

## FRACTIONAL CALCULUS OF $\mathcal{P}$ - TRANSFORMS

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*Dedicated to the memory of Professor Anatoly Kilbas*

### Abstract

The fractional calculus of the  $\mathcal{P}$ -transform or pathway transform which is a generalization of many well known integral transforms is studied. The Mellin and Laplace transforms of a  $\mathcal{P}$ -transform are obtained. The composition formulae for the various fractional operators such as Saigo operator, Kober operator and Riemann-Liouville fractional integral and differential operators with  $\mathcal{P}$ -transform are proved. Application of the  $\mathcal{P}$ -transform in reaction rate theory in astrophysics in connection with extended non-resonant thermonuclear reaction rate probability integral in the Maxwell-Boltzmann case and cut-off case is established. The behaviour of the kernel functions of type-1 and type-2  $\mathcal{P}$ -transform are also studied.

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### 1. Introduction

The paper is devoted to the study of the fractional calculus of  $\mathcal{P}$ -transform, called also a pathway transform, defined by

$$(\mathcal{P}_{\nu}^{\rho, \beta, \alpha} f)(x) = \int_0^{\infty} D_{\rho, \beta}^{\nu, \alpha}(xt) f(t) dt, \quad x > 0, \quad (1)$$

where  $D_{\rho,\beta}^{\nu,\alpha}(x)$  denotes the kernel-function

$$D_{\rho,\beta}^{\nu,\alpha}(x) = \int_0^{\left[\frac{1}{a(1-\alpha)}\right]^{\frac{1}{\rho}}} y^{\nu-1} [1 - a(1-\alpha)y^\rho]^{-\frac{1}{1-\alpha}} e^{-xy^{-\beta}} dy, \quad x > 0, \quad (2)$$

with  $\nu \in \mathbb{C}, \beta > 0, \rho > 0, a > 0, \alpha < 1$ . In this case we say that (2) is a type-1  $\mathcal{P}$ -transform. Instead of using the kernel function given in (2), if we use

$$D_{\rho,\beta}^{\nu,\alpha}(x) = \int_0^\infty y^{\nu-1} [1 + a(\alpha-1)y^\rho]^{-\frac{1}{\alpha-1}} e^{-xy^{-\beta}} dy, \quad x > 0, \quad (3)$$

for  $\nu \in \mathbb{C}, \beta > 0, a > 0, \rho \in \mathbb{R}, \alpha > 1$ , we obtain a type-2  $\mathcal{P}$ -transform. The  $\mathcal{P}$ -transform of both types are defined in the space  $\mathbb{L}_{\nu,r}(0, \infty)$  consisting of the Lebesgue measurable complex valued functions  $f$  for which

$$\|f\|_{\nu,r} = \left\{ \int_0^\infty |t^\nu f(t)|^r \frac{dt}{t} \right\}^{\frac{1}{r}} < \infty, \quad (4)$$

for  $1 \leq r < \infty, \nu \in \mathbb{R}$ . The  $\mathcal{P}$ -transforms of both types are obtained by using the pathway model of Mathai [17], Mathai and Haubold [20]. When  $\beta = 1, a = 1$  and  $\alpha \rightarrow 1$  we can observe that

$$\lim_{\alpha \rightarrow 1} D_{\rho,1}^{\nu,\alpha}(x) = Z_\rho^\nu(x). \quad (5)$$

where  $Z_\rho^\nu(x)$  is the kernel function of the Krätzel transform, introduced by Krätzel [14], and given by

$$\mathcal{K}_\nu^{(\rho)} f(x) = \int_0^\infty Z_\rho^\nu(xt) f(t) dt, \quad x > 0, \quad (6)$$

where

$$Z_\rho^\nu(x) = \int_0^\infty y^{\nu-1} e^{-y^\rho - xy^{-1}} dy. \quad (7)$$

The transform in (6) and its several modifications were considered by many authors. Glaeske et al. [5] considered a modified version of the Krätzel transform and its compositions with fractional calculus operators on the spaces of  $\mathcal{F}_{p,\mu}$  and  $\mathcal{F}'_{p,\mu}$ . Bonilla et al. [1, 2] studied the Krätzel transform in the space  $\mathcal{F}_{p,\mu}$  and  $\mathcal{F}'_{p,\mu}$ . Kilbas et al. [8] obtained the asymptotic representation for the modified Krätzel function, Liouville and Erdélyi-Kober type fractional integrals of the modified Krätzel function. Kilbas et al. [10] studied the Krätzel function in (7) for all values of  $\rho$  and established it in terms of Fox's  $H$ -function. When  $\beta = 1, a = 1, \rho = 1$  and  $\alpha \rightarrow 1$ , the  $\mathcal{P}$ -transform of both types reduces to the Meijer transform. For  $\beta = 1, a = 1, \rho = 1$  and  $\alpha \rightarrow 1$  along with  $x$  replaced by  $\frac{t^2}{4}$  in (2) and (3), we can see that

$$\lim_{\alpha \rightarrow 1} D_{1,1}^{\nu,\alpha} \left( \frac{t^2}{4} \right) = 2 \left( \frac{t}{2} \right)^\nu K_{-\nu}(t), \quad (8)$$

where  $K_{-\nu}(t)$  is the modified Bessel function of the third kind or the McDonald function (see [4], Sect. 7.2.2). Kilbas and Kumar [7] considered (3) for  $\beta = 1$  and established its composition with fractional operators and represented it in terms of various generalized special functions.

In this paper we present some fractional calculus of  $\mathcal{P}$ -transforms. The manuscript is organized as follows: In Section 2 the Mellin and Laplace transforms of the  $\mathcal{P}$ -transform are obtained. In Section 3 we obtain the composition formulae for left-hand sided Saigo fractional operator with  $\mathcal{P}$ -transform, where the composition formulae for the left-hand sided Kober and Riemann Liouville operators with  $\mathcal{P}$ -transform are proved as particular cases. Section 4 contains the composition formulae for the right-hand sided Riemann-Liouville fractional integrals and derivatives with  $\mathcal{P}$ -transforms. The application of the  $\mathcal{P}$ -transform in reaction rate theory in astrophysics and the behaviour of the kernel functions of the  $\mathcal{P}$ -transform have been studied in Section 5.

**2. Mellin and Laplace transform of the  $\mathcal{P}$ -transform**

Here we find the Mellin and Laplace transforms of the  $\mathcal{P}$ -transform.

**2.1. Mellin transform of the  $\mathcal{P}$ -transform**

The Mellin transform of the function is defined as

$$(\mathcal{M}f)(s) = \int_0^\infty x^{s-1} f(x) dx, \quad s \in \mathbb{C}, \quad x > 0, \tag{9}$$

whenever  $(\mathcal{M}f)(s)$  exists.

**THEOREM 1.** *Let  $f \in \mathbb{L}_{\nu,r}(0, \infty)$ ,  $s, \nu \in \mathbb{C}$ ,  $\beta > 0$ ,  $x > 0$  be such that  $\rho > 0$  in the case of a type-1  $\mathcal{P}$ -transform and  $\rho \in \mathbb{R}$ ,  $\rho \neq 0$  in the case of type-2  $\mathcal{P}$ -transform. Then the Mellin transform of a type-1  $\mathcal{P}$ -transform is given by*

$$(\mathcal{M} \mathcal{P}_\nu^{\rho,\beta,\alpha} f)(s) = \frac{\Gamma(s)\Gamma\left(\frac{\nu+\beta s}{\rho}\right)\Gamma\left(\frac{1}{1-\alpha} + 1\right)}{\rho[a(1-\alpha)]^{\frac{\nu+\beta s}{\rho}}\Gamma\left(\frac{1}{1-\alpha} + 1 + \frac{\nu+\beta s}{\rho}\right)} (\mathcal{M}f)(1-s), \tag{10}$$

where  $\Re\left(\frac{\nu+\beta s}{\rho}\right) > 0$  and the Mellin transform of a type-2  $\mathcal{P}$ -transform is given by

$$(\mathcal{M} \mathcal{P}_\nu^{\rho, \beta, \alpha} f)(s) = \frac{\Gamma(s) \Gamma\left(\frac{\nu + \beta s}{\rho}\right) \Gamma\left(\frac{1}{\alpha - 1} - \frac{\nu + \beta s}{\rho}\right)}{|\rho| [a(\alpha - 1)]^{\frac{\nu + \beta s}{\rho}} \Gamma\left(\frac{1}{\alpha - 1}\right)} (\mathcal{M}f)(1 - s), \quad (11)$$

where  $\Re\left(\frac{\nu + \beta s}{\rho}\right) > 0$  and  $\Re\left(\frac{1}{1 - \alpha} - \frac{\nu + \beta s}{\rho}\right) > 0$ .

