdot)]^{-1},$$

is M-pseudo-Lipschitz around $(0, x^*)$.

3. Convergence analysis. The main theorem of this study reads as follows:

Theorem 3.1. *Let x^* be a solution of (1), if we suppose that assumptions* **(H0)**-**(H3)** *are satisfied, then for every $C > \frac{ML}{6}$ one can find $\delta > 0$ such that for every starting point $x_0 \in B_\delta(x^*)$, there exists a sequence (x_k) for (1), defined by (2), which satisfies*

$$\| x_{k+1} - x^* \| \leq C \| x_k - x^* \|^3 . \tag{5}$$

In other words, (2) generates (x_k) with cubic order.

Before proving Theorem 3.1, we need to introduce a few notation. First, for $k \in \mathbb{N}$ and $x_k \in X$ we define the set-valued map Q from X to the subsets of Y by

$$Q(x) = f(x^*) + \nabla f(x^*)(x - x^*) + \frac{1}{2}\nabla^2 f(x^*)(x - x^*)^2 + G(x).$$

Then we set

$$Z_k(x) := f(x^*) + \nabla f(x^*)(x - x^*) + \frac{1}{2}\nabla^2 f(x^*)(x - x^*)^2 - f(x_k) - \nabla f(x_k)(x - x_k) - \frac{1}{2}\nabla^2 f(x_k)(x - x_k)^2.$$

Finally, we define the set-valued map $\phi_k : X \to X$ by

$$\phi_k(x) = Q^{-1}[Z_k(x)].$$

One can note that x_1 is a fixed point of ϕ_0 if and only if the following holds:

$$f(x^*) + \nabla f(x^*)(x_1 - x^*) + \frac{1}{2}\nabla^2 f(x^*)(x_1 - x^*)^2 - f(x_0) - \nabla f(x_0)(x_1 - x_0) - \frac{1}{2}\nabla^2 f(x_0)(x_1 - x_0)^2 \in Q(x_1).$$

Thus, it is easy to see that the previous assertion is equivalent to

$$0 \in f(x_0) + \nabla f(x_0)(x_1 - x_0) + \frac{1}{2}\nabla^2 f(x_0)(x_1 - x_0)^2 + G(x_1). \tag{6}$$

Once x_k is computed, we show that the function ϕ_k has a fixed point x_{k+1} in X. This process allows us to prove the existence of a sequence (x_k) satisfying (2).

Now, we state a result which is the starting point of our algorithm. It will be very usefull to prove Theorem 3.1 and reads as follows:

Proposition 3.1. *Under the hypotheses of Theorem* 3.1*, there exists* $\delta > 0$ *such that for all* $x_0 \in B_\delta(x^*)$ $(x_0 \neq x^*)$*, the map* ϕ_0 *has a fixed point* x_1 *in* $B_\delta(x^*)$ *satisfying* $\|x_1 - x^*\| \leq C\|x_0 - x^*\|^3$.

P r o o f. By hypothesis **(H3)** there exist positive numbers a and b such that

$$e(Q^{-1}(y') \cap B_a(x^*), Q^{-1}(y'')) \leq M \parallel y' - y'' \parallel, \ \forall y', y'' \in B_b(0). \tag{7}$$

Fix $\delta > 0$ such that

$$\delta < \min\left\{a, \left(\frac{2b}{3L}\right)^{\frac{1}{3}}, \frac{1}{\sqrt{C}}\right\}. \tag{8}$$

To prove Proposition 3.1 we intend to show that both assertions (a) and (b) of Lemma 2.1 hold; where $\eta_0 := x^*$, ϕ is the function ϕ_0 defined at the very begining of this section and where r and λ are numbers to be set.

According to the definition of the excess e, we have

$$\text{dist } (x^*, \phi_0(x^*)) \leq e\Big(Q^{-1}(0) \cap B_\delta(x^*), \phi_0(x^*)\Big). \tag{9}$$

Moreover, for all $x_0 \in B_\delta(x^*)$ such that $x_0 \neq x^*$ we have

$\|Z_0(x^*)\| = \|f(x^*) - f(x_0) - \nabla f(x_0)(x^* - x_0) - \frac{1}{2}\nabla^2 f(x_0)(x^* - x_0)^2\|$, so

$\|Z_0(x^*)\| \leq \frac{L}{6}\|x^* - x_0\|^3$.

Then (8) yields, $\|Z_0(x^*)\| < b$. Hence from (7) one has

$$e\Big(Q^{-1}(0)\cap B_\delta(x^*),\phi_0(x^*)\Big) = e\Big(Q^{-1}(0)\cap B_\delta(x^*),Q^{-1}[Z_0(x^*)]\Big) \leq \frac{ML}{6}\|x^*-x_0\|^3.$$

By (9), we get

$$\text{dist } (x^*,\phi_0(x^*)) \leq \frac{ML}{6}\|x^*-x_0\|^3. \tag{10}$$

Since $C > \frac{ML}{6}$ there exists $\lambda \in]0,1[$ such that $C(1-\lambda) \geq \frac{ML}{6}$. Hence,

$$\text{dist } (x^*,\phi_0(x^*)) \leq C(1-\lambda)\|x^*-x_0\|^3. \tag{11}$$

By setting $\eta_0 := x^*$ and $r := r_0 = C\|x^*-x_0\|^3$ we can deduce from the last inequalities that assertion (a) in Lemma 2.1 is satisfied.

