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CERTAIN DIFFERENTIAL SUBORDINATIONS USING A GENERALIZED SĂLĂGEAN OPERATOR AND RUSCHEWEYH OPERATOR

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Dedicated to Prof. H.M. Srivastava for the 70th anniversary

Abstract

In the present paper we define a new operator using the generalized Sălăgean operator and the Ruscheweyh operator. Denote by DR_λ^n the Hadamard product of the generalized Sălăgean operator D_λ^n and of the Ruscheweyh operator R^n, given by

$$DR_\lambda^n : A \to A, \quad DR_\lambda^n f(z) = (D_\lambda^n * R^n) f(z),$$

where $A_n = \{f \in \mathcal{H}(U) : \ f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\}$ is the class of normalized analytic functions in the unit disc, with $A_1 := A$. We study some differential subordinations regarding the operator DR_λ^n.

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1. Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : \ |z| < 1\}$ and by $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $A_n = \{f \in \mathcal{H}(U) : \; f(z) = z + a_{n+1}z^{n+1} + \ldots, \; z \in U\}$, for $n \in \mathbb{N}$ and $A_1 := A$.

Denote by $K = \left\{ f \in A : \; \mathrm{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, \; z \in U \right\}$ the class of the normalized convex functions in U.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \qquad \text{for} \quad z \in U, \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of U.

Definition 1.1. (Al Oboudi [2], generalized the Sălăgean operator) For $f \in A$, $z \in U$, $\lambda \geq 0$ and $n \in \mathbb{N}$, the operator $D_\lambda^n : A \to A$ is defined by:

$$\begin{aligned} D_\lambda^0 f(z) &= f(z) \\ D_\lambda^1 f(z) &= (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z) \\ &\;\;... \\ D_\lambda^n f(z) &= (1-\lambda) D_\lambda^{n-1} f(z) + \lambda z \left(D_\lambda^n f(z)\right)' = D_\lambda \left(D_\lambda^{n-1} f(z)\right). \end{aligned}$$

Remark 1.2. If $f \in A$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then

$$D_\lambda^n f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^n a_j z^j, \quad \text{for} \quad z \in U.$$

Remark 1.3. For $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator [5].

Definition 1.4. (Ruscheweyh [4]) For $f \in A$ and $n \in \mathbb{N}$, the operator R^n is defined by $R^n : A \to A$,

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= zf'(z) \\ &\;\;... \\ (n+1) R^{n+1} f(z) &= z \left(R^n f(z)\right)' + nR^n f(z), \quad \text{for} \;\; z \in U. \end{aligned}$$

REMARK 1.5. If $f \in A$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then

$$R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j, \quad \text{for } z \in U.$$

LEMMA 1.6. (Miller and Mocanu [3]) *Let g be a convex function in U and let $h(z) = g(z) + n\alpha z g'(z)$, for $z \in U$, where $\alpha > 0$ and n is a positive integer.*

If $p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \dots$, for $z \in U$, is holomorphic in U and

$$p(z) + \alpha z p'(z) \prec h(z),$$

for $z \in U$, then

$$p(z) \prec g(z)$$

and this result is sharp.

2. Main results

DEFINITION 2.1. Let $\lambda \geq 0$ and $n \in \mathbb{N}$. Denote by $DR_\lambda^n : A \to A$ the operator given by the Hadamard product (the convolution product) of the generalized Sălăgean operator D_λ^n and the Ruscheweyh operator R^n:

$$DR_\lambda^n f(z) = (D_\lambda^n * R^n) f(z),$$

for any $z \in U$ and each nonnegative integer n.

REMARK 2.2. If $f \in A$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then

$$DR_\lambda^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} \left[1 + (j-1)\lambda\right]^n a_j^2 z^j, \quad \text{for } z \in U.$$

REMARK 2.3. For $\lambda = 1$ we obtain the Hadamard product SR^n (see [1]) of the Sălăgean operator S^n and the Ruscheweyh operator R^n.

