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ON THE OPERATIONAL SOLUTION OF A SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

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This paper is dedicated to the 70th anniversary of Professor Srivastava

Abstract

We consider a linear system of differential equations with fractional derivatives, and its corresponding system in the field of Mikusiński operators, written in a matrix form, by using the connection between the fractional and the Mikusiński calculus. The exact and the approximate operational solution of the corresponding matrix equations, with operator entries are determined, and their characters are analyzed.

By using the packages *Scientific Workplace* and *GeoGebra*, the exact and the approximate solution of the given numerical example are constructed, and their dependence on the initial condition and the fractional derivatives is shown graphically.

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1. Introduction

In this paper we consider, in the frame of the Mikusiński calculus, the system of time-fractional differential equations of the form

$$\frac{d^{\boldsymbol{\beta}}\mathbf{u}(t)}{dt^{\beta}} = \mathbf{A}\mathbf{u}(t), \tag{1}$$

with the initial conditions:

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{2}$$

In (1), $\mathbf{A}$ is the matrix $[a_{ij}]$, $i,j = 1,2\dots n$, where a_{ij}, are numerical constants. The unknown $n-$dimensional vector, $\mathbf{u}$ is considered as $\mathbf{u}= [u_1\ u_2\ \dots\ u_n]^T$, where u_i, $i = 1,2,\dots n$, are the unknown functions. The vector $\boldsymbol{\beta} = [\beta_1\ \beta_2\ \dots\ \beta_n]^T$, where $0 < \beta_i < 1$, $i = 1,2,\dots n$, is given, and $0 < t < T$.

The fractional derivative, in equation (1),

$$\frac{d^{\boldsymbol{\beta}}\mathbf{u}(t)}{dt^{\boldsymbol{\beta}}} = \left[\frac{d^{\beta_1}u_1(t)}{dt^{\beta_1}}, \frac{d^{\beta_2}u_1(t)}{dt^{\beta_2}}, \dots, \frac{d^{\beta_n}u_1(t)}{dt^{\beta_n}}\right]^T,$$

is considered in the sense of Caputo ([1]).

The initial vector $\mathbf{u}_0 = [\phi_1\ \phi_2\ \dots\ \phi_n]^T$, $i = 1,2,\dots,n$, is given in (2), with the numerical constants ϕ_i, $i = 1,2,\dots,n$.

In this paper we construct the exact and the approximate solution of the system given by (1), (2), in the frame of Mikusiński calculus. In the paper [8], the special system of two fractional differential equations is considered and its exact and the approximate solution is determined. Analytic study on linear systems of fractional differential equations is given in the paper [5].

In Section 2, we give some notions from the fractional calculus and the Mikusiński operational calculus ([4]), with the accent on their connections.

In Section 3, we consider the operational differential equation in the field $\mathcal{F}$, corresponding to the time fractional differential equation appearing in (1), taking $n = 1$. We construct the exact and the approximate solution of such equation with the given initial condition, and express the error of approximation in the field of Mikusi'nski operators.

In Section 4, we constructed the system of operational differential equations in the field $\mathcal{F}$, corresponding to the system of fractional differential equations (1), (2). In fact, we considered the matrix equation in the field of Mikusiński operators, and construct its operational exact and the approximate solution. We analyzed the character of the obtained solution. Since, the operational solution represents the continuous function we express the exact and the approximate solution of the system (1), depending on the initial conditions (2), and the vector $\boldsymbol{\beta}$.

In Section 5, we constructed the approximate solution, of the given system of three time fractional differential equations, by using packages *Scientific Workplace* and *GeoGebra*. We presented the graphs of the obtained approximate solutions and discussed the dependence of the solution on the initial conditions given in (2), and the numbers in $\boldsymbol{\beta}$, appearing in fractional derivative, graphically.

In the papers [8], [9] and [10] the solution of the partial differential equations, mathematical models of a viscoelastic bar is constructed, by using the similar procedure.

