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ON FRACTIONAL HELMHOLTZ EQUATIONS

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Abstract

In this paper we derive an analytic solution for the fractional Helmholtz equation in terms of the Mittag-Leffler function. The solutions to the fractional Poisson and the Laplace equations of the same kind are obtained, again represented by means of the Mittag-Leffler function. In all three cases the solutions are represented also in terms of Fox's H-function.

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1. Introduction

The Helmholtz equation $\nabla^2\Psi(x, y, z) + k^2\Psi(x, y, z) = 0$ is named after Hermann Von Helmholtz (1821-1894). It represents the time-independent form of the wave equation or diffusion equation obtained while applying the technique of separation of variables to reduce the complexities of the solution procedure of the original equations. The two-dimensional Helmholtz equation appears in physical phenomena and engineering applications such as heat conduction, acoustic radiation, water wave propagation and even in biology. It plays essential role for estimating the geodesic sea floor properties, the proper prediction of acoustic propagation in shallow water as well as at low frequencies, for solutions provided to such problems, we refer to

Liu et al. [9]. The Helmholtz equation solves also problems in pattern formation of animal coating, see Murray and Myerscough [12]; in electromagnetics, where its two-dimensional form appears as the governing equation for waveguide problems, etc.

Different numerical methods like the Ritz-Galerkin method (Itoh [7]), the surface integral equation method and the finite element method have been employed to solve this equation. In the Ritz-Galerkin method and in the integral equation method, the application of the method of moments to the solution of the integral equation leads to homogeneous systems of linear equations. The matrix coefficients of these systems of equations are given by infinite summation in the case of the Ritz-Galerkin method, while in the surface integral method, integrals containing Hankel functions have to be numerically computed, and this consumes large central processing unit time. In the finite element method (we refer e.g. to Pregla [16]), the basis functions representing the field singularities lead to matrices in the generalized eigenvalue problem, whose coefficients have to be computed by means of numerical integration. If we ignore these field singularities when applying finite element method, inaccurate results may be obtained. To reduce the memory requirements and inaccuracy, it is advantageous to reformulate the boundary value problem for the Helmholtz equation as an initial value problem of fractional order α, where $1 < \Re(\alpha) \leq 2$.

In this paper, we introduce a model fractional Helmholtz equation in two-dimensions, in which both the space variables x and y are allowed to take fractional order changes. This model is suitable for the study of electromagnetic waves propagating in the upper halfspace of the cartesian plane and is defined as

$$ {}_{-\infty}D_x^\alpha N(x,y) + {}_0D_y^\alpha N(x,y) + k^2 N(x,y) = \Phi(x,y); \tag{1} $$

$k > 0,\ x \in \Re,\ y \in \Re^+,\ 1 < \Re(\alpha) \leq\ 2$, where k is the wave number given by $k = \frac{2\pi}{\lambda}$, λ is the wavelength, $N(x,y)$ is the field variable of interest, which could be acoustic pressure, wave elevation or electromagnetic potential, among many other possibilities and $\Phi(x,y)$ is a non-linear function in the field. Here ${}_{-\infty}D_x^\alpha$ denotes the Weyl fractional derivative of order α, and ${}_0D_y^\alpha$ is the Caputo fractional derivative of order α, which are two alternative forms of the Riemann-Liouville fractional derivative ${}_aD_x^\alpha$ of order α:

$$ {}_aD_t^\alpha N(x,t) := {}_a^RD_t^\alpha N(x,t) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_a^t \frac{N(x,u)}{(t-u)^{\alpha-m+1}}\, \mathrm{d}u; \tag{2} $$

that are defined as follows:

$$ {}_0D_t^\alpha N(x,t) := {}_0^C D_t^\alpha N(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{N^{(m)}(x,u)}{(t-u)^{\alpha-m+1}}\,\mathrm{d}u, \tag{3} $$

$$ {}_{-\infty}D_t^\alpha N(x,t) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d^m}}{\mathrm{d}t^m} \int_{-\infty}^t \frac{N(x,u)}{(t-u)^{\alpha-m+1}}\mathrm{d}u, \tag{4} $$

where $t>0$, $m-1<\alpha\leq m$, $m\in N$, and $\frac{\mathrm{d^m}}{\mathrm{d}t^m}N(x,t)$ is the m^{th} derivative of order m of the function $N(x,t)$ with respect to t.