*P r o o f.* First consider the case of a type-1  $\mathcal{P}$ -transform. Using (1) and (2) taking into account (9), we get

$$\begin{aligned} (\mathcal{M} \mathcal{P}_\nu^{\rho, \beta, \alpha} f)(s) &= \int_0^\infty x^{s-1} (\mathcal{P}_\nu^{\rho, \beta, \alpha} f)(x) dx \\ &= \int_0^\infty x^{s-1} \int_0^\infty D_{\rho, \beta}^{\nu, \alpha}(xt) f(t) dt dx \\ &= \int_0^\infty x^{s-1} \int_0^\infty \int_0^{\left[\frac{1}{a(1-\alpha)}\right]^{\frac{1}{\rho}}} y^{\nu-1} [1 - a(1-\alpha)y^\rho]^{\frac{1}{1-\alpha}} e^{-xty^{-\beta}} dy f(t) dt dx. \end{aligned}$$

Changing the order of integration, we obtain

$$= \int_0^\infty f(t) \int_0^{\left[\frac{1}{a(1-\alpha)}\right]^{\frac{1}{\rho}}} y^{\nu-1} [1 - a(1-\alpha)y^\rho]^{\frac{1}{1-\alpha}} \int_0^\infty x^{s-1} e^{-xty^{-\beta}} dx dy dt.$$

By putting  $xty^{-\beta} = u$  and using the gamma function ([3]1.1(1)), we get

$$= \Gamma(s) \int_0^\infty \frac{1}{t^s} f(t) \int_0^{\left[\frac{1}{a(1-\alpha)}\right]^{\frac{1}{\rho}}} y^{\nu + \beta s - 1} [1 - a(1-\alpha)y^\rho]^{\frac{1}{1-\alpha}} dy dt.$$

Using the substitution  $a(1-\alpha)y^\rho = v$ , we get

$$\begin{aligned} (\mathcal{M} \mathcal{P}_\nu^{\rho, \beta, \alpha} f)(s) &= \frac{\Gamma(s) \Gamma\left(\frac{\nu + \beta s}{\rho}\right) \Gamma\left(\frac{1}{1-\alpha} + 1\right)}{\rho [a(1-\alpha)]^{\frac{\nu + \beta s}{\rho}} \Gamma\left(\frac{1}{1-\alpha} + 1 + \frac{\nu + \beta s}{\rho}\right)} \int_0^\infty t^{-s} f(t) dt \\ &= \frac{\Gamma(s) \Gamma\left(\frac{\nu + \beta s}{\rho}\right) \Gamma\left(\frac{1}{1-\alpha} + 1\right)}{\rho [a(1-\alpha)]^{\frac{\nu + \beta s}{\rho}} \Gamma\left(\frac{1}{1-\alpha} + 1 + \frac{\nu + \beta s}{\rho}\right)} (\mathcal{M}f)(1 - s), \end{aligned}$$

where  $\Re(s) > 0$ ,  $\Re\left(\frac{\nu + \beta s}{\rho}\right) > 0$ . In the case of type-2  $\mathcal{P}$ -transform the proof is exactly in the same way as in the case of type-1  $\mathcal{P}$ -transform. By considering the function given in (3) and by using the type-2 beta integral we obtain the result given in (11) for  $\rho \in \mathbb{R}$ . Hence the theorem follows. ■

COROLLARY 1.1. *If the conditions of Theorem 1 are satisfied and if  $a = 1, \beta = 1$ , then*

$$\lim_{\alpha \rightarrow 1} (\mathcal{M} \mathcal{P}_{\nu}^{\rho, 1, \alpha} f)(s) = (\mathcal{M} \mathcal{K}_{\nu}^{\rho} f)(s) = \frac{\Gamma(s)\Gamma\left(\frac{\nu+s}{\rho}\right)}{|\rho|} (\mathcal{M}f)(1-s). \quad (12)$$

COROLLARY 1.2. *Let  $\rho \in \mathbb{R}, \rho \neq 0, \nu \in \mathbb{C}, a > 0, \beta > 0, \alpha > 1$  and  $s \in \mathbb{C}, \Re(s) > 0$  be such that for  $D_{\rho, \beta}^{\nu, \alpha}(x)$  as defined in (2), then we have*

$$(\mathcal{M}D_{\rho, \beta}^{\nu, \alpha})(s) = \frac{1}{\rho[a(1-\alpha)]^{\frac{\nu+\beta s}{\rho}}} \frac{\Gamma(s)\Gamma\left(\frac{\nu+\beta s}{\rho}\right)\Gamma\left(\frac{1}{1-\alpha} + 1\right)}{\Gamma\left(\frac{1}{1-\alpha} + \frac{\nu+\beta s}{\rho}\right)} \quad (13)$$

where  $\rho > 0$  and  $\Re(\nu + \beta s) > 0$  and for  $D_{\rho, \beta}^{\nu, \alpha}(x)$  is as defined in (3),

$$(\mathcal{M}D_{\rho, \beta}^{\nu, \alpha})(s) = \frac{1}{|\rho|[a(\alpha-1)]^{\frac{\nu+\beta s}{\rho}}} \frac{\Gamma(s)\Gamma\left(\frac{\nu+\beta s}{\rho}\right)\Gamma\left(\frac{1}{\alpha-1} - \frac{\nu+\beta s}{\rho}\right)}{\Gamma\left(\frac{1}{\alpha-1}\right)}, \quad (14)$$

where  $\Re(\nu + \beta s) > 0$  and  $\Re\left(\frac{1}{\alpha-1} - \frac{\nu+\beta s}{\rho}\right) > 0$  when  $\rho > 0$  and  $\Re(\nu + \beta s) < 0$  and  $\Re\left(\frac{1}{\alpha-1} - \frac{\nu+\beta s}{\rho}\right) > 0$  when  $\rho < 0$ .

P r o o f. The result can be obtained from the proof of Theorem 1. ■

COROLLARY 1.3. *If the conditions of Corollary 1.2 are satisfied with  $a = 1, \beta = 1$ , then*

$$\lim_{\alpha \rightarrow 1} (\mathcal{M}D_{\rho, 1}^{\nu, \alpha})(s) = (\mathcal{M}Z_{\rho}^{\nu})(s) = \frac{1}{|\rho|} \Gamma(s)\Gamma\left(\frac{\nu+s}{\rho}\right), \rho \neq 0, \nu \in \mathbb{C}. \quad (15)$$

Now we obtain the  $H$ -function representation of the functions defined in (2) and (3). Let  $m, n, p, q$  be integers such that  $1 \leq m \leq q, 0 \leq n \leq p$ , for  $a_i, b_j \in \mathbb{C}$  and for  $\alpha_i, \beta_j \in \mathbb{R}^+ = (0, \infty), i = 1, 2, \dots, p; j = 1, 2, \dots, q$  the  $H$ -function is defined as a Mellin-Barnes type integral

$$H_{p, q}^{m, n}(z) = H_{p, q}^{m, n}\left(z \mid \begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix}\right) = \frac{1}{2\pi i} \int_{\mathcal{L}} h(s) z^{-s} ds, \quad (16)$$

with

$$h(s) = \frac{\left\{ \prod_{j=1}^m \Gamma(b_j + \beta_j s) \right\} \left\{ \prod_{j=1}^n \Gamma(1 - a_j - \alpha_j s) \right\}}{\left\{ \prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j s) \right\} \left\{ \prod_{j=n+1}^p \Gamma(a_j + \alpha_j s) \right\}}. \quad (17)$$

Here

$$z^{-s} = \exp[-s\{\ln|z| + i \arg z\}], \quad z \neq 0, \quad i = \sqrt{-1}, \quad (18)$$

where  $\ln|z|$  represents the natural logarithm of  $|z|$  and  $\arg z$  is not necessarily the principal value. An empty product in (17), if it occurs, is taken to be one. The contour  $\mathcal{L}$  separates the poles of the gamma functions  $\Gamma(b_j + \beta_j s)$ ,  $j = 1, \dots, m$  from those of the gamma functions  $\Gamma(1 - a_j - \alpha_j s)$ ,  $j = 1, \dots, n$ . The theory of the  $H$ -function are well explained in the books of Mathai [16], Mathai and Saxena ([21], Ch.2), Srivastava, Gupta and Goyal ([24], Ch.1) and Kilbas and Saigo ([11], Ch.1 and Ch.2). It is to be noted that (16) and (17) mean  $H_{p,q}^{m,n}(z)$  in (16) is the inverse Mellin transform of  $h(s)$  in (17).