Now, we show that condition (b) of lemma 2.1 is satisfied. Since $\frac{1}{\sqrt{C}} \geq \delta$ and $\|x^*-x_0\| \leq \delta$, we have $r_0 \leq \delta \leq a$.

Moreover for $x \in B_\delta(x^*)$,

$$\begin{aligned}\|Z_0(x)\| &\leq \|f(x^*)-f(x)-\nabla f(x^*)(x-x^*)-\frac{1}{2}\nabla^2 f(x^*)(x-x^*)^2\| \\ &\quad + \|f(x)-f(x_0)-\nabla f(x_0)(x-x_0)-\frac{1}{2}\nabla^2 f(x_0)(x-x_0)^2\| \\ &\leq \frac{L}{6}\|x-x^*\|^3+\frac{L}{6}\|x-x_0\|^3 \\ &\leq \frac{3L}{2}\delta^3.\end{aligned}$$

Then by (8) we deduce that for all $x \in B_\delta(x^*)$, $Z_0(x) \in B_b(0)$. Then it follows that for all $x', x'' \in B_{r_0}(x^*)$, we have

$e(\phi_0(x') \cap B_{r_0}(x^*),\phi_0(x'')) \leq e(\phi_0(x') \cap B_\delta(x^*),\phi_0(x''))$, which yields by (7):

$$\begin{aligned}e(\phi_0(x') \cap B_{r_0}(x^*),\phi_0(x'')) &\leq M\|Z_0(x')-Z_0(x'')\| \\ &\leq M\|\nabla f(x^*)(x'-x'')-\nabla f(x_0)(x'-x'')\end{aligned}$$

$$+\frac{1}{2}\nabla^2 f(x^*)(x'-x^*)^2 - \frac{1}{2}\nabla^2 f(x^*)(x''-x^*)^2$$

$$+\frac{1}{2}\nabla^2 f(x_0)(x''-x_0)^2 - \frac{1}{2}\nabla^2 f(x_0)(x'-x_0)^2\|$$

$$\leq M\|\nabla f(x^*)(x'-x'') - \nabla f(x_0)(x'-x'')$$

$$+\frac{1}{2}\nabla^2 f(x^*)(x'-x''+x''-x^*)^2 - \frac{1}{2}\nabla^2 f(x^*)(x''-x^*)^2$$

$$+\frac{1}{2}\nabla^2 f(x_0)(x''-x_0)^2 - \frac{1}{2}\nabla^2 f(x_0)(x'-x''+x''-x_0)^2\|.$$

Assumption **(H2)** ensures the existence of $L_1 > 0$ such that $\|\nabla^2 f\| \leq L_1$ on $B_\delta(x^*)$. Then an easy computation yields:

$$e(\phi_0(x') \cap B_{r_0}(x^*), \phi_0(x'')) \leq 5ML_1\delta\|x'-x''\|. \tag{12}$$

Without loss of generality we may assume that $\delta < \dfrac{\lambda}{5ML_1}$ thus condition (b) of Lemma 2.1 is satisfied. Since both conditions of Lemma 2.1 are fulfilled, we can deduce the existence of a fixed point $x_1 \in B_{r_0}(x^*)$ for the map ϕ_0. Then the proof of Proposition 3.1 is complete. □

Now that we proved Proposition 3.1, the proof of Theorem 3.1 is straightforward as it is shown below.

Proof of Theorem 3.1. Proceeding by induction, keeping $\eta_0 = x^*$ and setting $r_k = C\|x_k - x^*\|^3$, the application of proposition 3.1 to the map ϕ_k gives the existence of a fixed point x_{k+1} for ϕ_k, which is an element of $B_{r_k}(x^*)$. This last fact implies that :

$$\| x_{k+1} - x^* \| \leq C \| x_k - x^* \|^3 . \tag{13}$$

In others words, (2) generates a sequence (x_k) with cubic order and the proof of theorem 3.1 is complete. □

Corollary 3.1. *Let x^* be an isolated solution of* (1)*, if assumptions* **(H0)**-**(H3)** *are satisfied, then for every $C > \dfrac{ML}{6}$ one can find $\delta > 0$ such that any sequence (x_k) generated by* (2) *with $x_k \in B_\delta(x^*)$ satisfies* (5).