The Mittag-Leffler function is a special function having an essential role in the solutions of fractional order integral and differential equations. This function is frequently used recently in modeling phenomena of fractional order appearing in physics, biology, engineering and applied other sciences. After being introduced and studied by Mittag-Leffler (1903-1905), Wiman (1905) and Agarwal (1953), the Mittag-Leffler function, in its two forms:

$$ E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)},\ \alpha\in\mathbb{C},\ \Re(\alpha)>0, \tag{5} $$

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)},\ \alpha,\beta\in\mathbb{C},\ \Re(\alpha)>0,\ \Re(\beta)>0, \tag{6} $$

has been studied in details by Dzherbashyan [1]. Both functions (5)-(6) are entire functions of order $\rho=\frac{1}{\alpha}$ and type $\sigma=1$. Firstly in 1920, Hille and Tamarkin [6] have presented a solution of the Abel-Volterra type integral equation

$$ \phi(x) - \frac{\lambda}{\Gamma(\alpha)} \int_0^x \frac{\phi(t)}{(x-t)^{1-\alpha}}\mathrm{d}t = f(x),\quad 0<x<1, $$

in terms of Mittag-Leffler function. We are motivated by the recent works of Saxena-Mathai-Haubold ([18], [19], [20]) on fractional kinetic and fractional diffusion equations to obtain solutions in terms of the Mittag-Leffler function. For an extensive bibliography on these kind of studies, see e.g. in Povstenko [14].

The objective of this paper is to represent a solution of the fractional Helmholtz equation (1) in terms of the Mittag-Leffler function as well as in Fox's H-function, using the Laplace and Fourier transforms and their inverse transforms. Mathematically, the Poisson and the Laplace equations are two special cases of the Helmholtz equation. We apply this fact also to the fractional case. In Section 3, we derive the solution of the fractional Laplace equation in Mittag-Leffler function and Fox's H-function. Section 4 is devoted to the fractional Poisson equation. The solution of the fractional Helmholtz equation is given in Section 5. Its proof and the convergence and the series representation of the H-function are given in the Appendix.

2. Mathematical preliminaries

The main results on Mittag-Leffler functions (5), (6) are available in the handbook of Erdélyi et al. [4], the monographs by Dzherbashyan ([1], [2]), some recent books as by Kiryakova (1994), Podlubny [15], Kilbas, Srivastava and Trujillo (2006), Mainardi (2010), among them see e.g. the book of Mathai-Saxena-Haubold [10]. The H-function is defined by means of a Mellin-Barnes type integral in the following manner (we refer to Mathai-Saxena-Haubold [10]),

$$H^{m,n}_{p,q}(z)\left[z\,\Big|^{(a_p,A_p)}_{(b_q,B_q)}\right] = \frac{1}{2\pi i}\int_L \frac{\left\{\prod_{j=1}^{m}\Gamma(b_j+B_js)\right\}\left\{\prod_{j=0}^{n}\Gamma(1-b_j-B_js)\right\}}{\left\{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\right\}\left\{\prod_{j=n+1}^{p}\Gamma(a_j+A_js)\right\}}\, z^{-s}\,\mathrm{d}s, \tag{7}$$

where: an empty product is always interpreted as unity; $m,\ n,\ p,\ q \in N_0$ with $0 \le n \le p,\ 1 \le m \le q,\ A_j,\ B_j \in \Re_+,\ a_i,\ b_j \in \Re$ or $\mathbb{C},\ i = 1,\dots,\ p;\ j = 1,\dots,\ q$ such that $A_i(b_j+k) \neq B_j(a_i-l-1),\ k,\ l \in N_0;\ i = 1,\dots,\ n;\ j = 1,\dots,\ m$, and we employ the usual notations: $N_0 = (0,1,\dots,);\ \Re = (-\infty,\infty),\ \Re_+ = (0,\infty)$ and $\mathbb{C}$ being the complex number field. For the details about the contour L and the existence conditions see, Mathai-Saxena-Haubold [10], Prudnikov-Brychkov-Marichev [17] and Kilbas-Saigo [8].

The Wright's generalized hypergeometric function (Wright ([21], [22]) is defined by the series representation

$${}_p\psi_q(z) ={}_p \psi_q\left[z\Big|^{(a_p,A_p)}_{(b_q,B_q)}\right] = \sum_{r=0}^{\infty}\frac{\left[\prod_{j=1}^{p}\Gamma(a_j+A_js)\right]z^s}{\left[\prod_{j=1}^{q}\Gamma(b_j+B_js)\right]\ s!},$$

where $z\in\mathbb{C},\ a_j,\ b_j\in\mathbb{C},\ A_j,\ B_j\in\Re_+;\ i=1,...,p;\ j=1,...,q;\ \sum_{j=1}^{q}b_j - \sum_{j=1}^{p}A_j > -1$; $\mathbb{C}$ is the complex number field. The Mittag-Leffler function is a special case of this function,

$$E_{\alpha,\beta}(z) = {}_1\psi_1\left[z\,\Big|^{(1,1)}_{(\beta,\alpha)}\right] = H^{1,1}_{1,2}\left[-z\,\Big|^{(0,1)}_{(0,1)(1-\beta,\alpha)}\right]. \tag{8}$$

