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A NOTE ON UNIVALENT FUNCTIONS WITH FINITELY MANY COEFFICIENTS

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Abstract

The main object of this article is to introduce sufficient conditions of univalency for a class of analytic functions with finitely many coefficients defined by approximate functions due to Suffridge on the unit disk of the complex plane whose image is saddle-shaped. Sandwich theorem is also discussed.

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1. Introduction

In the theory of univalent function, it is known that Riemann mapping theorem plays an important role. It shows the existence of a unique conformal univalent map f of the open unit disk $U := \{z : |z| < 1\}$ onto each simply connected domain $\mathcal{G}$ such that $f(z_0) = g_0$ and $f'(z_0) > 0$. If the boundary of $\mathcal{G}$ is piecewise analytic and g_1 is a point on the boundary of $\mathcal{G}$, then the uniqueness assertion of the Riemann mapping theorem can be reformulated alternately as the statement that there exists a unique conformal mapping f of U onto $\mathcal{G}$ such that $f(z_0) = g_0$ and $f(1) = g_1$.

One of the major branches of complex analysis is univalent function theory: the study of one-to-one analytic functions f of the unit disk U normalized to Taylor series

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots ,$$

and all this class of function denoted by $\mathcal{A}$. Many papers and books have been written about the properties of the class $\mathcal{S}$ of such functions. An important result in this area is Bieberbach's Conjecture (1916), then famously known as de Branges Theorem (1985): for any $f \in \mathcal{S}$, the Taylor coefficients satisfy $|a_n| \le n$ (see [1]-[3]). The well known result due to Nevanlinna (1921) stated that if f is holomorphic in $|z| < 1$ and satisfies $f(0) = 0, f'(0) \neq 0$, then f is univalent and maps the unit disk onto a starlike domain (with respect to 0) if and only if $Re[zf'(z)/f(z)] > 0$ everywhere. Later, Wald (1978) gave a characterization of those functions which are starlike with respect to another center. Observe that although the classes of starlike, spirallike and convex functions were studied very extensively, little was known about functions that are holomorphic on the unit disk U and starlike with respect to a boundary point. It was first known that in 1981 Robertson [4] introduced two classes of univalent functions and conjectured that they coincide. In (1984) his conjecture was proved by Lyzzaik [5]. Finally, in (1990) Silverman and Silvia [6], used similar method, and gave a full description of the class of univalent functions on U, the image of which is starlike with respect to a boundary point.

The Koebe function $k(z) = z/(1-z)^2$ is extremal for a variety of problems for univalent functions and a sequence of polynomials constructed by Suffridge [7] provides a good approximation to $k(z)$. Suffridge defined and studied the classes of univalent polynomials

$$S_m(j;z) = z + \sum_{n=2}^{m} \frac{m-n+1}{m} \frac{\sin nj\pi/(m+1)}{\sin j\pi/(m+1)} z^n, \;\; j \in \mathcal{N},$$

establishing various extremal properties. It is interesting that $S^m(1;z)$ is the desired approximation to $k(z)$.

Consider the subclass $\mathcal{A}(e_m)$ of the class $\mathcal{A}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{m} \frac{m-n+1}{m} \frac{\sin nj\pi/(m+1)}{\sin j\pi/(m+1)} e_n z^n + \sum_{k=m+1}^{\infty} a_k z^k, \qquad (1)$$

where

$$e_n := \frac{m}{m-n+1} \frac{\sin j\pi/(m+1)}{\sin nj\pi/(m+1)} a_n.$$

Also we consider the subclass $\mathcal{T}(e_m)$ of the class $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{m} \frac{m-n+1}{m} \frac{\sin nj\pi/(m+1)}{\sin j\pi/(m+1)} e_n z^n - \sum_{k=m+1}^{\infty} a_k z^k. \tag{2}$$

In this work, we introduce the following definition:

Definition 1.1. For functions $f(z) \in \mathcal{A}$ are called in the class saddle-like, denoted by $\mathcal{SD}$, if satisfy:

$$\Re \left\{ e^{a^2+ib} \frac{zf'(z)}{f(z)} \right\} > 0,$$

where

$$\frac{-2}{\pi} \le a \le \frac{2}{\pi},\ -\frac{\pi}{2} < b < \frac{\pi}{2},\ z \in U.$$

Remark 1.1.

(i) If $a = b = 0$, then we obtain the starlike subclass.
(ii) If $a = 0$, then Definition 1.1 reduces to the class of spiral-like.