2. Notions and notations

In this paper we use the Riemann-Liouville fractional integral operator J^α, of order $\alpha > 0$, defined by the convolution

$$\begin{aligned} J^\alpha f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \\ J^0 f(t) &= f(t), \end{aligned} \tag{3}$$

and the so-called Caputo derivative (originated from [1]) because it is more suitable for applications to problems with initial and boundary conditions (see [3], [6]). The definition used for the Caputo derivative is as follows ([6], [7]):

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau, \quad t > 0. \tag{4}$$

The elements of the field of Mikusiński operator, $\mathcal{F}$, are called *operators*; they are quotients of the form

$$\frac{f}{g}, \quad f \in \mathcal{C}_+,\ 0 \not\equiv g \in \mathcal{C}_+,$$

where the last division is meant in the sense of convolution (see [4]).

We shall denote by $\mathcal{F}_c$ the proper subset of $\mathcal{F}$ consisting of the operators representing continuous functions. As examples of such operators, we have the integral operator $\ell \in \mathcal{F}_c$ representing the constant function 1 on $[0, \infty)$, and the α powers of ℓ, ℓ^α :

$$\ell = \{1\}, \quad \ell^\alpha = \left\{ \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right\}, \quad \alpha \geq 1. \tag{5}$$

Also, among the most important operators are the differential operator s (the inverse operator to ℓ), and I, the identity operator, i.e.,

$$\ell s \;=\; I.$$

Neither s nor I are operators from $\mathcal{F}_c$.

For the use of the Mikusiński operational calculus in the theory of differential equations, the following relation, connecting the operator representing the n-th derivative of an n times derivable function $a = a(t)$ with the operator a is essential:

$$\{a^{(n)}(t)\} \quad = \quad s^n a - a(0)s^{n-1} - \cdots - a^{(n-1)}(0)\, I. \tag{6}$$

In the following we shall connect the Mikusiński and the fractional calculus. From relations (3) it follows that in the field $\mathcal{F}$ the operator ℓ^α corresponds to the Riemann-Liouville fractional integral operator of order α, J^α. In fact, for every continuous function f it holds:

$$\ell^\alpha f = \{J^\alpha f(t)\}\,, \quad 0 < \alpha < 1. \tag{7}$$

On the other hand, the Caputo fractional derivative $D^\alpha f(t)$, of order α, $0 < \alpha < 1$, applied to function f, corresponds to the operator $s^\alpha f - f(0)s^{\alpha-1}$, i.e.

$$s^\alpha f - f(0)s^{\alpha-1} = \{D^\alpha f(t)\}. \tag{8}$$

3. An operational equation

In the system (1), for $n = 1$, it appears the time-fractional differential equation of the form

$$\frac{d^\beta u(t)}{dt} = Au, \tag{9}$$

with the initial conditions:

$$u(0^+) = \phi, \tag{10}$$

where u is the unknown function, $0 < \beta < 1$, and A and ϕ are numerical constants. The equation (9) with (10) corresponds in the field of Mikusiński operators to the equation:

$$(s^\beta - A)u = s^{1-\beta}\phi. \tag{11}$$

The exact solution of equation (11) has the form:

$$u = \frac{\ell^{1-\beta}\phi}{s^\beta - A} = \frac{\ell\phi}{I - A\ell^\beta} = \ell\phi \sum_{i=0}^{\infty} (A\ell^\beta)^i, \tag{12}$$

and represent the continuous function, which can be treated as the solution of the time fractional differential equation (9):

$$u(t) = \phi \sum_{i=0}^{\infty} \frac{t^{i\beta} A^i}{\Gamma(1+i\beta)}. \tag{13}$$

The approximate solution of equation (11) can be written in a form:

$$u_N = \ell\phi \sum_{i=0}^{N} (A\ell^\beta)^i, \tag{14}$$

and the approximate solution of the time fractional differential equation (9) can be considered in the form:

$$u(t) = \phi \sum_{i=0}^{N} \frac{t^{i\beta} A^i}{\Gamma(1+i\beta)}. \tag{15}$$

Since the solution of equation (11) represent the continuous function the error of approximation can be estimated as:

$$\begin{aligned} |u - u_N| &= \ell|\phi| \left| \sum_{i=0}^{\infty} (A\ell^\beta)^i - \sum_{i=0}^{N} (A\ell^\beta)^i \right| = \ell|\phi| \left| \sum_{i=N+1}^{\infty} (A\ell^\beta)^i \right| \\ &\leq_T \ell|\phi| \sum_{i=N+1}^{\infty} \frac{T^{i\beta} A^i}{\Gamma(1+i\beta)}. \end{aligned} \tag{16}$$

From (16), it follows that in order to get good approximate solution one has to take big number N, because $0 < \beta < 1$.