The Laplace transform of the function $N(x,t)$ with respect to t is

$$L\left[N(x,t)\right] = \int_0^\infty e^{-st} N(x,t)\mathrm{d}t = \tilde{N}(x,s),\ x \in \Re,\ \Re(s) > 0, \tag{9}$$

and its inverse transform with respect to s is given by

$$L^{-1}\left[\tilde{N}(x,s)\right] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\tilde{N}(x,s)\mathrm{d}s = N(x,t), \tag{10}$$

γ being a fixed real number. The Fourier transform of a function $N(x,t)$ with respect to x is defined as

$$F\left[N(x,t)\right] = \int_{-\infty}^\infty e^{ipx} N(x,t)\mathrm{d}x = N^*(p,t),\quad x \in \Re, \tag{11}$$

the inverse Fourier transform with respect to p:

$$F^{-1}\left[N^*(p,t)\right] = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-ipx} N^*(p,t)\ \mathrm{d}p = N(x,t). \tag{12}$$

The space of functions for the above two transforms is $LF = L(\Re_+) \times F(\Re)$, where $L(\Re_+)$ is the space of summable functions on $\Re_+$ with norm $||f|| = \int_0^\infty |f(t)|\mathrm{d}t\ <\ \infty$ and $F(\Re)$ is the space of summable functions on $\Re$ with norm $||f|| = \int_{-\infty}^\infty |f(t)|\mathrm{d}t\ <\ \infty$. From Saxena-Mathai-Haubold [10] and Prudnikov-Brychkov-Marichev [17], the cosine transform of the H-function is given by

$$\int_0^\infty t^{\rho-1}\cos(kt) H_{p,q}^{m,n}\left[at^\mu \Big|_{(b_q,B_q)}^{(a_p,A_p)}\right]\mathrm{d}t \tag{13}$$
$$= \frac{\pi}{k^\rho} H_{q+1,p+2}^{n+1,m}\left[\frac{p^\mu}{a}\ \Big|_{(\rho,\mu),\ (1-a_p,A_p),(\frac{1+\rho}{2},\frac{\mu}{2})}^{(1-b_q,B_q)(\frac{1+\rho}{2},\ \frac{\mu}{2})}\right],$$

where $\Re\left[\rho + \mu \min_{1\le j\le m}(\frac{b_j}{B_j})\right]\ >\ 0,\ \Re\left[\rho + \mu \min_{1\le j\le n}(\frac{a_j-1}{A_j})\right]\ <\ 0,\ |arg\ a| < \frac{1}{2}\pi\theta; \theta = \sum_{j=1}^n A_j - \sum_{j=n+1}^p A_j + \sum_{j=1}^m B_j - \sum_{j=m+1}^q B_j > 0.$

The Laplace transform of the Caputo fractional derivative is (see e.g. Podlubny [15])

$$L[{}_0D_t^\alpha N(x,t)] = s^\alpha N(x,s) - \sum_{r=1}^n s^{r-1}{}_0D_t^{\alpha-r}N(x,t)|_{t=0},\ n-1 < \Re(\alpha) \le n. \tag{14}$$

The Fourier transform of the Weyl fractional derivative, as given by Metzler-Klafter [11] is

$$F[{}_{-\infty}D_t^\alpha N(x,t)] = (ik)^\alpha \tilde{N}(p,t),\ \Re(\alpha) > 0, \tag{15}$$

where $\tilde{N}(p,t)$ is the Fourier transform of $N(x,t)$ with respect to the variable x of $N(x,t)$. By adopting Saxena-Mathai-Haubold [20] and also from Gorenflo-Luchko-Umarov [5],

$$F[{}_{-\infty}D_t^{\alpha}N(x,t)] = |k|^{\alpha}\tilde{N}(p,t),\ \Re(\alpha) > 0. \tag{16}$$

We also need the following property of H-function, well known from the recent books on special functions (see e.g. in Mathai-Saxena-Haubold [10]):

$$H_{p,q}^{m,n}\left[z^{\delta}\left|\begin{matrix}(a_p,A_p)\\ \\ (b_q,B_q)\end{matrix}\right.\right] = \frac{1}{\delta}H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p,\frac{A_p}{\delta})\\ \\ (b_q,\frac{B_q}{\delta})\end{matrix}\right.\right], \tag{17}$$

and the following result (see e.g. Podlubny [15])

$$L^{-1}\left(\frac{s^{\beta-1}}{s^{\alpha}+a};t\right) = t^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(-at^{\alpha}),\ \Re(s)>0, \Re(\alpha-\beta)>-1, \left|\frac{a}{s^{\alpha}}\right|<1. \tag{18}$$