Definition 1.2. For functions $f(z) \in \mathcal{A}(e_m)$ we define the subclass $\mathcal{SD}(e_m)$, if they satisfy:

$$\Re \left\{ e^{a^2+ib} \frac{zf'(z)}{f(z)} \right\} > 0,$$

where

$$\frac{-2}{\pi} \le a \le \frac{2}{\pi},\ -\frac{\pi}{2} < b < \frac{\pi}{2},\ z \in U.$$

Our aim is to introduce the sufficient and necessary conditions for functions belong to the class $\mathcal{SD}(e_m)$. For this purpose we need to the following preliminaries.

Let $\phi : \mathcal{C}^2 \to \mathcal{C}$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\phi(p(z)), zp'(z)) \prec h(z)$, then p is called a solution of the differential subordination. The univalent function

q is called a dominant of the solutions of the differential subordination, $p \prec q$. If p and $\phi(p(z)), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \phi(p(z)), zp'(z))$, then p is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$.

LEMMA 1.1. ([8]) *Let q be convex (univalent) in U and R, S be analytic in $\mathcal{C}$ and T analytic in a domain $D \supset q(U)$. Suppose that*

$$(a)\ \Re\{\frac{S((1+t)zq'(z))T'(q(z))}{R'((1+t)zq'(z)) + S'((1+t)zq'(z))T(q(z))}\} \geq 0,\ \forall z \in U, \text{ and } t \geq 0.$$

(b) $Q(z) = zq'(z)R'(zq'(z)) + S'(zq'(z))T(q(z))$ is starlike (univalent) in U.

If p is analytic in U with $p(0) = q(0), p(U) \subset D$ and

$$R(zp'(z)) + S(zp'(z))T(p(z)) \prec R(zq'(z)) + S(zq'(z))T(q(z)),$$

then $p(z) \prec q(z)$.

LEMMA 1.2. [9] *Let ϕ be convex univalent in U and ω analytic in U with $\Re\{\omega\} \geq 0$. If p is analytic in U with $p(0) = \phi(0)$, then*

$$p(z) + \omega(z)zp'(z) \prec \phi(z)$$

implies that

$$p(z) \prec \phi(z),\ \ z \in U.$$

LEMMA 1.3. ([10]) *Let P and p be analytic in U with $p(0) = P(0) = 1$ and satisfy*

$$P(z) \prec 1 + \frac{\lambda\mu z}{\mu + \alpha},\ \ z \in U,$$

and

$$P(z)[1 - \alpha + \alpha(\nu + (1 - \nu)p(z))] \prec 1 + \lambda z,\ \ z \in U, \nu < 1,$$

then $\Re\{p(z)\} > 0$ in U, where

$$|\lambda| \leq (\mu + \alpha)\sqrt{[2\alpha(1 - \nu) - 1]/[\alpha^2 + 2\alpha(\mu + (1 - \nu)\mu^2)]},$$

with $\alpha > 1/[2(1 - \nu)]$.

2. Sufficient and necessary conditions

The object of this section is to pose the sufficient conditions for functions in the class $\mathcal{SD}(e_m)$.

THEOREM 2.1. *Let the function f defined by (1). Then $f \in \mathcal{SD}(e_m)$, if*

$$\sum_{k=m+1}^{\infty} k|a_k| \leq 1 - \sum_{n=2}^{m} n|e_n|,$$

where

$$e_n = \frac{m}{m-n+1} \frac{\sin j\pi/(m+1)}{\sin nj\pi/(m+1)} a_n.$$

P r o o f. Let the function $f \in \mathcal{A}(e_m)$. By the assumption of the theorem, we have

$$1 - \sum_{n=2}^{m} n|e_n| - \sum_{k=m+1}^{\infty} k|a_k| > 0, \quad \forall n, k \in \mathcal{N}.$$

Consequently, this yields, $\forall n, k \in \mathcal{N}$,

$$\frac{1 + \sum_{n=2}^{m} n|e_n| + \sum_{k=m+1}^{\infty} k|a_k|}{1 - \sum_{n=2}^{m} |e_n| - \sum_{k=m+1}^{\infty} |a_k|} \geq \frac{1 - \sum_{n=2}^{m} n|e_n| - \sum_{k=m+1}^{\infty} k|a_k|}{1 - \sum_{n=2}^{m} |e_n| - \sum_{k=m+1}^{\infty} |a_k|} > 0,$$

which implies

$$\Re\{e^{a^2+ib} \frac{zf'(z)}{f(z)}\} > 0.$$

Hence $f \in \mathcal{SD}(e_m)$. ■

THEOREM 2.2. *Let p be analytic in U. Then we have the following:*

(i) Let $\lambda \in \mathcal{C}$ such that $\Re\{\frac{\lambda}{1+\lambda z}\} > 0$, then

$$\frac{zp'(z)}{p(z)} \prec \frac{\lambda z}{1+\lambda z} \quad \textit{implies } p(z) \prec 1 + \lambda z. \tag{3}$$