**THEOREM 2.** Let  $\nu, z \in \mathbb{C}$ ,  $a > 0$ ,  $\rho > 0$ ,  $\beta > 0$  and  $\alpha < 1$ , then

$$D_{\rho,\beta}^{\nu,\alpha}(z) = \frac{\Gamma\left(\frac{1}{1-\alpha} + 1\right)}{\rho[a(1-\alpha)]^{\frac{\nu}{\rho}}} H_{1,2}^{2,0} \left[ [a(1-\alpha)]^{\frac{\beta}{\rho}} z \middle|_{(0,1), \left(\frac{\nu}{\rho}, \frac{\beta}{\rho}\right)}^{\left(\frac{1}{1-\alpha} + 1 + \frac{\nu}{\rho}, \frac{\beta}{\rho}\right)} \right], \quad (19)$$

where  $D_{\rho,\beta}^{\nu,\alpha}(z)$  is as defined in (2).

**P r o o f.** The result can be obtained by taking the inverse Mellin transform of (13) in Corollary 1.2 and using (16) and (17). ■

**THEOREM 3.** Let  $\rho \in \mathbb{R}$ ,  $\nu \in \mathbb{C}$ ,  $a > 0$ ,  $\beta > 0$  and  $\alpha > 1$ . If  $\rho > 0$ , then

$$D_{\rho,\beta}^{\nu,\alpha}(z) = \frac{1}{\rho[a(\alpha-1)]^{\frac{\nu}{\rho}} \Gamma\left(\frac{1}{\alpha-1}\right)} H_{1,2}^{2,1} \left[ [a(\alpha-1)]^{\frac{\beta}{\rho}} z \middle|_{(0,1), \left(\frac{\nu}{\rho}, \frac{\beta}{\rho}\right)}^{\left(1 - \frac{1}{\alpha-1} + \frac{\nu}{\rho}, \frac{\beta}{\rho}\right)} \right]. \quad (20)$$

If  $\rho < 0$ , then

$$D_{\rho,\beta}^{\nu,\alpha}(z) = -\frac{1}{\rho[a(\alpha-1)]^{\frac{\nu}{\rho}} \Gamma\left(\frac{1}{\alpha-1}\right)} H_{1,2}^{2,1} \left[ [a(\alpha-1)]^{\frac{\beta}{\rho}} z \middle|_{(0,1), \left(\frac{1}{\alpha-1} - \frac{\nu}{\rho}, -\frac{\beta}{\rho}\right)}^{\left(1 - \frac{\nu}{\rho}, -\frac{\beta}{\rho}\right)} \right], \quad (21)$$

where  $D_{\rho,\beta}^{\nu,\alpha}(z)$  is as defined in (3).

**P r o o f.** The result can be obtained by taking the inverse Mellin transform of (14) in Corollary 1.2 and using (16) and (17). ■

## 2.2. Laplace transform of a $\mathcal{P}$ -transform

The Laplace transform of a function  $f$  is defined as

$$(Lf)(\omega) = \int_0^{\infty} e^{-\omega t} f(t) dt, \quad (22)$$

whenever  $(Lf)(\omega)$  exists. The following result establishes the Laplace transform of a  $\mathcal{P}$ -transform.

THEOREM 4. Let  $f \in \mathbb{L}_{\nu,r}(0, \infty), \omega, \nu \in \mathbb{C}, \beta > 0$  be such that  $\rho > 0$  in the case of a type-1  $\mathcal{P}$ -transform and  $\rho \in \mathbb{R}$  in the case of type-2  $\mathcal{P}$ -transform. Then the Laplace transform of a type-1  $\mathcal{P}$ -transform is given by

$$\begin{aligned} & \left( L \mathcal{P}_{\nu}^{\rho,\beta,\alpha} f \right) (\omega) \tag{23} \\ &= \frac{\Gamma\left(\frac{1}{1-\alpha} + 1\right)}{\omega \rho [a(1-\alpha)]^{\frac{\nu}{\rho}}} \int_0^{\infty} H_{2,2}^{2,1} \left[ \frac{[a(1-\alpha)]^{\frac{\beta}{\rho}} u}{\omega} \middle|_{(0,1), \left(\frac{\nu}{\rho}, \frac{\beta}{\rho}\right)}^{(0,1), \left(\frac{1}{1-\alpha} + 1 + \frac{\nu}{\rho}, \frac{\beta}{\rho}\right)} \right] f(u) du, \end{aligned}$$

and the Laplace transform of a type-2  $\mathcal{P}$ -transform is given by

$$\begin{aligned} & \left( L \mathcal{P}_{\nu}^{\rho,\beta,\alpha} f \right) (\omega) \tag{24} \\ &= \frac{1}{\omega \rho [a(\alpha-1)]^{\frac{\nu}{\rho}}} \int_0^{\infty} H_{2,2}^{2,2} \left[ \frac{[a(\alpha-1)]^{\frac{\beta}{\rho}} u}{\omega} \middle|_{(0,1), \left(\frac{\nu}{\rho}, \frac{\beta}{\rho}\right)}^{(0,1), \left(1 - \frac{1}{\alpha-1} + \frac{\nu}{\rho}, \frac{\beta}{\rho}\right)} \right] f(u) du \end{aligned}$$

for  $\rho > 0$ , and

$$\begin{aligned} & \left( L \mathcal{P}_{\nu}^{\rho,\beta,\alpha} f \right) (\omega) \tag{25} \\ &= -\frac{1}{\omega \rho [a(\alpha-1)]^{\frac{\nu}{\rho}}} \int_0^{\infty} H_{2,2}^{2,2} \left[ \frac{[a(\alpha-1)]^{\frac{\beta}{\rho}} u}{\omega} \middle|_{(0,1), \left(\frac{1}{\alpha-1} - \frac{\nu}{\rho}, -\frac{\beta}{\rho}\right)}^{(0,1), \left(1 - \frac{\nu}{\rho}, -\frac{\beta}{\rho}\right)} \right] f(u) du \end{aligned}$$

for  $\rho < 0$ .

P r o o f. Using (22), (1), (2) and by taking the inverse Mellin transform of (13), we have

$$\begin{aligned} & \left( L \mathcal{P}_{\nu}^{\rho,\beta,\alpha} f \right) (\omega) = \int_0^{\infty} e^{-\omega t} \int_0^{\infty} D_{\rho,\beta}^{\nu,\alpha}(tu) f(u) du dt = \frac{\Gamma\left(\frac{1}{1-\alpha} + 1\right)}{\rho [a(1-\alpha)]^{\frac{\nu}{\rho}}} \\ & \times \frac{1}{2\pi i} \int_0^{\infty} e^{-\omega t} \int_0^{\infty} \int_{\mathcal{L}} \frac{\Gamma(s)\Gamma\left(\frac{\nu+\beta s}{\rho}\right)}{\Gamma\left(\frac{1}{1-\alpha} + 1 + \frac{\nu+\beta s}{\rho}\right)} \{[a(1-\alpha)]^{\frac{\beta}{\rho}} ut\}^{-s} ds f(u) du dt. \end{aligned}$$