4. A system of operational equations

In this section we shall determine the exact and the approximate solution of operator differential equations corresponding to the system of equations (1), (2).

In the field $\mathcal{F}$, the system of equations

$$\begin{bmatrix} s^{\beta_1} - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & s^{\beta_2} - a_{22} & \dots & -a_{2n} \\ \dots & \dots & \dots & \\ -a_{2n} & -a_{22} & \dots & s^{\beta_n} - a_{2n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} = \begin{bmatrix} \phi_1 \ell^{1-\beta_1} \\ \phi_2 \ell^{1-\beta_2} \\ \dots \\ \phi_n \ell^{1-\beta_n} \end{bmatrix} \tag{17}$$

corresponds to the problem (1).

In (17) we got the matrix equation in the field of Mikusiński operators with the non integer exponent of s and ℓ. In the paper ([2]) the matrix equation with the integer exponent of s and ℓ was considered and its exact and approximate solution id determined.

In this paper we consider the numerical constants $0 < \beta_i < 1, \quad i = 1, 2 \ldots, n$ as follows:

1. $\beta_1 = \beta_2 = \ldots = \beta_n = \beta$;
2. $0 < \beta_i < 1, \;\; i = 1, 2 \ldots, n$ are rational numbers of the form

$$\beta_1 = \frac{p_1}{r}, \;\; \beta_2 = \frac{p_2}{r} \;\; \ldots . \;\; \beta_n = \frac{p_n}{r}, \tag{18}$$

where $p_1, \; p_2, \; \ldots, \; p_n, \; r$ are natural numbers.

In the first case we shall consider the exact solution of matrix equation (17) in a form of infinite series:

$$u_i = \ell \sum_{k=0}^{\infty} u_{ik} \ell^{k\beta}, \quad i = 1, \; 2, \tag{19}$$

where numerical constants $u_{ik}, \quad i = 1, 2, \ldots n, \quad k = 1, 2, \ldots,$ are to be determined in the following.

In order to determine the numerical coefficients u_{ik}, $i = 1, 2, \ldots n$, $k = 1, 2, \ldots$, from (19), the matrix equation can be written in the form of the following system of equations:

$$s^{\beta} \left(\sum_{k=0}^{\infty} u_{ik} \ell^{1+k\beta} \right) - \sum_{j=1}^{n} a_{ij} \left(\sum_{k=0}^{\infty} u_{jk} \ell^{1+k\beta} \right) = \phi_i \ell^{1-\beta}, \tag{20}$$

for $i = 1, 2, \ldots n$, wherefrom we get:

$$u_{i0} = \phi_i, \quad u_{i1} = \sum_{j=1}^{n} a_{ij} \phi_j, \quad u_{ik} = \sum_{j=1}^{n} a_{ij} u_{jk-1}, \quad i = 1, 2, \ldots, n. \tag{21}$$

If $\beta_i, \quad i = 1, 2 \ldots n$, are different, and has the form (18), then the solution can be considered as:

$$u_i = \ell \sum_{k=0}^{\infty} u_{ik} \ell^{\frac{k}{r}}, \quad i = 1, \; 2. \tag{22}$$

The numerical coefficients u_{ik}, $i = 1, 2, \ldots n$, $k = 1, 2, \ldots$, can be determined from the system:

$$s^{\frac{p_i}{r}} \left(\sum_{k=0}^{\infty} u_{ik} \ell^{1+\frac{k}{r}} \right) - \sum_{j=1}^{n} a_{ij} \left(\sum_{k=0}^{\infty} u_{jk} \ell^{1+\frac{k}{r}} \right) = \phi_i \ell^{1-\frac{p_i}{r}}, \tag{23}$$

for $i = 1, 2, \ldots n$, wherefrom we get:

$$u_{i0} = \phi_i, \quad u_{ij_i} = \sum_{j=1}^{n} a_{ij} \phi_i, \quad i = 1, 2, \ldots, n. \tag{24}$$

Further on, by equating the coefficients we got the other finite number of u_{ik}, $k \in N$, $i = 1, 2, \ldots n$.