3. Fractional Laplace equation

Let us start with the fractional Laplace equation. By replacing the integer order of the standard Laplace equation

$$\frac{\partial^2}{\partial x^2}N(x,y) + \frac{\partial^2}{\partial y^2}N(x,y) = 0, \tag{19}$$

by a fractional order α, where $1 < \Re(\alpha) \leq 2$, we get the fractional Laplace equation

$${}_{-\infty}D_x^{\alpha}N(x,y) + {}_0D_y^{\alpha}N(x,y) = 0.$$

In this section we derive a solution of the fractional Laplace equation in terms of the Mittag-Leffler function and derive also its Fox's H-function form. The case we consider is related to a Cauchy problem.

3.1. Cauchy problem. The solution to the fractional Laplace equation

$${}_{-\infty}D_x^{\alpha}N(x,y) + {}_0D_y^{\alpha}N(x,y) = 0;\ y \geqslant 0,\ x\in\Re,\ 1<\Re(\alpha)\leq 2, \tag{20}$$

with the initial conditions ${}_0D_y^{\alpha-1}N(x,0) = f(x)$, ${}_0D_y^{\alpha-2}N(x,0) = g(x)$, $x\in\Re$, $\lim\limits_{x\to\pm\infty} N(x,y) = 0$ is:

$$\begin{aligned} N(x,y) &= \frac{y^{\alpha-2}}{2\pi}\int_{-\infty}^{\infty}\tilde{g}(p)E_{\alpha,\alpha-1}[-(|p|^{\alpha}y^{\alpha})]e^{-ipx}\mathrm{d}p \\ &+ \frac{y^{\alpha-1}}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(p)E_{\alpha,\alpha}[-(|p|^{\alpha}y^{\alpha})]e^{-ipx}\mathrm{d}p, \end{aligned} \tag{21}$$

where $\sim$ indicates the Fourier transform with respect to space variable x.

Solution. If we apply the Laplace transform with respect to the space variable y and use (14), then (20) becomes

$$ {}_{-\infty}D_x^{\alpha} N^*(x,s) + s^{\alpha} N^*(x,s) - f(x) - sg(x) = 0, \tag{22}$$

where $*$ depicts the Laplace transform with respect to the space variable y. Now applying the Fourier transform with respect to the space variable x and using initial conditions and (16), the above equation takes the form

$$\tilde{N}^*(p,s) \quad = \quad \frac{\tilde{f(p)}}{s^{\alpha} + |p|^{\alpha}} + \frac{s\tilde{g}(p)}{s^{\alpha} + |p|^{\alpha}}.$$

Using (18), it is seen that

$$\tilde{N}(p,y) = \tilde{f}(p) y^{\alpha-1} E_{\alpha,\alpha}\left[-(|p|^{\alpha} y^{\alpha})\right] + \tilde{g}(p) y^{\alpha-2} E_{\alpha,\alpha-1}\left[-(|p|^{\alpha} y^{\alpha})\right].$$

Taking the inverse Fourier transform on both sides, the solution is

$$\begin{aligned} N(x,y) \quad &= \quad \frac{y^{\alpha-1}}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(p) E_{\alpha,\alpha}[-(|p|^{\alpha} y^{\alpha})] e^{-ipx} \mathrm{d}p \\ &+ \frac{y^{\alpha-2}}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(p) E_{\alpha,\alpha-1}[-(|p|^{\alpha} y^{\alpha})] e^{-ipx} \mathrm{d}p. \end{aligned}$$

In the initial conditions of Cauchy problem 3.1, put $g(x) = 0$, and with the help of the convolution theorem of the Fourier transforms, see the changes in the solution of the above problem.