THEOREM 2.4. *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $\lambda \geq 0$, $n \in \mathbb{N}$, $f \in A$ and the differential subordination*

$$\frac{n+1}{\lambda z} DR_\lambda^{n+1} f(z) - \frac{n(1-\lambda)}{\lambda z} DR_\lambda^n f(z) - \left(n - 1 + \frac{1}{\lambda}\right) \left(DR_\lambda^n f(z)\right)' \prec h(z), \tag{2}$$

holds for $z \in U$, then

$$\left(DR_\lambda^n f(z)\right)' \prec g(z), \quad \text{for } z \in U, \tag{3}$$

and this result is sharp.

P r o o f. With the notation

$$p(z) = \left(DR_{\lambda}^{n} f(z)\right)' = 1 + \sum_{j=2}^{\infty} C_{n+j-1}^{n} \left[1 + (j-1)\lambda\right]^{n} j a_j^2 z^{j-1}$$

and $p(0) = 1$, we obtain for $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$:

$$p(z) + zp'(z) = 1 + \sum_{j=2}^{\infty} C_{n+j-1}^{n} \left[1 + (j-1)\lambda\right]^{n} j^2 a_j^2 z^{j-1}$$

$$= \frac{n+1}{\lambda z}\left[z + \sum_{j=2}^{\infty} C_{n+j}^{n+1} \left[1 + (j-1)\lambda\right]^{n+1} a_j^2 z^j\right] + \frac{\lambda - n - 1}{\lambda}$$

$$- \sum_{j=2}^{\infty} C_{n+j-1}^{n} \left[1 + (j-1)\lambda\right]^{n} a_j^2 z^{j-1} \left(n - 1 + \frac{1}{\lambda}\right) j$$

$$- \sum_{j=2}^{\infty} C_{n+j-1}^{n} \left[1 + (j-1)\lambda\right]^{n} a_j^2 z^{j-1} \frac{n(1-\lambda)}{\lambda}$$

$$= \frac{n+1}{\lambda z} DR_{\lambda}^{n+1} f(z) - \left(n - 1 + \frac{1}{\lambda}\right)\left(DR_{\lambda}^{n} f(z)\right)' - \frac{n(1-\lambda)}{\lambda z} DR_{\lambda}^{n} f(z).$$

We have $p(z) + zp'(z) \prec h(z)$, for $z \in U$. By using Lemma 1.6 we obtain $p(z) \prec g(z)$, for $z \in U$, i.e. $\left(DR_{\lambda}^{n} f(z)\right)' \prec g(z)$, for $z \in U$ and this result is sharp. ∎

COROLLARY 2.5. (see [1]) *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $n \in \mathbb{N}$, $f \in A$ and the differential subordination*

$$\frac{1}{z} SR^{n+1} f(z) + \frac{n}{n+1} z \left(SR^{n} f(z)\right)'' \prec h(z) \quad \text{for } z \in U, \tag{4}$$

holds, then $\left(SR^{n} f(z)\right)' \prec g(z)$ for $z \in U$ and this result is sharp.

THEOREM 2.6. *Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $n \in \mathbb{N}$ and $f \in A$ verifies the differential subordination*

$$\left(DR_{\lambda}^{n} f(z)\right)' \prec h(z) \quad \text{for } z \in U, \tag{5}$$

then

$$\frac{DR_{\lambda}^{n} f(z)}{z} \prec g(z) \quad \text{for } z \in U, \tag{6}$$

and this result is sharp.

P r o o f. For $f \in A$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ we have

$$DR_{\lambda}^{n} f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} \left[1 + (j-1)\lambda\right]^{n} a_j^2 z^j, \quad \text{for} \quad z \in U.$$

Consider

$$p(z) = \frac{DR_{\lambda}^{n} f(z)}{z} = \frac{z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} \left[1 + (j-1)\lambda\right]^{n} a_j^2 z^j}{z}$$
$$= 1 + \sum_{j=2}^{\infty} C_{n+j-1}^{n} \left[1 + (j-1)\lambda\right]^{n} a_j^2 z^{j-1}.$$

We have $p(z) + zp'(z) = (DR_{\lambda}^{n} f(z))'$, for $z \in U$.