Changing the order of integration which is possible because of the uniform continuity of the integral and using gamma function, we get

$$\begin{aligned} \left( L \mathcal{P}_{\nu}^{\rho,\beta,\alpha} f \right) (\omega) &= \frac{\Gamma\left(\frac{1}{1-\alpha} + 1\right)}{\rho [a(1-\alpha)]^{\frac{\nu}{\rho}}} \frac{1}{2\pi i} \int_0^{\infty} \int_{\mathcal{L}} \frac{\Gamma(s)\Gamma\left(\frac{\nu+\beta s}{\rho}\right)}{\Gamma\left(\frac{1}{1-\alpha} + 1 + \frac{\nu+\beta s}{\rho}\right)} \\ & \times \{[a(1-\alpha)]^{\frac{\beta}{\rho}} u\}^{-s} \int_0^{\infty} e^{-\omega t} t^{-s} dt ds f(u) du \end{aligned}$$

$$= \frac{\Gamma\left(\frac{1}{1-\alpha}+1\right)}{\omega\rho[a(1-\alpha)]^{\frac{\nu}{\rho}}} \frac{1}{2\pi i} \int_0^\infty \int_{\mathcal{L}} \frac{\Gamma(s)\gamma(1-s)\Gamma\left(\frac{\nu+\beta s}{\rho}\right)}{\Gamma\left(\frac{1}{1-\alpha}+1+\frac{\nu+\beta s}{\rho}\right)} \left\{ \frac{[a(1-\alpha)]^{\frac{\beta}{\rho}} u}{\omega} \right\}^{-s} ds f(u) du,$$

where  $\Re(s) > 0, \Re(1-s) > 0, \Re\left(\frac{\nu+\beta s}{\rho}\right) > 0$ , which yields the result in (23). Proceeding exactly in the same way as in the case of type-1  $\mathcal{P}$ -transform and using the inverse Mellin transform of (14) the results in (24) and (25) follows. Hence the theorem follows.  $\blacksquare$

**COROLLARY 4.1.** *If the conditions of Theorem 4 are satisfied with  $a = 1, \beta = 1$ , then the Laplace transform of the Krätzel transform is given by*

$$\lim_{\alpha \rightarrow 1} \left( L \mathcal{P}_\nu^{\rho, \beta, \alpha} f \right) (s) = (L \mathcal{K}_\nu^\rho f)(x) = \frac{1}{\omega\rho} \int_0^\infty H_{1,2}^{2,1} \left[ \frac{u}{\omega} \middle| \begin{matrix} (0,1) \\ (0,1), (\frac{\nu}{\rho}, \frac{1}{\rho}) \end{matrix} \right] f(u) du \tag{26}$$

for  $\rho > 0$  and

$$\begin{aligned} \lim_{\alpha \rightarrow 1} \left( L \mathcal{P}_\nu^{\rho, \beta, \alpha} f \right) (s) &= (L \mathcal{K}_\nu^\rho f)(x) \tag{27} \\ &= -\frac{1}{\omega\rho} \int_0^\infty H_{2,1}^{1,2} \left[ \frac{u}{\omega} \middle| \begin{matrix} (0,1), (1-\frac{\nu}{\rho}, -\frac{1}{\rho}) \\ (0,1) \end{matrix} \right] f(u) du, \end{aligned}$$

for  $\rho > 0$ .

### 3. Saigo fractional operator and $\mathcal{P}$ -transforms

The fractional integral operator introduced by Saigo [22] for  $\Re(\gamma) > 0$  with  $\gamma, \delta, \eta \in \mathbb{C}$  is defined as

$$I_{0+}^{\gamma, \delta, \eta} f(x) = \frac{x^{-\gamma-\delta}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} {}_2F_1(\gamma+\delta, -\eta; \gamma; 1-\frac{t}{x}) f(t) dt, \tag{28}$$

and

$$I_-^{\gamma, \delta, \eta} f(x) = \frac{1}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\gamma-\delta} {}_2F_1(\gamma+\delta, -\eta; \gamma; 1-\frac{x}{t}) f(t) dt, \tag{29}$$

where  ${}_2F_1(\alpha, \beta; \gamma; z)$  is the Gauss hypergeometric series defined for  $\alpha, \beta, \gamma \in \mathbb{C}, \gamma \neq 0, -1, -2, \dots$  by the series ([3], 2.1(2))

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{k=0}^\infty \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{z^k}{k!}, \tag{30}$$

where  $(\alpha)_k, (\beta)_k$  and  $(\gamma)_k$  is the Pochhammer symbol defined for  $a \in \mathbb{C}$  by

$$(a)_0 = 1, (a)_k = a(a+1) \cdots (a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}, k = 1, 2, \dots, a \neq 0 \tag{31}$$



whenever  $\Gamma(a)$  exists. The series in (30) is absolutely convergent for  $|z| < 1, \Re(\gamma - \alpha - \beta) > 0$ . Moreover,

$${}_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}, \quad \Re(\gamma - \alpha - \beta) > 0 \quad (32)$$

When  $\delta = -\gamma$ , the operator in (28) coincides with the classical Liouville fractional integrals ([23])

$$I_{0+}^{\gamma, -\gamma, \eta} f(x) = (\mathcal{I}_{0+}^{\gamma} f)x = \frac{1}{\Gamma(\gamma)} \int_0^x (x - t)^{\gamma-1} f(t) dt, \quad \Re(\gamma) > 0 \quad (33)$$

and

$$I_{-}^{\gamma, -\gamma, \eta} f(x) = (\mathcal{I}_{-}^{\gamma} f)x = \frac{1}{\Gamma(\gamma)} \int_x^{\infty} (t - x)^{\gamma-1} f(t) dt, \quad \Re(\gamma) > 0, \quad (34)$$

while when  $\delta = 0_{-}$ , we get the Erdélyi-Kober fractional integrals

$$I_{0+}^{\gamma, 0, \eta} f(x) = (\mathcal{I}_{\eta, \gamma}^{+} f)x = \frac{x^{-\gamma-\eta}}{\Gamma(\gamma)} \int_0^x (x - t)^{\gamma-1} t^{\eta} f(t) dt, \quad x \in \mathbb{R}_{+} \quad (35)$$

and

$$I_{-}^{\gamma, 0, \eta} f(x) = (K_{\eta, \gamma}^{-} f)(x) = (I_{-; 1, \eta}^{\gamma} f)(x) = \frac{x^{\eta}}{\Gamma(\gamma)} \int_x^{\infty} (t - x)^{\gamma-1} t^{-\gamma-\eta} f(t) dt, \quad (36)$$

$x \in \mathbb{R}_{+}$ . Kilbas and Sebastian [12] studied the composition of some of the above generalized fractional integrals with the Bessel function of the first kind. Various generalized fractional integral and differential operators, including as special cases (28), (33)-(34), (35)-(36) and others, and their properties can be seen in the book of Kiryakova [13].

The main result in this section is obtained by using the left-hand sided Saigo operator of a power function which is established by the following lemma.

LEMMA 1. *Let  $\gamma, \delta, \eta, \lambda \in \mathbb{C}$  be such that  $\Re(\gamma) > 0, \Re(\lambda + 1) > 0$  and  $\Re(\lambda + 1 + \eta - \delta) > 0$ , then we have*

$$I_{0+}^{\gamma, \delta, \eta} x^{\lambda} = \frac{\Gamma(\lambda + 1)\Gamma(\lambda + 1 + \eta - \delta)}{\Gamma(\lambda + 1 - \delta)\Gamma(\lambda + 1 + \gamma + \eta)} x^{\lambda - \delta}. \quad (37)$$

The proof can be found in many papers dedicated to the Saigo operator.

COROLLARY 4.2. *If the conditions of Lemma 1 are satisfied with  $\delta = 0_{-}$ , we get the left-hand sided Erdélyi-Kober fractional operator of a power function, as*

$$I_{0+}^{\gamma, 0, \eta} x^{\lambda} = \mathcal{I}_{\eta, \gamma}^{+} x^{\lambda} = \frac{\Gamma(\lambda + 1 + \eta)}{\Gamma(\lambda + 1 + \gamma + \eta)} x^{\lambda}, \quad (38)$$

and with  $\delta = -\gamma$  we get the left-hand sided Riemann-Liouville fractional operator of a power function as

$$I_{0+}^{\gamma, -\gamma, \eta} x^\lambda = \mathcal{I}_{0+}^\gamma x^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 + \gamma)} x^{\lambda + \gamma}. \quad (39)$$

Now we obtain a composition formula for the left-hand sided Saigo fractional integral with the  $\mathcal{P}$ -transform.