COROLLARY 3.1. *The solution of the fractional Laplace equation*

$$\begin{gathered} {}_{-\infty}D_x^{\alpha} N(x,y) + {}_0D_y^{\alpha} N(x,y) = 0, \quad y \geqslant 0, \\ {}_0D_y^{\alpha-1} N(x, y=0) = f(x), \; {}_0D_y^{\alpha-2} N(x, y=0) = 0, \end{gathered} \tag{23}$$

$x \in R$, $\lim\limits_{x \to \pm\infty} N(x,y) = 0, 1 < \Re(\alpha) \leq 2$, *is given by*

$$N(x,y) = \int_{-\infty}^{\infty} G(x-\tau, y) f(\tau) \mathrm{d}\tau, \tag{24}$$

where

$$G(x,y) = \frac{y^{\alpha-1}}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} E_{\alpha,\alpha}(-|p|^{\alpha} y^{\alpha}) \mathrm{d}p$$

$$= \frac{y^{\alpha-1}}{\pi\alpha} \int_0^{\infty} \cos(px) H_{1,2}^{1,1}\left[|p|y \left| \begin{matrix} (0,\frac{1}{\alpha}) \\ (0,\frac{1}{\alpha}),(1-\alpha,1) \end{matrix} \right. \right] \mathrm{d}p, \quad \text{using property (17)},$$

$$= \frac{y^{\alpha-1}}{\alpha|x|} H_{3,3}^{2,1}\left[\frac{|x|}{y} \left| \begin{matrix} (1,\frac{1}{\alpha}),(\alpha,1),(1,\frac{1}{2}) \\ (1,1),(1,\frac{1}{\alpha}),(1,\frac{1}{2}) \end{matrix} \right. \right], \quad \text{using eq. (13), where } 1 < \Re(\alpha) \leq 2.$$

Again, by changing the initial conditions of the Cauchy problem 1, as $f(x) = \delta(x)$, where $\delta(x)$ is the Dirac-delta function defined by
$$\delta(x) = \begin{cases} 1, & x = 0, \\ 0, & \text{elsewhere,} \end{cases}$$
and $g(x) = 0$, we can express the above solution in terms of a Fox's H-function. We shall consider this in the following result.

Corollary 3.2. *The solution to the fractional Laplace equation*

$$_{-\infty}D_x^{\alpha} N(x,y) + {_0D_y^{\alpha}} N(x,y) = 0; \quad y \geqslant 0, \ x \in \Re, \ 1 < \Re(\alpha) \leq 2, \tag{25}$$

with the initial conditions $_0D_y^{\alpha-1}N(x,0) = \delta(x)$, $_0D_y^{\alpha-2}N(x,0) = 0$, $x \in R$, $\lim_{x\to\pm\infty} N(x,y) = 0$, is:

$$\begin{aligned} N(x,y) &= \frac{y^{\alpha-1}}{2\pi} \int_{-\infty}^{\infty} e^{-ipx} E_{\alpha,\alpha}(-|p|^{\alpha} y^{\alpha}) \mathrm{d}p, \\ &= \frac{y^{\alpha-1}}{\alpha|x|} H_{3,3}^{2,1} \left[\frac{|x|}{y} \middle|_{(1,1),(1,\frac{1}{\alpha}),(1,\frac{1}{2})}^{(1,\frac{1}{\alpha}),(\alpha,1)(1,\frac{1}{2})} \right], \ 1 < \Re(\alpha) \leq 2. \end{aligned} \tag{26}$$

4. Fractional Poisson equation

To generalize Cauchy problem 3.1, we consider the Mittag-Leffler solution of the fractional Poisson equation

$$_{-\infty}D_x^{\alpha} N(x,y) + {_0D_y^{\alpha}} N(x,y) = \Phi(x,y). \tag{27}$$

Namely, we consider the following

4.1. Cauchy problem. The solution to the fractional Poisson equation

$$_{-\infty}D_x^{\alpha} N(x,y) + {_0D_y^{\alpha}} N(x,y) = \Phi(x,y); \ y \geqslant 0, \ x \in \Re, \ 1 < \Re(\alpha) \leq 2, \tag{28}$$

where $\Phi(x,y)$ is a non-linear function, with the initial conditions $_0D_y^{\alpha-1}N(x,0) = f(x)$, $_0D_y^{\alpha-2}N(x,0) = g(x), x \in R$, $\lim_{x\to\pm\infty} N(x,y) = 0$,

$$\begin{aligned} N(x,y) &= \frac{y^{\alpha-1}}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(p) E_{\alpha,\alpha}[-(|p|^{\alpha} y^{\alpha})] e^{-ipx} \mathrm{d}p, \\ &+ \frac{y^{\alpha-2}}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(p) E_{\alpha,\alpha-1}[-(|p|^{\alpha} y^{\alpha})] e^{-ipx} \mathrm{d}p, \\ &+ \frac{1}{2\pi} \int_0^y \xi^{\alpha-1} \int_{-\infty}^{\infty} \tilde{\Phi}(k, y-\xi) E_{\alpha,\alpha}[-(|p|^{\alpha} \xi^{\alpha})] e^{-ipx} \mathrm{d}p d\xi. \end{aligned} \tag{29}$$