Let us remark that the approximate operational solution (finite number of numerical coefficients) of the matrix equation (17) can be obtain algebraically, by using some programme package, because the field Mikusiński operators has very good algebraic properties. After that it is important to analyze the character of the obtained approximate and the exact solution. In this case the solution of the considered system (17) given by (19) and (22) represent continuous function and in this case we can write the exact solution $\mathbf{u} = \{u_i\}$, $i = 1, 2, \ldots n$, as:

$$u_i = \sum_{k=0}^{\infty} u_{ik} \frac{t^{k\beta}}{\Gamma(1 + k\beta)}, \quad i = 1,\ 2,\ \ldots,\ n, \tag{25}$$

for $\beta_1 = \beta_2 = \ldots = \beta_n = \beta$. The coefficients u_{ik} are given by (21). In this case the approximate solution has the form:

$$u_i = \sum_{k=0}^{N} u_{ik} \frac{t^{k\beta}}{\Gamma(1 + k\beta)}, \quad i = 1,\ 2,\ \ldots,\ n. \tag{26}$$

5. An application

Let us consider the system of three fractional differential equations of the form

$$\begin{aligned} \frac{d^{\beta_1} u_1(t)}{dt} &= a_{11} u_1(t) + a_{13} u_3(t), \\ \frac{d^{\beta_2} u_2(t)}{dt} &= 2u_1(t) - u_2(t), \\ \frac{d^{\beta_3} u_2(t)}{dt} &= -u_2(t) + u_3(t), \end{aligned} \tag{27}$$

with the initial conditions:

$$u_1(0^+) = a, \quad u_2(0^+) = b, \quad u_2(0^+) = c. \tag{28}$$

Taking $\beta_1 = 4/5$, $\beta_2 = 3/5$, $\beta_3 = 2/5$, and $a_{11} = 2, \quad a_{13} = 1, \quad a_{21} = 2 \quad a_{22} = -1, \quad a_{32} = -1, \quad a_{33} = 1, \quad a_{12} = a_{23} = a_{31} = 0$, the system (27), (28) can be written, in the field of Mikusiński operators, in the matrix form:

$$\begin{bmatrix} s^{4/5} - 2 & 0 & -1 \\ -2 & s^{3/5} + 1 & 0 \\ 0 & 1 & s^{2/5} - 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} as^{4/5-1} \\ bs^{3/5-1} \\ cs^{2/5-1} \end{bmatrix}, \tag{29}$$

where s is a differential operator. Using the package *Scientific Workplace*, we obtain the solution of previous equation in the form:

$$u\_1 = \ell \left(a + a\ell^{3/5} - a\ell - a\ell^{7/5} - b\ell^{6/5} + c\ell^{4/5} + c\ell^{7/5}\right)(1 - 2\ell^{2/5} + \ell^{3/5}$$

$$+3\ell^{4/5} - 6\ell - \ell^{6/5} + 17\ell^{7/5} - 13\ell^{8/5} - 35\ell^{9/5} + 61\ell^2 + 35\ell^{11/5} + ...),$$

$$u\_2 = \ell \left(b + 2a\ell^{3/5} - b\ell^{2/5} - 2a\ell - 2b\ell^{4/5} + 2b\ell^{6/5} + 2c\ell^{7/5}\right)(1 - 2\ell^{2/5}$$

$$+\ell^{3/5} + 3\ell^{4/5} - 6\ell - \ell^{6/5} + 17\ell^{7/5} - 13\ell^{8/5} - 35\ell^{9/5} + 61\ell^2 + 35\ell^{11/5} + ...),$$

$$u\_3 = \ell(c - b\ell^{2/5} - 2a\ell + c\ell^{3/5} + 2b\ell^{6/5} - 2c\ell^{4/5} - 2c\ell^{7/5})(1 - 2\ell^{2/5} + \ell^{3/5}$$

$$+3\ell^{4/5} - 6\ell - \ell^{6/5} + 17\ell^{7/5} - 13\ell^{8/5} - 35\ell^{9/5} + 61\ell^2 + 35\ell^{11/5} + ...). \tag{30}$$

Using the previous form of the solution and the approximate solution of the matrix equation, (29) can be considered in the form:

$$u_i^{18} = \ell \sum_{k=0}^{18} u_{ik} \ell^{\frac{k}{5}}, \quad i = 1,\ 2,\ 3, \tag{31}$$

where the numerical coefficients $u_{ik}, \quad i = 1,\ 2,\ 3, \quad k = 1, 2, \ldots$, are given in (24).