**THEOREM 5.** Let  $f \in \mathbb{L}_{\nu, r}(0, \infty)$ ,  $\gamma, \nu, \eta \in \mathbb{C}$ ,  $\beta > 0$ ,  $\Re(\gamma) > 0$  be such that  $\rho > 0$  in case of type-1  $\mathcal{P}$ -transform and  $\rho \in \mathbb{R}$  in case of type-2  $\mathcal{P}$ -transform, then for type-1  $\mathcal{P}$ -transform we have

$$\begin{aligned} \left( I_{0+}^{\gamma, \delta, \eta} \mathcal{P}_\nu^{\rho, \beta, \alpha} f \right) (x) &= \frac{\Gamma\left(\frac{1}{1-\alpha} + 1\right)}{x^\delta \rho [a(1-\alpha)]^{\frac{\nu}{\rho}}} \\ &\times \int_0^\infty H_{3,4}^{2,2} \left[ [a(1-\alpha)]^{\frac{\beta}{\rho}} ux \middle|_{(0,1), \left(\frac{\nu}{\rho}, \frac{\beta}{\rho}\right), (\delta, 1), (-\gamma-\eta, 1)}^{(0,1), (\delta-\eta, 1), \left(\frac{1}{1-\alpha} + 1 + \frac{\nu}{\rho}, \frac{\beta}{\rho}\right)} \right] f(u) du, \end{aligned} \quad (40)$$

and in the case of type-2  $\mathcal{P}$ -transform, we get

$$\begin{aligned} \left( I_{0+}^{\gamma, \delta, \eta} \mathcal{P}_\nu^{\rho, \beta, \alpha} f \right) (x) &= \frac{1}{x^\delta \rho [a(\alpha-1)]^{\frac{\nu}{\rho}} \Gamma\left(\frac{1}{\alpha-1}\right)} \\ &\times \int_0^\infty H_{3,4}^{2,3} \left[ [a(\alpha-1)]^{\frac{\beta}{\rho}} ux \middle|_{(0,1), \left(\frac{\nu}{\rho}, \frac{\beta}{\rho}\right), (\delta, 1), (-\gamma-\eta, 1)}^{(0,1), (\delta-\eta, 1), \left(1 - \frac{1}{\alpha-1} + \frac{\nu}{\rho}, \frac{\beta}{\rho}\right)} \right] f(u) du \end{aligned} \quad (41)$$

for  $\rho > 0$  and

$$\begin{aligned} \left( I_{0+}^{\gamma, \delta, \eta} \mathcal{P}_\nu^{\rho, \beta, \alpha} f \right) (x) &= -\frac{1}{x^\delta \rho [a(1-\alpha)]^{\frac{\nu}{\rho}} \Gamma\left(\frac{1}{\alpha-1}\right)} \\ &\times \int_0^\infty H_{3,4}^{2,3} \left[ [a(\alpha-1)]^{\frac{\beta}{\rho}} ux \middle|_{(0,1), \left(\frac{1}{\alpha-1} - \frac{\nu}{\rho}, -\frac{\beta}{\rho}\right), (\delta, 1), (-\gamma-\eta, 1)}^{(0,1), (\delta-\eta, 1), \left(1 - \frac{\nu}{\rho}, -\frac{\beta}{\rho}\right)} \right] f(u) du. \end{aligned} \quad (42)$$

**P r o o f.** Using (28) and taking into account (1) and (2), and applying the inverse Mellin transform of (13), we have

$$\begin{aligned} & \left( I_{0+}^{\gamma, \delta, \eta} \mathcal{P}_\nu^{\rho, \beta, \alpha} f \right) (x) = \frac{x^{-\gamma-\delta}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} \\ & \times {}_2F_1\left(\gamma + \delta, -\eta; \gamma; 1 - \frac{t}{x}\right) \int_0^\infty D_{\rho, \beta}^{\nu, \alpha}(tu) f(u) du dt \\ & = \frac{x^{-\gamma-\delta} \Gamma\left(\frac{1}{1-\alpha} + 1\right)}{\Gamma(\gamma) \rho [a(1-\alpha)]^{\frac{\nu}{\rho}}} \int_0^x (x-t)^{\gamma-1} {}_2F_1\left(\gamma + \delta, -\eta; \gamma; 1 - \frac{t}{x}\right) \\ & \times \int_0^\infty \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s) \Gamma\left(\frac{\nu+\beta s}{\rho}\right)}{\Gamma\left(\frac{1}{1-\alpha} + 1 + \frac{\nu+\beta s}{\rho}\right)} \{[a(1-\alpha)]^{\frac{\beta}{\rho}} ut\}^{-s} ds f(u) du dt, \end{aligned}$$

where  $\Re(s) > 0, \Re\left(\frac{\nu+\beta s}{\rho}\right) > 0$ . Changing the order of integration due to the uniform continuity of the integral and using Lemma 1, we get

$$\begin{aligned} & \left( I_{0+}^{\gamma, \delta, \eta} \mathcal{P}_\nu^{\rho, \beta, \alpha} f \right) (x) = \frac{\Gamma\left(\frac{1}{1-\alpha} + 1\right)}{\rho [a(1-\alpha)]^{\frac{\nu}{\rho}}} \int_0^\infty \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s) \Gamma\left(\frac{\nu+\beta s}{\rho}\right)}{\Gamma\left(\frac{1}{1-\alpha} + 1 + \frac{\nu+\beta s}{\rho}\right)} \\ & \times \{[a(1-\alpha)]^{\frac{\beta}{\rho}} u\}^{-s} f(u) du \left( I_{0+}^{\gamma, \delta, \eta} t^{-s} \right) (x) ds \\ & = \frac{\Gamma\left(\frac{1}{1-\alpha} + 1\right)}{x^\delta \rho [a(1-\alpha)]^{\frac{\nu}{\rho}}} \int_0^\infty \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s) \Gamma\left(\frac{\nu+\beta s}{\rho}\right)}{\Gamma\left(\frac{1}{1-\alpha} + 1 + \frac{\nu+\beta s}{\rho}\right)} \\ & \times \frac{\Gamma(1-s) \Gamma(1+\eta-\delta-s)}{\Gamma(1-\delta-s) \Gamma(1+\gamma+\eta-s)} \{[a(1-\alpha)]^{\frac{\beta}{\rho}} ux\}^{-s} ds f(u) du, \end{aligned}$$

where  $\Re(s) > 0, \Re(1-s) > 0, \Re(\nu+\beta s) > 0, \Re(1+\eta-\delta-s) > 0$  which yields the result in (40) for type-1  $\mathcal{P}$ -transform. The result for type-2  $\mathcal{P}$ -transform can also be proved similarly by considering the inverse Mellin transform of (14) and using (16). ■

**COROLLARY 5.1.** *If the conditions of Theorem 5 are satisfied with  $\delta = 0_-$ , then the left-hand sided Erdélyi-Kober fractional operator in the case of type-1  $\mathcal{P}$ -transform is*

$$\begin{aligned} & \left( I_{0+}^{\gamma, -\gamma, \eta} \mathcal{P}_\nu^{\rho, \beta, \alpha} f \right) (x) = \left( \mathcal{I}_{\eta, \gamma}^+ \mathcal{P}_\nu^{\rho, \beta, \alpha} f \right) (x) = \frac{\Gamma\left(\frac{1}{1-\alpha} + 1\right)}{\rho [a(1-\alpha)]^{\frac{\nu}{\rho}}} \\ & \times \int_0^\infty H_{2,3}^{2,1} \left[ [a(1-\alpha)]^{\frac{\beta}{\rho}} ux \Big|_{(0,1), \left(\frac{\nu}{\rho}, \frac{\beta}{\rho}\right), (-\gamma-\eta, 1)}^{(-\eta, 1), \left(\frac{1}{1-\alpha} + 1 + \frac{\nu}{\rho}, \frac{\beta}{\rho}\right)} \right] f(u) du, \end{aligned} \tag{43}$$

and in the case of type-2  $\mathcal{P}$ -transform we get

$$\begin{aligned} \left( I_{0+}^{\gamma, \delta, \eta} \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) &= \left( \mathcal{I}_{\eta, \gamma}^+ \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) = \frac{1}{\rho [a(\alpha - 1)]^{\frac{\nu}{\rho}} \Gamma \left( \frac{1}{\alpha - 1} \right)} \\ &\times \int_0^{\infty} H_{2,3}^{2,2} \left[ [a(\alpha - 1)]^{\frac{\beta}{\rho}} u x \middle|_{(0,1), \left( \frac{\nu}{\rho}, \frac{\beta}{\rho} \right), (-\gamma - \eta, 1)}^{(-\eta, 1), \left( 1 - \frac{1}{\alpha - 1} + \frac{\nu}{\rho}, \frac{\beta}{\rho} \right)} \right] f(u) du \end{aligned} \quad (44)$$

for  $\rho > 0$ , and

$$\begin{aligned} \left( I_{0+}^{\gamma, \delta, \eta} \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) &= \left( \mathcal{I}_{\eta, \gamma}^+ \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) = -\frac{1}{\rho [a(1 - \alpha)]^{\frac{\nu}{\rho}} \Gamma \left( \frac{1}{\alpha - 1} \right)} \\ &\times \int_0^{\infty} H_{2,3}^{2,2} \left[ [a(\alpha - 1)]^{\frac{\beta}{\rho}} u x \middle|_{(0,1), \left( \frac{1}{\alpha - 1} - \frac{\nu}{\rho}, -\frac{\beta}{\rho} \right), (-\gamma - \eta, 1)}^{(-\eta, 1), \left( 1 - \frac{\nu}{\rho}, -\frac{\beta}{\rho} \right)} \right] f(u) du \end{aligned} \quad (45)$$

for  $\rho < 0$ .