Solution. Applying the Laplace transform with respect to the space variable y and Fourier transform with respect to the space variable x and using the initial conditions and (14), we have

$$[s^\alpha + |p|^\alpha]\,\tilde{N}^*(p,s) \quad = \quad \tilde{f(p)} - s\tilde{g}(p) + \tilde{\phi}^*(p,s).$$

Using the result (16),

$$\begin{aligned}\tilde{N}(p,y) \quad = \quad & \tilde{f}(p)y^{\alpha-1}E_{\alpha,\alpha}\left[-(|p|^\alpha y^\alpha)\right] + \tilde{g}(p)y^{\alpha-2}E_{\alpha,\alpha-1}\left[-(|p|^\alpha y^\alpha)\right] \\ & + \int_0^\infty \tilde{\phi}(p,y-\xi)\xi^{\alpha-1}E_{\alpha,\alpha}[-(|p|^\alpha)\xi^\alpha]\mathrm{d}\xi.\end{aligned}$$

Taking the inverse Fourier transform, the solution is

$$\begin{aligned}N(x,y) \quad = \quad & \frac{y^{\alpha-1}}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(p)E_{\alpha,\alpha}[-|p|^\alpha y^\alpha]e^{-ipx}\mathrm{d}p \\ & + \frac{y^{\alpha-2}}{2\pi}\int_{-\infty}^{\infty}\tilde{g}(p)E_{\alpha,\alpha-1}[-|p|^\alpha y^\alpha]e^{-ipx}\mathrm{d}p \\ & + \frac{1}{2\pi}\int_0^y \xi^{\alpha-1}\int_{-\infty}^{\infty}\tilde{\phi}(k,y-\xi)E_{\alpha,\alpha}[-(|p|^\alpha)\xi^\alpha]e^{-ipx}\mathrm{d}p\mathrm{d}\xi.\end{aligned}$$

Now we shall try to get an H-function solution.

Corollary 4.1. *The solution of the fractional Poisson equation*

$$_{-\infty}D_x{}^\alpha N(x,y) + {}_0D_y{}^\alpha N(x,y) = \Phi(x,y),\ \ y \geqslant 0, \tag{30}$$

$_0D_y{}^{\alpha-1}N(x,y{=}0) = f(x),\ {}_0D_y{}^{\alpha-2}N(x,y{=}0) = 0$ *for* $x \in \Re,\ \lim_{x\to\pm\infty} N(x,y) = 0,\ 1 < \Re(\alpha) \le 2$ *and* $\Phi(x,y)$ *a non-linear function of* x *and* y, *is given by*

$$N(x,y) = \int_{-\infty}^{\infty} G_1(x-\tau,y)f(\tau)d\tau + \int_0^y (y-\xi)^{\alpha-1}\int_0^x G_2(x-\tau,y-\xi)\Phi(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi, \tag{31}$$

where

$$G_1(x,y) = \frac{t^{\alpha-1}}{\alpha|x|}H^{2,1}_{3,3}\left[\frac{|x|}{y}\Big|^{(1,\frac{1}{\alpha}),(\alpha,1),(1,\frac{1}{2})}_{(1,1),(1,\frac{1}{\alpha}),(1,\frac{1}{2})}\right],\quad 1 < \Re(\alpha) \le 2,$$

Similarly,

$$G_2(x,y) = \frac{1}{\alpha|x|}H^{2,1}_{3,3}\left[\frac{|x|}{y}\Big|^{(1,\frac{1}{\alpha}),(\alpha,1),(1,\frac{1}{2})}_{(1,1),(1,\frac{1}{\alpha}),(1,\frac{1}{2})}\right],\quad 1 < \Re(\alpha) \le 2.$$

5. Fractional Helmhotz equation

In this section, we solve the fractional Helmholtz equation (1) to generate a solution in terms of the Mittag-Leffler function, using the Laplace and Fourier transforms and their inverses.

THEOREM 5.1. *Consider the fractional Helmholtz equation*

$$ {}_{-\infty}D_x^{\alpha}N(x,y) + {}_0D_y^{\alpha}N(x,y) + k^2N(x,y) = \Phi(x,y); \quad k>0,\ x\in\Re,\ y\geqslant 0, \tag{32}$$