Then $(DR_{\lambda}^{n} f(z))' \prec h(z)$, for $z \in U$, becomes $p(z) + zp'(z) \prec h(z) = g(z) + zg'(z)$, for $z \in U$. By using Lemma 1.6 we obtain $p(z) \prec g(z)$, for $z \in U$, i.e. $\frac{DR_{\lambda}^{n} f(z)}{z} \prec g(z)$, for $z \in U$. ∎

Corollary 2.7. (see [1]) *Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $n \in \mathbb{N}$ and $f \in A$ verifies the differential subordination*

$$(SR^{n} f(z))' \prec h(z), \quad \text{for} \quad z \in U, \tag{7}$$

then $\frac{SR^{n} f(z)}{z} \prec g(z)$, for $z \in U$, and this result is sharp.

Theorem 2.8. *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $n \in \mathbb{N}$ and $f \in A$ verifies the differential subordination*

$$\left(\frac{zDR_{\lambda}^{n+1} f(z)}{DR_{\lambda}^{n} f(z)}\right)' \prec h(z), \quad \text{for} \quad z \in U, \tag{8}$$

then

$$\frac{DR_{\lambda}^{n+1} f(z)}{DR_{\lambda}^{n} f(z)} \prec g(z), \quad \text{for} \quad z \in U, \tag{9}$$

and this result is sharp.

P r o o f. For $f \in A$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ we have

$$DR_{\lambda}^{n} f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} \left[1 + (j-1)\lambda\right]^{n} a_j^2 z^j, \quad \text{for} \quad z \in U.$$

Consider

$$p(z) = \frac{DR_\lambda^{n+1} f(z)}{DR_\lambda^n f(z)} = \frac{z + \sum_{j=2}^{\infty} C_{n+j}^{n+1} \left[1 + (j-1)\lambda\right]^{n+1} a_j^2 z^j}{z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} \left[1 + (j-1)\lambda\right]^{n} a_j^2 z^j}$$
$$= \frac{1 + \sum_{j=2}^{\infty} C_{n+j}^{n+1} \left[1 + (j-1)\lambda\right]^{n+1} a_j^2 z^{j-1}}{1 + \sum_{j=2}^{\infty} C_{n+j-1}^{n} \left[1 + (j-1)\lambda\right]^{n} a_j^2 z^{j-1}}.$$

We have $p'(z) = \frac{\left(DR_\lambda^{n+1} f(z)\right)'}{DR_\lambda^n f(z)} - p(z) \cdot \frac{\left(DR_\lambda^n f(z)\right)'}{DR_\lambda^n f(z)}$.

Then $p(z) + zp'(z) = \left(\frac{zDR_\lambda^{n+1} f(z)}{DR_\lambda^n f(z)}\right)'$. Relation (8) becomes $p(z) + zp'(z) \prec h(z) = g(z) + zg'(z)$, for $z \in U$, and, by using Lemma 1.6 we obtain $p(z) \prec g(z)$, for $z \in U$, i.e. $\frac{DR_\lambda^{n+1} f(z)}{DR_\lambda^n f(z)} \prec g(z)$, for $z \in U$. ■

Corollary 2.9. (see [1]) *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $n \in \mathbb{N}$ and $f \in A$ verifies the differential subordination*

$$\left(\frac{zSR^{n+1} f(z)}{SR^n f(z)}\right)' \prec h(z) \quad \text{for } z \in U, \tag{10}$$

then $\frac{SR^{n+1} f(z)}{SR^n f(z)} \prec g(z)$, for $z \in U$, and this result is sharp.

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