(ii) For $0 < |\lambda| \leq 1$ and

$$\frac{zp'(z)}{p(z)} \prec \frac{-2\lambda z}{1-\lambda^2 z^2} \quad \textit{implies } p(z) \prec \frac{1-\lambda z}{1+\lambda z}. \tag{4}$$

(iii) Let $\lambda \in \mathcal{C}$ *such that* $\Re(\frac{\lambda}{p(z)}) \geq 0$, *then*

$$\lambda[p(z) + \frac{zp'(z)}{p(z)}] \prec \lambda[1-z] \;\; \textit{implies} \;\; \lambda p(z) \prec \lambda[1-z]. \tag{5}$$

(iv) Let $\Re(\lambda) \geq 0$ *and*

$$p(z) - \lambda z p'(z) \prec \frac{\lambda z}{1+\lambda z} \;\; \textit{implies} \;\; p(z) \prec \frac{\lambda z}{1+\lambda z}. \tag{6}$$

P r o o f. If we take $R(\theta) = 0, S(\theta) = \theta, T(\theta) = \frac{1}{\theta}$, $\theta \in \mathcal{C}$ with $q(z) = 1+\lambda z$ in (i) and $R(\theta) = 0, S(\theta) = \theta, T(\theta) = \frac{1}{\theta}$, $\theta \in \mathcal{C}$ with $q(z) = \frac{1-\lambda z}{1+\lambda z}$ in (ii), then (3) and (4) follow from Lemma 1.1. If we chose $\omega(\theta) = \frac{\lambda}{\theta}$, $\theta \in \mathcal{C}\backslash\{0\}$ with $\phi(z) = \lambda z$ in (iii) and $\omega(\theta) = -\lambda$, $\phi(z) = \frac{\lambda z}{1+\lambda z}$ in (iv), then (5) and (6) follow from Lemma 1.2. ■

As applications of Theorem 2.2, we have the following examples.

Example 2.1. Let $f \in \mathcal{A}(e_m)$, $p(z) := \frac{zf'(z)}{f(z)}$, with $\Re\{\frac{\lambda}{1+\lambda z}\} > 0$ in Theorem 2.1 (i), we obtain that

$$\frac{zf''(z)}{f'(z)} + 1 - \frac{zf'(z)}{f(z)} \prec \frac{\lambda z}{1+\lambda z}$$

implies

$$\frac{zf'(z)}{f(z)} \prec 1 + \lambda z.$$

Example 2.2. Let $p(z) := \alpha f(z)$, $\alpha \in \mathcal{C}$ with $\Re(\alpha) > 0$ and $f \in \mathcal{A}(e_m)$ in Theorem 2.1 (ii), we pose

$$|\alpha||\frac{zf'(z)}{f(z)}| \leq 2 \;\; \text{implies} \;\; |\alpha||f(z)| \leq 1.$$

Example 2.3. Let $f \in \mathcal{A}(e_m)$, $p(z) := \lambda\frac{zf'(z)}{f(z)}$, such that $\Re(\lambda\frac{f(z)}{zf'(z)}) \geq 0$ in Theorem 2.1 (iii), we have that

$$\lambda[\frac{zf''(z)}{f'(z)} + 1] \prec \lambda[1-z]$$

implies

$$\lambda\frac{zf'(z)}{f(z)} \prec \lambda[1-z].$$

EXAMPLE 2.4. Let $f \in \mathcal{A}(e_m)$, $p(z) := \lambda\frac{zf'(z)}{f(z)}$, such that $\Re(\lambda) \geq 0$ in Theorem 2.1 (iv), we have that

$$\lambda\frac{zf'(z)}{f(z)}[\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}] \prec \frac{\lambda z}{1+\lambda z}$$