P r o o f. As we recalled it in the proof of Proposition 3.1, there exists $L_1 > 0$ such that $\|\nabla^2 f(x)\| \leq L_1$. Then, we fix δ satisfying both relation (8) and the following:

$$\delta < \min\left\{\frac{1}{3ML_1}, \frac{6C - ML}{18CML_1}\right\}. \tag{14}$$

Without loss of generality we may assume that the solution of (1) is unique in $B_{4\delta}(x^*)$. Let (x_k) be a sequence generated by (2) with $x_k \in B_\delta(x^*)$, then x^* is the only point in $B_{4\delta}(x^*)$ satisfying (1), i.e., $x^* = Q^{-1}(0) \cap B_{4\delta}(x^*)$. Moreover, for all $k \in \mathbb{N}$, by Theorem 3.1 we have:

$$x_{k+1} \in Q^{-1}[Z_k(x_{k+1})].$$

Hence,

$$\|x_{k+1} - x^*\| = \text{dist } (x_{k+1}, Q^{-1}(0)) \text{ then,}$$

$$\|x_{k+1} - x^*\| \leq e\Big(Q^{-1}[Z_k(x_{k+1})] \cap B_\delta(x^*), Q^{-1}(0)\Big),$$

$$\|x_{k+1} - x^*\| \leq M\|Z_k(x_{k+1})\|,$$

$$\|x_{k+1} - x^*\| \leq M\|f(x^*) + \nabla f(x^*)(x_{k+1} - x^*) + \frac{1}{2}\nabla^2 f(x^*)(x_{k+1} - x^*)^2$$
$$-f(x_k) - \nabla f(x_k)(x_{k+1} - x_k) - \frac{1}{2}\nabla^2 f(x_k)(x_{k+1} - x_k)^2\|.$$

Then, an easy computation shows that

$$\|x_{k+1} - x^*\| \leq M\left(\frac{L}{6}\|x^* - x_k\|^3 + 3L_1\delta\|x_{k+1} - x^*\|\right).$$

Thus,

$$\|x_{k+1} - x^*\| \leq \frac{ML}{6(1 - 3ML_1\delta)}\,\|x_k - x^*\|^3.$$

Thanks to (14), we have $C > \dfrac{ML}{6(1 - 3ML_1\delta)}$ so $\|x_{k+1} - x^*\| \leq C\,\|x_k - x^*\|^3$ and then the proof is complete. □

Acknowledgement. The authors thank the referee for his valuable remarks and comments on this work.

REFERENCES

[1] J. P. Aubin. Lipschitz behavior of solutions to convex minimization problems. *Math. Oper. Res.* **9** (1984) 87–111.

[2] J. P Aubin, H. Frankowska. Set-valued Analysis. Birkhäuser, Boston, 1990.

[3] A. L. Dontchev. Local analysis of a Newton-type method based on partial linearization. In: The mathematics of numerical analysis (Eds Renegar, James et al.) 1995 AMS-SIAM summer seminar in applied mathematics, Providence, RI: AMS. Lect. Appl. Math. vol. **32** (1996), 295–306.

[4] A. L. Dontchev. Local convergence of the Newton method for generalized equation, *C. R. Acad. Sci. Paris Ser. I Math.* **322**, Serie I, (1996), 327–331.

[5] A. L. Dontchev, W. W. Hager. An inverse function theorem for set-valued maps. *Proc. Amer. Math. Soc.* **121** (1994), 481–489.

[6] A. L. Dontchev, R. T. Rockafellar. Characterizations of strong regularity for variational inequalities over polyhedral convex sets. *SIAM J. Optim.* **6**, *4* (1996), 1087–1105.

[7] A. D. Ioffe, V. M. Tikhomirov. Theory of Extremal Problems. North Holland, Amsterdam, 1979.

[8] B. S. Mordukhovich. Complete characterization of openess metric regularity and Lipschitzian properties of multifunctions. *Trans. Amer. Math. Soc* **340** (1993), 1–36.

[9] A. Pietrus. Generalized equation under mild differentiability conditions. *Rev. R. Acad. Cienc. Exactas Fis. Nat. (Esp.)* 94, **(1)** (2000), 15–18.

[10] S. M. Robinson. Strong regular generalized equations. *Math. of Oper. Res.* **5** (1980), 43–62.

[11] R. T. Rockafellar. Lipschitzian properties of multifunctions. *Nonlinear Anal.* **9** (1984), 867–885.

[12] R. T. Rockafellar, R. Wets. Variational Analysis. A Series of comprehensive studies in mathematics, Springer, vol. **317**, 1998.

M. Geoffroy
A. Pietrus
Laboratoire Analyse, Optimisation, Contrôle
Université des Antilles et de la Guyane
Département de Mathématiques
et Informatique
Campus de Fouillole
F-97159 Pointe-à-Pitre
France
e-mail: michel.geoffroy@univ-ag.fr
e-mail: alain.pietrus@univ-ag.fr

S. Hilout
Département de Mathématiques
Appliquées et Informatique
Faculté des Sciences et Techniques
B.P. 523, Beni-Mellal
Maroc

Received August 13, 2002
Revised January 28, 2003