**COROLLARY 5.2.** *If the conditions of Theorem 5 are satisfied with  $\delta = -\gamma$ , then the left-hand sided Riemann-Liouville fractional operator in the case of type-1  $\mathcal{P}$ -transform is*

$$\begin{aligned} \left( I_{0+}^{\gamma, -\gamma, \eta} \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) &= \left( \mathcal{I}_{0+}^{\gamma} \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) = \frac{x^{\gamma} \Gamma \left( \frac{1}{1 - \alpha} + 1 \right)}{\rho [a(1 - \alpha)]^{\frac{\nu}{\rho}}} \\ &\times \int_0^{\infty} H_{2,3}^{2,1} \left[ [a(1 - \alpha)]^{\frac{\beta}{\rho}} u x \middle|_{(0,1), \left( \frac{\nu}{\rho}, \frac{\beta}{\rho} \right), (-\gamma, 1)}^{(0,1), \left( \frac{1}{1 - \alpha} + 1 + \frac{\nu}{\rho}, \frac{\beta}{\rho} \right)} \right] f(u) du, \end{aligned} \quad (46)$$

and in the case of type-2  $\mathcal{P}$ -transform, we get

$$\begin{aligned} \left( I_{0+}^{\gamma, \delta, \eta} \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) &= \left( \mathcal{I}_{0+}^{\gamma} \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) = \frac{x^{\gamma}}{\rho [a(\alpha - 1)]^{\frac{\nu}{\rho}} \Gamma \left( \frac{1}{\alpha - 1} \right)} \\ &\times \int_0^{\infty} H_{2,3}^{2,2} \left[ [a(\alpha - 1)]^{\frac{\beta}{\rho}} u x \middle|_{(0,1), \left( \frac{\nu}{\rho}, \frac{\beta}{\rho} \right), (-\gamma, 1)}^{(0,1), \left( 1 - \frac{1}{\alpha - 1} + \frac{\nu}{\rho}, \frac{\beta}{\rho} \right)} \right] f(u) du \end{aligned} \quad (47)$$

for  $\rho > 0$ , and

$$\begin{aligned} \left( I_{0+}^{\gamma, \delta, \eta} \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) &= \left( \mathcal{I}_{0+}^{\gamma} \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) = - \frac{x^{\gamma}}{\rho [a(1-\alpha)]^{\frac{\nu}{\rho}} \Gamma\left(\frac{1}{\alpha-1}\right)} \\ &\times \int_0^{\infty} H_{2,3}^{2,2} \left[ [a(\alpha-1)]^{\frac{\beta}{\rho}} ux \Big|_{(0,1), \left(\frac{1}{\alpha-1} - \frac{\nu}{\rho}, -\frac{\beta}{\rho}\right)}^{(0,1), \left(1 - \frac{\nu}{\rho}, -\frac{\beta}{\rho}\right)} \right] f(u) du \end{aligned} \tag{48}$$

for  $\rho < 0$ .

COROLLARY 5.3. *If the conditions of Theorem 5 are satisfied with  $a = 1, \beta = 1$ , then*

$$\begin{aligned} \lim_{\alpha \rightarrow 1} \left( I_{0+}^{\gamma, \delta, \eta} \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) &= \left( I_{0+}^{\gamma, \delta, \eta} \mathcal{K}_{\nu}^{\rho} f \right) (x) \\ &= \frac{1}{x^{\delta} \rho} \int_0^{\infty} H_{2,4}^{2,2} \left[ ux \Big|_{(0,1), \left(\frac{\nu}{\rho}, \frac{\beta}{\rho}\right), (\delta, 1), (-\gamma - \eta, 1)}^{(0,1), (\delta - \eta, 1)} \right] f(u) du \end{aligned} \tag{49}$$

for  $\rho > 0$ , and

$$\begin{aligned} \lim_{\alpha \rightarrow 1} \left( I_{0+}^{\gamma, \delta, \eta} \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) &= \left( I_{0+}^{\gamma, \delta, \eta} \mathcal{K}_{\nu}^{\rho} f \right) (x) \\ &= - \frac{1}{x^{\delta} \rho} \int_0^{\infty} H_{3,3}^{1,3} \left[ ux \Big|_{(0,1), (\delta, 1), (-\gamma - \eta, 1)}^{(0,1), (\delta - \eta, 1), \left(1 - \frac{\nu}{\rho}, -\frac{\beta}{\rho}\right)} \right] f(u) du \end{aligned} \tag{50}$$

for  $\rho < 0$ .

COROLLARY 5.4. *If the conditions of Theorem 5 are satisfied with  $\delta = 0_-, a = 1, \beta = 1$ , then*

$$\lim_{\alpha \rightarrow 1} \left( \mathcal{I}_{\eta, \gamma}^+ \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) \tag{51}$$

$$= \left( \mathcal{I}_{\eta, \gamma}^+ \mathcal{K}_{\nu}^{\rho} f \right) (x) = \frac{1}{\rho} \int_0^{\infty} H_{1,3}^{2,1} \left[ ux \Big|_{(0,1), \left(\frac{\nu}{\rho}, \frac{\beta}{\rho}\right), (-\gamma - \eta, 1)}^{(-\eta, 1)} \right] f(u) du$$

for  $\rho > 0$ , and

$$\lim_{\alpha \rightarrow 1} \left( \mathcal{I}_{\eta, \gamma}^+ \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) \tag{52}$$

$$= \left( \mathcal{I}_{\eta, \gamma}^+ \mathcal{K}_{\nu}^{\rho} f \right) (x) = - \frac{1}{\rho} \int_0^{\infty} H_{2,2}^{1,2} \left[ ux \Big|_{(0,1), (-\gamma - \eta, 1)}^{(-\eta, 1), \left(1 - \frac{\nu}{\rho}, -\frac{\beta}{\rho}\right)} \right] f(u) du$$

for  $\rho < 0$ .

COROLLARY 5.5. *If the conditions of Theorem 5 are satisfied with  $\delta = -\gamma, a = 1, \beta = 1$ , then*

$$\lim_{\alpha \rightarrow 1} \left( \mathcal{I}_{0+}^{\gamma} \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) \quad (53)$$

$$= (\mathcal{I}_{0+}^{\gamma} \mathcal{K}_{\nu}^{\rho} f)(x) = \frac{x^{\gamma}}{\rho} \int_0^{\infty} H_{1,3}^{2,1} \left[ ux \middle| \begin{matrix} (0,1) \\ (0,1), (\frac{\nu}{\rho}, \frac{\beta}{\rho}), (-\gamma, 1) \end{matrix} \right] f(u) du$$

for  $\rho > 0$ , and

$$\lim_{\alpha \rightarrow 1} \left( \mathcal{I}_{0+}^{\gamma} \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) \quad (54)$$

$$= (\mathcal{I}_{0+}^{\gamma} \mathcal{K}_{\nu}^{\rho} f)(x) = -\frac{x^{\gamma}}{\rho} \int_0^{\infty} H_{2,2}^{1,2} \left[ ux \middle| \begin{matrix} (0,1), (1-\frac{\nu}{\rho}, -\frac{\beta}{\rho}) \\ (0,1), (-\gamma, 1) \end{matrix} \right] f(u) du$$

for  $\rho < 0$ .