$1<\Re(\alpha)\leq 2$, *with initial conditions* ${}_0D_y^{\alpha-1}N(x,0)=f(x),\ {}_0D_y^{\alpha-2}N(x,0)=g(x),\ x\in\Re,$
$\lim_{x\to\pm\infty} N(x,y)=0$, *where* ${}_0D_y^{\alpha-1}N(x,0)$ *means the Riemann-Liouville fractional derivative of order* $\alpha-1$ *with respect to* y *and* ${}_0D_y^{\alpha-2}N(x,0)$ *means the Riemann-Liouville fractional derivative of order* $\alpha-2$ *with respect to* y*, when* $y=0$*. The quantity* k *is a constant and* $\Phi(x,y)$ *is a nonlinear function. Then the solution of (32) subject to the initial conditions, is:*

$$\begin{aligned} N(x,y) &= \frac{y^{\alpha-1}}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(p)E_{\alpha,\alpha}[-(|p|^{\alpha}+k^2)y^{\alpha}]e^{-ipx}\mathrm{d}p \\ &+ \frac{y^{\alpha-2}}{2\pi}\int_{-\infty}^{\infty}\tilde{g}(p)E_{\alpha,\alpha-1}[-(|p|^{\alpha}+k^2)y^{\alpha}]e^{-ipx}\mathrm{d}p \\ &+ \frac{1}{2\pi}\int_0^y \xi^{\alpha-1}\int_{-\infty}^{\infty}\tilde{\Phi}(k,y-\xi)E_{\alpha,\alpha}[-(|p|^{\alpha}+k^2)\xi^{\alpha}]e^{-ipx}\mathrm{d}p\mathrm{d}\xi, \end{aligned} \tag{33}$$

where $\sim$ *indicates the Fourier transform with respect to the space variable* x.

The proof of this theorem and the H-function solution of the fractional Helmholtz equation are discussed in the Appendix.

6. Conclusion

The closed form solutions in terms of the Mittag-Leffler function and Fox's H-function are obtained for the fractional Laplace and the fractional Poisson equations. It is seen that, the solutions in terms of the Mittag-Leffler function as well as in the H-function to the fractional Helmholtz equation are the master solutions to the solutions of the fractional Laplace equation and the fractional Poisson equation.

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Appendix

A.1. The Mittag-Leffler solution of the fractional Helmholtz equation

If we apply the Laplace transform with respect to the space variable y and use (14), the equation (32) becomes

$$ {}_{-\infty}D_x^{\alpha} N^*(x,s) + s^{\alpha} N^*(x,s) - f(x) - sg(x) + k^2 N^*(x,s) = \Phi^*(x,s). \quad (34)$$

Now applying the Fourier transform with respect to the space variable x and use initial conditions and (16) to obtain

$$\tilde{N}^*(p,s) \;\; = \;\; \frac{\tilde{f(p)}}{s^{\alpha}+|p|^{\alpha}+k^2} + \frac{s\tilde{g}(p)}{s^{\alpha}+|p|^{\alpha}+k^2} + \frac{\tilde{\Phi}^*(p,s)}{s^{\alpha}+|p|^{\alpha}+k^2}.$$

Using the result (18), it is seen that

$$\tilde{N}(p,y) = \tilde{f}(p) y^{\alpha-1} E_{\alpha,\alpha}\left[-(|p|^{\alpha}+k^2)y^{\alpha}\right] + \tilde{g}(p) y^{\alpha-2} E_{\alpha,\alpha-1}\left[-(|p|^{\alpha}+k^2)y^{\alpha}\right] + \int_0^{\infty} \tilde{\phi}(p, y-\xi)\xi^{\alpha-1} E_{\alpha,\alpha}[-(|p|^{\alpha}+k^2)\xi^{\alpha}]\mathrm{d}\xi.$$

Taking the inverse Fourier transform on both sides, the solution is

$$\begin{aligned} N(x,y) \;\; = \;\; & \frac{y^{\alpha-1}}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(p) E_{\alpha,\alpha}[-(|p|^{\alpha}+k^2)y^{\alpha}] e^{-ipx}\mathrm{d}p \\ & + \frac{y^{\alpha-2}}{2\pi}\int_{-\infty}^{\infty} \tilde{g}(p) E_{\alpha,\alpha-1}[-(|p|^{\alpha}+k^2)y^{\alpha}] e^{-ipx}\mathrm{d}p \\ & + \frac{1}{2\pi}\int_0^{y} \xi^{\alpha-1}\int_{-\infty}^{\infty} \tilde{\phi}(k, y-\xi) E_{\alpha,\alpha}[-(|p|^{\alpha}+k^2)\xi^{\alpha}] e^{-ipx}\mathrm{d}p\mathrm{d}\xi. \end{aligned}$$

Hence Theorem 5.1 follows. Using the equation (8), the solution (33) of the fractional Helmholtz equation can be given in terms of the Fox's H-function.

We shall consider the case as follows.