implies

$$\lambda\frac{zf'(z)}{f(z)} \prec \frac{\lambda z}{1+\lambda z}.$$

EXAMPLE 2.5. Let $\lambda = 1$ in Example 2.3, then we get that

$$|\frac{zf''(z)}{f'(z)} - 1| < 1$$

implies

$$|\frac{zf'(z)}{f(z)} - 1| < 1.$$

EXAMPLE 2.6. Let $\lambda = 1$ and $|z| \leq \frac{1}{2}$ in Example 2.4, then we get that

$$|\frac{zf'(z)}{f(z)}[\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)}]| < 1$$

implies

$$|\frac{zf'(z)}{f(z)} - 1| < 1.$$

REMARK 2.1. Note that when $\lambda := e^{a^2+ib}$ in Example 2.3 and Example 2.4, we have $f \in \mathcal{SD}(e_m)$.

THEOREM 2.3. *Let α, λ, μ and ν be defined as in Lemma 1.3. And let $\beta \in \mathcal{C}$ such that $\Re(\beta) > 0$. Then for $f \in \mathcal{A}(e_m)$, we have*

$$(1-\alpha(1-\nu))(\frac{f(z)}{z})^{\mu} + \alpha\beta(1-\nu)\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^{\mu} \prec 1+\lambda z$$

implies $f \in \mathcal{SD}(e_m)$.

P r o o f. Setting

$$\beta := e^{a^2+ib},\ p(z) := \frac{zf'(z)}{f(z)} \text{ and } P(z) := (\frac{f(z)}{z})^{\mu}.$$

The result follows from Lemma 1.3. ∎

Note that Theorem 2.3, for the case $\mu = 1$, $\nu = 0$, $\beta = 1$ is due to Mocanu [11].

Next we establish the necessary conditions for analytic functions in the class $\mathcal{SD}(e_m)$.

THEOREM 2.4. *Let the function f defined by (2). Then $f \in \mathcal{SD}(e_m)$ if and only if*

$$\sum_{k=m+1}^{\infty} k|a_k| \leq 1 - \sum_{n=2}^{m} n|e_n|,$$

where

$$e_n = \frac{m}{m-n+1} \frac{\sin j\pi/(m+1)}{\sin nj\pi/(m+1)} a_n.$$

Other properties are studied in the next results.

THEOREM 2.5. *Let $f_1(z), ..., f_l(z)$ defined by (1) be in the class $\mathcal{SD}(e_m)$. Then for numbers g_j, not all of them vanish, the function*

$$G(z) := \sum_{j=1}^{l} g_j f_j(z)$$

is also in the class $\mathcal{SD}(e_m)$.

P r o o f. Assume that $\Re\{e^{a^2+ib}\frac{zf_j'(z)}{f_j(z)}\} = M_j > 0$, $\forall\, j = 1, ..., l$ and $M := \min(M_j)$

$$\Re\Big(e^{a^2+ib}\frac{zG'(z)}{G(z)}\Big) = \Re\Big(e^{a^2+ib}\frac{z(\sum_{j=1}^{l} g_j f_j(z))'}{\sum_{j=1}^{l} g_j f_j(z)}\Big)$$

$$= \Re\Big(e^{a^2+ib}\frac{\sum_{j=1}^{l} g_j (zf_j'(z))}{\sum_{j=1}^{l} g_j f_j(z)}\Big) = \Re\Big(e^{a^2+ib}\frac{\sum_{j=1}^{l} g_j f_j(z)\frac{zf_j'(z)}{f_j(z)}}{\sum_{j=1}^{l} g_j f_j(z)}\Big)$$

$$= \Re\Big(\frac{\sum_{j=1}^{l} g_j f_j(z)}{\sum_{j=1}^{l} g_j f_j(z)}\Big)M_j \geq M\Re\Big(\frac{\sum_{j=1}^{l} g_j f_j(z)}{\sum_{j=1}^{l} g_j f_j(z)}\Big) > 0.$$

Hence $G \in \mathcal{SD}(e_m)$. ∎

It is well known that the Koebe function $f(z) = \frac{z}{(1-z)^2}$ is the extremal function for the class of starlike functions in U and the function $g(z) = \frac{z}{1-z}$ is the extremal function for the class of convex functions in U. But the partial sum f_k of f is not starlike in U and the partial sum g_k of g is not convex in U. Next we introduce the sufficient condition for the partial sum f_m of functions $f \in \mathcal{SD}(e_m)$ to be in the same class.