#### 4. Right-hand sided Riemann-Liouville fractional operators and $\mathcal{P}$ -transform

In this section we present composition formulas of the  $\mathcal{P}$ -transform with the right-hand sided Riemann-Liouville fractional integral  $\mathcal{I}_{-}^{\gamma}$  defined in (34) and the differential operator  $\mathcal{D}_{-}^{\gamma}$  of complex order  $\gamma \in \mathbb{C}$  defined for  $x > 0$  by ([23], Section 5.1):

$$(\mathcal{D}_{-}^{\gamma} f)x = \left( -\frac{d}{dx} \right)^n (\mathcal{I}_{-}^{n-\gamma} f)(x), \quad x > 0, n = [\Re(\gamma)] + 1 \quad (55)$$

respectively with  $\gamma \in \mathbb{C}$  and  $\Re(\gamma) > 0$ , where  $[\Re(\gamma)]$  is the integral part of  $\Re(\gamma)$ .

The first statement is given by the following result.

THEOREM 6. *Let  $f \in \mathbb{L}_{\nu, r}(0, \infty), \gamma, \nu \in \mathbb{C}, \beta > 0, \Re(\gamma) > 0, \alpha > 1, x > 0$  be such that and  $\rho > 0$  in case of type-1  $\mathcal{P}$ -transform and  $\rho \in \mathbb{R}$  in case of type-2  $\mathcal{P}$ -transform. Then*

$$\left( \mathcal{I}_{-}^{\gamma} \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) = (\mathcal{P}_{\nu+\beta\gamma}^{\rho, \beta, \alpha} x^{-\gamma} f)(x). \quad (56)$$

P r o o f. Consider first the case of type-1  $\mathcal{P}$ -transform. Using (34), (1) and (2), we have

$$\begin{aligned} \left( \mathcal{I}_{-}^{\gamma} \mathcal{P}_{\nu}^{\rho, \beta, \alpha} f \right) (x) &= \frac{1}{\Gamma(\gamma)} \int_x^{\infty} (t-x)^{\gamma-1} \int_0^{\infty} D_{\rho, \beta}^{\nu, \alpha}(tu) f(u) dudt \\ &= \frac{1}{\Gamma(\gamma)} \int_x^{\infty} (t-x)^{\gamma-1} \int_0^{\infty} f(u) \\ &\quad \times \int_0^{\left[ \frac{1}{a(1-\alpha)} \right]^{\frac{1}{\rho}}} y^{\nu-1} [1 - a(1-\alpha)y^{\rho}]^{\frac{1}{1-\alpha}} e^{-tuy^{-\beta}} dy dudt. \end{aligned}$$

Changing the order of integration which is possible because of the uniform continuity of the integral, we get

$$\begin{aligned} (\mathcal{I}_-^\gamma \mathcal{P}_\nu^{\rho,\beta,\alpha} f)(x) &= \int_0^\infty f(u) \int_0^{\left[\frac{1}{a(1-\alpha)}\right]^{\frac{1}{\rho}}} y^{\nu-1} [1 - a(1-\alpha)y^\rho]^{\frac{1}{1-\alpha}} \\ &\times \frac{1}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} e^{-tuy^{-\beta}} dt dy du \\ &= \int_0^\infty f(u) \int_0^{\left[\frac{1}{a(1-\alpha)}\right]^{\frac{1}{\rho}}} y^{\nu-1} [1 - a(1-\alpha)y^\rho]^{\frac{1}{1-\alpha}} (\mathcal{I}_-^\gamma (e^{-tuy^{-\beta}}))(x) dy du. \end{aligned}$$

Using the formula ([23], (5.20))

$$(\mathcal{I}_-^\gamma e^{-\lambda t})(x) = \lambda^{-\gamma} e^{-\lambda x}, \quad \Re(\gamma) > 0, \Re(\lambda) > 0, \tag{57}$$

we have

$$\begin{aligned} (\mathcal{I}_-^\gamma \mathcal{P}_\nu^{\rho,\beta,\alpha} f)(x) &= \int_0^\infty f(u) u^{-\gamma} \int_0^{\left[\frac{1}{a(1-\alpha)}\right]^{\frac{1}{\rho}}} y^{\nu+\beta\gamma-1} [1 - a(1-\alpha)y^\rho]^{\frac{1}{1-\alpha}} \\ &\times e^{-xuy^{-\beta}} dy du = \int_0^\infty f(u) u^{-\gamma} D_{\rho,\beta}^{\nu+\beta\gamma,\alpha}(xu) du = (\mathcal{P}_{\nu+\beta\gamma}^{\rho,\beta,\alpha} x^{-\gamma} f)(x). \end{aligned}$$

According to (34) this yields the result in (56). The result for type-2  $\mathcal{P}$ -transform can be proved similarly. This completes the proof of the theorem. ■

**COROLLARY 6.1.** *If the conditions of Theorem 6 are satisfied with  $a = 1, \beta = 1$ , then*

$$\lim_{\alpha \rightarrow 1} (\mathcal{I}_-^\gamma \mathcal{P}_\nu^{\rho,1,\alpha} f)(x) = (\mathcal{I}_-^\gamma \mathcal{K}_\nu^\rho f)(x) = (\mathcal{K}_{\nu+\gamma}^\rho x^{-\gamma} f)(x). \tag{58}$$

The following theorem gives the composition formula of fractional derivative (55) with  $\mathcal{P}$ -transform (1).

**THEOREM 7.** *Let  $\gamma, \nu \in \mathbb{C}, \Re(\gamma) > 0, \rho > 0, \beta > 0$  and  $\alpha \geq 1$ . Then*

$$(\mathcal{D}_-^\gamma \mathcal{P}_\nu^{\rho,\beta,\alpha} f)(x) = (\mathcal{P}_{\nu-\beta\gamma}^{\rho,\beta,\alpha} x^\gamma f)(x). \tag{59}$$

**P r o o f.** Let  $n = [\Re(\gamma)] + 1$ . Using (55) and (1) and applying (56), with  $\gamma$  replaced by  $n - \gamma$ , we have

$$\begin{aligned} (\mathcal{D}_-^\gamma \mathcal{P}_\nu^{\rho,\beta,\alpha} f)(x) &= \left(-\frac{d}{dx}\right)^n (\mathcal{I}_-^{n-\gamma} \mathcal{P}_\nu^{\rho,\beta,\alpha} f)(x) = \left(-\frac{d}{dx}\right)^n \\ &\times (\mathcal{P}_{\nu+\beta n-\beta\gamma}^{\rho,\beta,\alpha} x^{-(n-\gamma)} f)(x) = \left(-\frac{d}{dx}\right)^n \int_0^\infty f(u) u^{-(n-\gamma)} D_{\rho,\beta}^{\nu+\beta n-\beta\gamma,\alpha}(xu) du \end{aligned}$$

$$\begin{aligned}
 &= \left(-\frac{d}{dx}\right)^n \int_0^\infty f(u)u^{-(n-\gamma)} \\
 &\quad \times \int_0^{\left[\frac{1}{a(1-\alpha)}\right]^{\frac{1}{\rho}}} y^{\nu+\beta n-\beta\gamma-1} [1-a(1-\alpha)y^\rho]^{\frac{1}{1-\alpha}} e^{-xuy^{-\beta}} dy du.
 \end{aligned}$$

Putting the differentiation inside the integral, we obtain

$$\begin{aligned}
 (\mathcal{D}_-^\gamma \mathcal{P}_\nu^{\rho,\beta,\alpha} f)(x) &= \int_0^\infty f(u)u^\gamma \int_0^{\left[\frac{1}{a(1-\alpha)}\right]^{\frac{1}{\rho}}} y^{\nu-\beta\gamma-1} [1-a(1-\alpha)y^\rho]^{\frac{1}{1-\alpha}} \\
 &\quad \times e^{-xuy^{-\beta}} dy du = \int_0^\infty D_{\rho,\beta}^{\nu-\beta\gamma,\alpha}(xu)u^\gamma f(u)du.
 \end{aligned}$$

In accordance with (1), this yields the result in (59) and hence the theorem. ■

**COROLLARY 7.1.** *If the conditions of Theorem 7 are satisfied and if  $a = 1, \beta = 1$ , then*

$$\lim_{\alpha \rightarrow 1} (\mathcal{D}_-^\gamma \mathcal{P}_\nu^{\rho,1,\alpha} f)(x) = (\mathcal{D}_-^\gamma \mathcal{K}_\nu^\rho f)(x) = (\mathcal{K}_{\nu-\gamma}^\rho x^\gamma f)(x). \tag{60}$$

### 5. Application of $\mathcal{P}$ -transform in Astrophysics

The thermonuclear reaction rate  $r_{ij}$  in the non-degenerate environment with non-resonant thermonuclear reactions between the particles of type  $i$  and  $j$  is given by [18],

$$r_{ij} = n_i n_j \left(\frac{8}{\pi\mu}\right)^{\frac{1}{2}} \sum_{\nu=0}^2 \left(\frac{1}{kT}\right)^{-\nu+\frac{1}{2}} \frac{S^{(\nu)}(0)}{\nu!} \times \int_0^\infty y^\nu e^{-y-xy^{-\frac{1}{2}}} dy \tag{61}$$

where  $y = \frac{E}{kT}$  and  $b = 2\pi \left(\frac{\mu}{2kT}\right)^{\frac{1}{2}} \frac{z_i z_j e^2}{\hbar}$ .