We can say that the exact solution of the matrix equation (29) in the field $\mathcal{F}$ has the form

$$u_i = \ell \sum_{k=0}^{\infty} u_{ik} \ell^{\frac{k}{5}}, \quad i = 1,\ 2,\ 3. \tag{32}$$

From (32) it follows that the exact solution solution of the problem (27), (28) has the form

$$u_i = \sum_{k=0}^{\infty} u_{ik} \frac{t^{\frac{k}{5}}}{\Gamma(1+\frac{k}{5})}, \quad i = 1,\ 2,\ 3. \tag{33}$$

The approximate solution of the problem (27), (28) has the form

$$u_i = \sum_{k=0}^{18} u_{ik} \frac{t^{\frac{k}{5}}}{\Gamma(1+\frac{k}{5})}, \quad i = 1,\ 2,\ 3, \tag{34}$$

where the coefficients u_{ik}, $k = 1, 2, \dots,\ \ i = 1,\ 2,\ 3$, are given in relation (32).

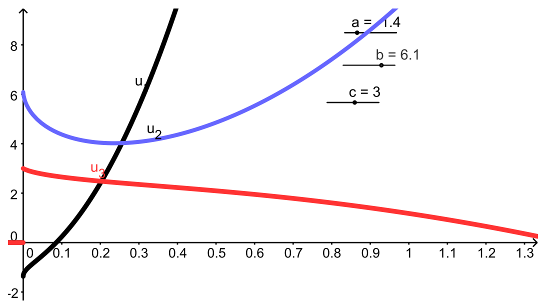

Figure 1.

Let us remark that the initial conditions (28), are given as parameters a, b, and c, and we used the package *GeoGebra* to draw the graph of the approximate solution of the system of fractional differential equations. Namely the parameters a, b, and c, can be continuously changed by using three sliders, and we obtain different curves representing the approximate solution. On Figure 1 the graphs of functions u_1, u_2, and u_3, are drawn for $a = -1.4$, $b = 6.1$ and $c = 3$, but these values can be changed by using the sliders a, b, and c the corresponding solutions u_1, u_2 u_3 can be obtained. Next, we consider the system of fractional differential equations (27), with the conditions (28), by taking

$$0 \leq \beta_1 = \beta_2 = \beta_3 = p < 1.$$

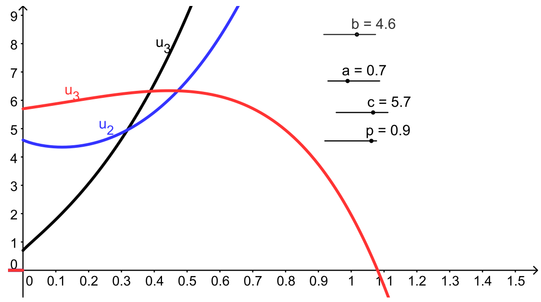

Figure 2.

Then the approximate solution solution of the system (27), (28) has the form

$$u_i^N = \ell \sum_{k=0}^{N} u_{ik} \ell^{kp}, \quad i = 1,\ 2,\ 3, \tag{35}$$

where the numerical coefficients u_{ik}, $i = 1,\ 2,\ 3,$ $k = 1, 2, \ldots,$ are given by the relations (21), and in this particular case they have the form:

$$u_{10} = a, \quad u_{11} = 2a + c, \quad u_{12} = 4a - b + 3c) \quad u_{13} = 6a - 2b + 7c,$$

$$u_{14} = 8a - 5b + 13c, \quad u_{15} = 6a - 8b + 21c, \quad u_{16} = 27c - 13b - 4a, \ldots$$

$$u_{20} = b, \quad u_{21} = 2a - b, \quad u_{22} = 2a + b + 2c, \quad u_{23} = 6a - 3b + 4c,$$

$$u_{24} = 6a - b + 10c, \quad u_{25} = 10a - 9b + 16c, \quad u_{26} = 10a - 9b + 16c, \ldots$$

$$u_{30} = c, \quad u_{31} = c - b, \quad u_{32} = c - 2, \quad u_{33} = -4a - b - c,$$

$$u_{34} = 2b - 10a - 5c, \quad u_{35} = 2b - 10a - 5c, \quad u_{36} = 12b - 26a - 31c, \ldots$$

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