THEOREM 2.6. *Let f defined by (1) be in the class $\mathcal{SD}(e_m)$. Then its partial sums defined by*

$$f_m(z) = z + \sum_{n=2}^{m} \frac{m-n+1}{m} \frac{\sin nj\pi/(m+1)}{\sin j\pi/(m+1)} e_n z^n$$

are in the class $\mathcal{SD}(e_m)$ if $\sum_{n=2}^{m} n|e_n| \leq 1$.

3. Sandwich theorem

By employing the concept of the subordination and superordination given previously in the introduction, we pose the sandwich theorem containing functions $f \in \mathcal{A}$ to satisfy the sandwich relation

$$q_1(z) \prec e^u \frac{zf'(z)}{f(z)} \prec q_2(z), \quad u := a^2 + ib.$$

In order to obtain our results, we need the following lemmas.

LEMMA 3.1. ([12]) *Let $q(z)$ be univalent in the unit disk U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) := zq'(z)\phi(q(z))$, $h(z) := \theta(q(z)) + Q(z)$. Suppose that:*

1. *$Q(z)$ is starlike univalent in U, and*
2. *$\Re\{\frac{zh'(z)}{Q(z)}\} > 0$ for $z \in U$.*

If $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$, then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

DEFINITION 3.1. ([13]) Denote by $\mathbf{Q}$ the set of all functions $f(z)$ that are analytic and injective on $\overline{U} - E(f)$, where $E(f) := \{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\}$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$.

LEMMA 3.2. ([14]) *Let $q(z)$ be convex univalent in the unit disk U and ϑ and φ be analytic in a domain D containing $q(U)$. Suppose that:*

1. *$zq'(z)\varphi(q(z))$ is starlike univalent in U, and*
2. $\Re\{\dfrac{\vartheta'(q(z))}{\varphi(q(z))}\} > 0$ *for* $z \in U$.

If $p(z) \in \mathcal{H}[q(0),1] \cap \boldsymbol{Q}$, *with* $p(U) \subseteq D$ *and* $\vartheta(p(z)) + zp'(z)\varphi(z)$ *is univalent in* U *and* $\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z))$, *then* $q(z) \prec p(z)$ *and* $q(z)$ *is the best subordinant.*

THEOREM 3.1. *Let* $q, q(z) \neq 0$, *be a univalent function in* U *such that*

$$\Re\Big[\frac{zq''(z)}{q'(z)} + 1 - \frac{zq''(z)}{q'(z)}\Big] > \max[0, \Re(\frac{(e^u - 1)q(z)}{e^u}]. \tag{7}$$

If $e^u \frac{zf'(z)}{f(z)} \neq 0$, *satisfies the differential subordination*

$$-1 + e^u[e^u \frac{zf'(z)}{f(z)} + 2 + \frac{zf''(z)}{f'(z)}] \prec (1 - e^u(q(z) - 1)) + e^u \frac{zq'(z)}{q(z)},$$

then $e^u \frac{zf'(z)}{f(z)} \prec q(z)$ *and* q *is the best dominant.*

P r o o f. Let $p(z) := e^u \frac{zf'(z)}{f(z)}$, $\theta(w) := (1 - e^u(w - 1))$,

$\phi(w) := \frac{e^u}{w} \quad \Rightarrow Q(q) := \frac{zq'(z)}{q(z)} e^u$, $h(z) = (1 - e^u(q - 1)) + \frac{zq'(z)}{q(z)} e^u$.

Then all the above functions satisfy the assumptions of Lemma 3.1. Hence the proof is over. ■

THEOREM 3.2. *Let* $q, q(z) \neq 0$, *be a convex univalent function in* U *such that* $\Re(1 - e^u) > 0$, $-1 + e^u[e^u \frac{zf'(z)}{f(z)} + 2 + \frac{zf''(z)}{f'(z)}]$ *is univalent in* U, *and*

$$(1 - e^u(q(z) - 1)) + e^u \frac{zq'(z)}{q(z)} \prec -1 + e^u[e^u \frac{zf'(z)}{f(z)} + 2 + \frac{zf''(z)}{f'(z)}]$$

are satisfied, then $q(z) \prec e^u \frac{zf'(z)}{f(z)}$ *and* q *is the best subordinant.*

P r o o f. Let $p(z) := e^u \frac{zf'(z)}{f(z)}$, $\vartheta(w) := (1 - e^u(w - 1))$, $\varphi(w) := \frac{e^u}{w}$.