The reaction rate probability integral for non-resonant thermonuclear reactions in the Maxwell-Boltzmann case is given by

$$I_1(\nu, 1, b, \frac{1}{2}) = \int_0^\infty y^\nu e^{-y-xy^{-\frac{1}{2}}} dy, \quad x > 0. \tag{62}$$

If there appears a cut-off of the high energy tail in the Maxwell-Boltzmann distribution function then the integral is

$$I_2^{(d)}(\nu, 1, b, \frac{1}{2}) = \int_0^d y^\nu e^{-y-xy^{-\frac{1}{2}}} dy, \quad x > 0, \quad d < \infty. \tag{63}$$

The evaluation of the above integrals in closed forms and for detailed physics, see [18, 19]. The particular case of the kernel functions of the type-1 and



type-2  $\mathcal{P}$ -transforms given in (2) and (3) are the extended non-resonant thermonuclear function in reaction rate theory. As  $\nu - 1$  is replaced by  $\nu, a = 1, \beta = \frac{1}{2}, \rho = 1$  in (2) and (3) we get the extended non-resonant thermonuclear reaction rate probability integral in the Maxwell-Boltzmann case given by

$$I_{1\alpha} = \int_0^\infty y^\nu [1 + (\alpha - 1)y]^{-\frac{1}{\alpha-1}} e^{-xy^{-\frac{1}{2}}} dy, \alpha > 1 \tag{64}$$

and cut-off case given by

$$I_{2\alpha}^d = \int_0^d y^\nu [1 - (1 - \alpha)y]^{-\frac{1}{1-\alpha}} e^{-xy^{-\frac{1}{2}}} dy, x > 0, \alpha < 1 \tag{65}$$

where  $d = \frac{1}{(1-\alpha)}$ . For the evaluation of the integral in closed form via Meijer's  $G$ -function and for the physical interpretation, see Haubold and Kumar [6]. The possible series representations of the above integrals can be seen in the Kumar and Haubold [15]. As  $\alpha \rightarrow 1$  the integrals given in (64) and (65) will reduce to the standard reaction rate probability integrals given in (62) and (63). Using the pathway parameter  $\alpha$  one goes into a wider class of integrals which show similar behaviour to that of the standard case.

**5.1. Behaviour of the kernel function of the  $\mathcal{P}$ -transform**

The behaviour of the integral  $D_{\rho,\beta}^{\nu,\alpha}(x)$  defined in (2) is such that as the value of the pathway parameter  $\alpha$  increases the curve will move away from the Standard Krätzel case and comes closer to the origin (see **Figure 1** below). The graphs of the integral  $D_{\rho,\beta}^{\nu,\alpha}(x)$  defined in (2) when  $\nu = 2, \beta = 1$  for  $\rho = 2, \rho = 3$  and at  $\alpha = 0.55, \alpha = 0.65, \alpha = 0.75, \alpha = 0.85$  are plotted in **Figure 1**. We can take other value for  $\nu, \beta$  and  $\rho$  depending upon the condition for the existence of (2).

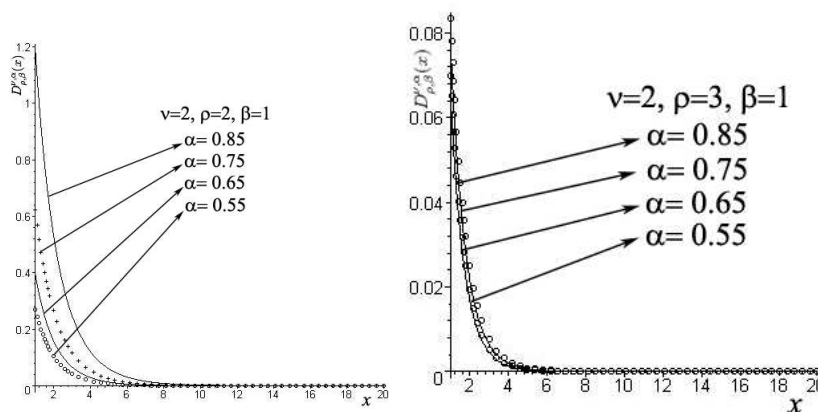


Figure 1. Behaviour of  $D_{\rho,\beta}^{\nu,\alpha}(x)$  defined in (2) for various values of  $\alpha < 1$

Similarly, the behaviour of the integrals  $D_{\rho,\beta}^{\nu,\alpha}(x)$  defined in (3) is such that the function moves away from the standard Krätzel case and moves away from the origin. The graphs of the integral  $D_{\rho,\beta}^{\nu,\alpha}(x)$  defined in (3) when  $\nu = 2$ ,  $\beta = 1$  for  $\rho = 2, rho = 3$  and at  $\alpha = 1$ ,  $\alpha = 1.15$ ,  $\alpha = 1.25$ ,  $\alpha = 1.35$ ,  $\alpha = 1.45$  are plotted in **Figure 2**.

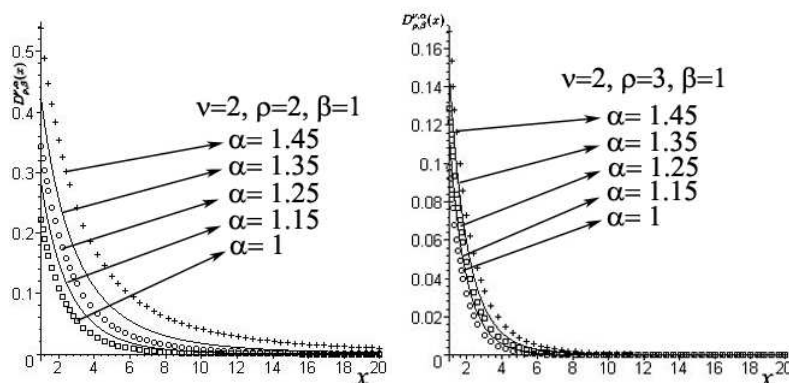


Figure 2. Behaviour of  $D_{\rho,\beta}^{\nu,\alpha}(x)$  defined in (3) for various values of  $\alpha > 1$

As  $\alpha \rightarrow 1$  in both integrals (2) and (3), we get the standard Krätzel integral in (7) which has been studied by many authors, see [1, 2, 10]. As  $\alpha \rightarrow 1$ , the two integrals will come close to the following limiting situation. It should be noted that in both the case as we increase the value of  $\rho$  the function shows a depletion towards the origin.

## 6. Conclusion

The  $\mathcal{P}$ -transform and the results in this article generalizes many existing results in the literature. As the pathway parameter  $\alpha$  varies, we get different integrals and integral transforms which shows similar behaviour. The particular case of the kernel function of the  $\mathcal{P}$ -transform is the reaction rate probability integral in the non-resonant case in the Maxwell-Boltzmann and cut-off case. It should be noted that the kernel functions and the  $\mathcal{P}$ -transform defined here can be used in any situation where the integrals or transforms of similar structure arise. The graphs for the kernel function of the type-1 and type-2  $\mathcal{P}$ -transform are plotted and the behaviour is observed. The plotting is done by using *Maple 9*.

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