A.2. The Fox's H-function solution of the fractional Helmholtz equation

The solution of the fractional Helmholtz equation

$$ {}_{-\infty}D_x^{\alpha}N(x,y) + {}_0D_y^{\alpha}N(x,y) + k^2N(x,y) = \Phi(x,y), \tag{35}$$

$y>0,\ k>0,\ x\in\Re,\ 1<\Re(\alpha)\le 2$ with the initial conditions ${}_0D_y^{\alpha-1}N(x,y) = f(x)$, ${}_0D_x^{\alpha-2}N(x,y)=0$, for $x\in R$, $\lim\limits_{x\to\pm\infty} N(x.y)=0$, and $\Phi(x,y)$ is a non-linear function, given by

$$N(x,y)=\int_{-\infty}^{\infty} G_1(x-\tau,y)f(\tau)d\tau+\int_0^y (y-\xi)^{\alpha-1}\int_0^x G_2(x-\tau,y-\xi)\Phi(\tau,\xi)\mathrm{d}\tau\mathrm{d}\xi, \tag{36}$$

where

$$G_1(x,y) = \frac{y^{\alpha-1}}{2\pi}\int_{-\infty}^{\infty} e^{-ipx}[H_{1,2}^{1,1}\left[(|p|^{\alpha}+k^2)y^{\alpha}\Big|_{(0,1),(1-\alpha,\alpha)}^{0,1}\right]\mathrm{d}p, \tag{37}$$

$$G_2(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx}H_{1,2}^{1,1}\left[(|p|^{\alpha}+k^2)y^{\alpha}\Big|_{(0,1),(1-\alpha,\alpha)}^{(0,1)}\right]\mathrm{d}p. \tag{38}$$

Convergence and the series representation of the solution (26)

By Mathai-Saxena-Haubold [10], the $H_{3,3}^{2,1}\left[\frac{|x|}{y}\Big|_{(1,1),(1,\frac{1}{\alpha}),(1,\frac{1}{2})}^{(1,\frac{1}{\alpha}),(\alpha,1),(1,\frac{1}{2})}\right]$ converges for $0<|\frac{|x|}{y}|<1$. Again, by Mathai-Saxena-Haubold [10], the series expansion of the H-function is given as follows: We have:

$$H_{3,3}^{2,1}\left[\frac{|x|}{y}\Big|_{(1,1),(1,\frac{1}{\alpha}),(1,\frac{1}{2})}^{(1,\frac{1}{\alpha}),(\alpha,1),(1,\frac{1}{2})}\right] = \frac{1}{2\pi i}\int_L \frac{\Gamma(1+s)\Gamma(1+\frac{s}{\alpha})\Gamma(\frac{-s}{\alpha})}{\Gamma(\frac{-s}{2})\Gamma(\alpha+s)\Gamma(1+\frac{s}{2})}\left[\frac{|x|}{y}\right]^{-s}\mathrm{d}s. \tag{39}$$

Let us assume that the poles of the integrand are simple. The region of convergence is $\mathrm{L}=\mathrm{L}_{\mathrm{i}\gamma\infty}$ is a contour starting at the point $\gamma-i\infty$ and terminating at the point $\gamma+i\infty$, where $\gamma\in\Re=(-\infty,+\infty)$ such that all the poles of $\Gamma(b_j+B_js),\ j=1,\cdots,m$ are separated from those of $\Gamma(1-a_j-A_js),\ j=1,\cdots,n$. By calculating the residues at the poles of $\Gamma(1+s)$ and $\Gamma(1+\frac{s}{\alpha})$ where the poles are given by $1+s=-\nu,\ \nu=0,1,2,\cdots$ and $1+\frac{s}{\alpha}=-\nu,\ \nu=0,1,2,\cdots$ we will get the series representation of (26).

$$N(x,y) = \tfrac{y^{\alpha-1}}{\alpha}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\Gamma(1-\frac{1+\nu}{\alpha})\Gamma(\frac{1+\nu}{\alpha})}{(\nu)!\Gamma(\frac{1+\nu}{2})\Gamma(\alpha-1-\nu)\Gamma(1-\frac{1+\nu}{2})}\left[\frac{|x|}{y}\right]^{\nu}$$
$$+\tfrac{|x|^{\alpha-1}}{y}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}\Gamma(1-\alpha(1+\nu))\Gamma((1+\nu))}{(\nu)!\Gamma((1+\nu)\frac{\alpha}{2})\Gamma(\alpha-\alpha(1+\nu))\Gamma(1-(1+\nu)\frac{\alpha}{2})}\left[\frac{|x|}{y}\right]^{\alpha\nu},$$

where $\Re(1-\frac{1+\nu}{\alpha})>0,\ 0<\left[\left|\frac{|x|}{y}\right|\right]<1.$

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