Then in view of Lemma 3.2, we obtain the desired assertion. ■

Combining Theorem 3.1 and Theorem 3.2, we get the following sandwich theorem:

THEOREM 3.3. *Let* $q_1(z), q_2 \neq 0$ *be convex and univalent in* U *respectively. Suppose that* $\Re(1 - e^u) > 0$,

$$\Re\Big[\frac{zq_2''(z)}{q_2'(z)}+1-\frac{zq_2''(z)}{q_2'(z)}\Big]>\max[0,\Re(\frac{(e^u-1)q_2(z)}{e^u}],$$
$$-1+e^u[e^u\frac{zf'(z)}{f(z)}+2+\frac{zf''(z)}{f'(z)}] \quad \textit{is univalent in } U.$$

If the subordination

$$(1-e^u(q_1(z)-1))+e^u\frac{zq_1'(z)}{q_1(z)}\prec -1+e^u[e^u\frac{zf'(z)}{f(z)}+2+\frac{zf''(z)}{f'(z)}]$$
$$\prec (1-e^u(q_2(z)-1))+e^u\frac{zq_2'(z)}{q_2(z)}$$

holds, then

$$q_1(z)\prec e^u\frac{zf'(z)}{f(z)}\prec q_2(z)$$

and q_1 and q_2 are the best dominant and subordinant respectively.

By letting $u=0$ in Theorem 3.3, we have the following result which can be found in [15]:

COROLLARY 3.1. *Let $q_1(z), q_2 \neq 0$ be convex and univalent in U respectively. Suppose that $\Re\Big[\frac{zq_2''(z)}{q_2'(z)}+1-\frac{zq_2''(z)}{q_2'(z)}\Big]>0$, $-1+[\frac{zf'(z)}{f(z)}+2+\frac{zf''(z)}{f'(z)}]$ is univalent in U. If the subordination*

$$(q_1(z)-1)+\frac{zq_1'(z)}{q_1(z)}\prec -1+[\frac{zf'(z)}{f(z)}+2+\frac{zf''(z)}{f'(z)}]\prec (q_2(z)-1)+\frac{zq_2'(z)}{q_2(z)}$$

holds, then

$$q_1(z)\prec \frac{zf'(z)}{f(z)}\prec q_2(z)$$

and q_1 and q_2 are the best dominant and subordinant respectively.

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References

[1] L. de Branges, A proof of the Bieberbach conjecture. *Acta Math.* **154** (1985), 137-152.

[2] W.K. Hayman, *Multivalent Functions.* Cambridge University Press, Cambridge, 1994.

[3] P. Henrici, *Applied and Computational Complex Analysis*, vol. III. Wiley, New York, 1986.

[4] M.S. Robertson, Univalent functions starlike with respect to a boundary point. *J. Math. Anal. Appl.* **81** (1981), 327-345.

[5] A. Lyzzaik, On a conjecture of M.S. Robertson. *Proc. Amer. Math. Soc.* **91** (1984), 108-110.

[6] H. Silverman, E.M. Silvia, Subclasses of univalent functions starlike with respect to a boundary point. *Houston J. Math.* **16** (1990), 289-299.

[7] T. Suffridge, On univalent polynomials. *J. London Math. Soc.* **44** (1969), 496-504.

[8] T. Bulboacà, On some classes of differential subordnations. *Studia Univ. Babes-Bolyai Math.* **31** (1986), 13-21.

[9] S. S. Miller and P. T. Mocanu, Differential subordination and univalent functions. *Michigan J. Math.* **28** (1981), 157-171.

[10] S. Ponnusamy, S. Rajasekaran, New sufficient conditions for starlike and univalent functions. *Soochow J. Math.* **21**, No 2 (1995), 193-201.

[11] P.T. Mocanu, Some starlikeness conditions for analytic functions. *Rev. Roumaine Math. Pures Appl.* **33** (1988), 117-124.

[12] S.S. Miller and P.T. Mocanu, *Differential Subordinantions: Theory and Applications.* Pure and Applied Math. No 225, Dekker, N. York, 2000.

[13] S.S. Miller and P.T.Mocanu, Subordinants of differential superordinations. *Complex Variables* **48**, No 10 (2003), 815-826.

[14] T. Bulboaca, Classes of first-order differential superordinations. *Demonstr. Math.* **35**, No 2 (2002), 287-292.

[15] R.M. Ali, V. Ravichandran, M. Hussain Khan and K.G. Subramanian, Differential sandwich theorems for certain analytic functions. *Far East J. Math. Sci.* **15**, No 1 (2005), 87-94.

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