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ANTI-PERIODIC BOUNDARY VALUE PROBLEM FOR IMPULSIVE FRACTIONAL INTEGRO DIFFERENTIAL EQUATIONS

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Abstract

In this paper we prove the existence of solutions for fractional impulsive differential equations with antiperiodic boundary condition in Banach spaces. The results are obtained by using fractional calculus' techniques and the fixed point theorems.

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Key Words and Phrases: impulsive fractional differential equations, contraction principle, antiperiodic boundary conditions, Sadovskii theorem

1. Introduction

Recently fractional differential equations have arisen in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electro-dynamics of complex medium, polymer rheology, etc. (see [8], [11]-[13]), involving derivatives of fractional order. The fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. The theory of fractional differential equations has been extensively studied by many authors, among them: Lakshmikantham et al. [17]-[19]. In the paper [21], it is proved the existence of solutions of abstract differential equations by using semigroup theory and fixed point theorem. Many partial fractional differential equations can be expressed as fractional differential equations in some Banach spaces.

The following equation

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad t \in J = [0,1]/t_1, t_2, ...t_k, \\ \Delta x(t_k) = I_k(x(t_k^-), \quad \Delta x'(t_k) = J_k(x(t_k^-) \quad t_k \in (0,1), k = 1,2,3....p. \\ x(0) + x'(0) = 0, x(1) + x'(1) = 0, \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative with $1 < q \leq 2$ was studied by Ahmad et al. [4] and the existence of positive solutions was obtained using classical fixed point theorems.

Recently, Alsaedi [1] has studied the integrodifferential equations of fractional order with antiperiodic boundary conditions

$$\begin{cases} {}^cD^q x(t) = f(t, x(t), Bx(t)), \quad t \in [0,T], 1 < q \leq 2, \\ x(0) = -x(T), x'(0) = -x'(T), \end{cases} \tag{1.2}$$

in general Banach space X with $0 < q < 1$. By means of the Krasnoselskii theorem, existence of solutions has been also obtained.

Subsequently several authors have investigated the problem for different types of nonlinear differential equations and integrodifferential equations including functional differential equations in Banach spaces.

Antiperiodic boundary value problems have recently received considerable attention, as antiperiodic boundary conditions appear in numerous situation, for instance, see [2], [3], [9], [10].

In [2], the existence of solutions to the equation

$$\begin{cases} {}^cD^q x(t) = f(t, x(t),), \quad t \in [0,T], \\ x(0) = -x(T), \quad x'(0) = -x'(T), x"(0) = -x"(T). \end{cases} \tag{1.3}$$

is studied. The results are obtained via construction and the contraction mapping principle and Krasnoselskii's fixed point theorem. Very recently, Ahmad et al. [3] have discussed the existence of solutions of fractional differential equations with antiperiodic boundary condition via the Leray-Schauder degree theory.

The paper is organized as follows. In Section 2 we introduce some preliminary results needed in the following sections. In Section 3 we present an existence result for antiperiodic boundary value problem for fractional impulsive differential equations in Banach spaces by using the fractional calculus' techniques and the Sadovskii fixed point theorem.

2. Preliminaries

First, we recall some basic definitions.

Consider the set of functions

$$PC(I,X) = \{x : I \to X : x \in C((t_k, t_{k+1}], X),\ k = 0, \dots, m \text{ and there exist } x(t_k^-) \text{ and } x(t_k^+),\ k = 1, \dots, m \text{ with } x(t_k^-) = x(t_k)\}.$$

This set is a Banach space with the norm

$$\|x\|_{PC} = \sup_{t \in I} |x(t)|.$$

Set $J' := [0,T] \backslash \{t_1, \dots, t_m\}$.

For some basic facts about fractional derivatives and fractional calculus, one can refer to the books [15], [20], [22], [23].

Definition 2.1. ([15], [22]) The fractional (arbitrary) order integral of the function $f \in L^1([a,b], R_+)$ of order $q \in R_+$ is defined by

$$I_a^q f(t) = \int_a^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^q f(t) = f(t) * \varphi_q(t)$, where $\varphi_q(t) = \frac{t^{q-1}}{\Gamma(q)}$ for $t > 0$, and $\varphi_q(t) = 0$ for $t \leq 0$, and $\varphi_q(t) \to \delta(t)$ as $q \to 0$, where δ is the delta function.

Definition 2.2. The Riemann-Liouville fractional integral of order $q > 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \quad q > 0, \quad t > 0$$
$$I^0 f(x) = f(x),$$

provided that the integral exists.

Definition 2.3. ([15], [22]) The Riemann-Liouville fractional derivative of order q for a function $f(t)$, is defined by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{q-n+1}} ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall use the modified fractional differential operator ${}^cD^q$ proposed by M. Caputo in his work [8] on the theory of viscoelasticity.

DEFINITION 2.4. ([14], [22]) The Caputo fractional-order derivative of f, is defined by

$$^{c}D^{q}f(t)=\frac{1}{\Gamma(n-q)}\int_{t_0}^{t}(t-s)^{n-q-1}f^{(n)}(s)ds,\quad t>t_0,$$

where $n=[q]+1$, $[q]$ denotes the integer part of real number q.

LEMMA 2.1. ([25]) *For $q>0$, the general solution of the fractional differential equation $^{c}D^{q}x(t)=0$ is given by*

$$x(t)=c_0+c_1t+c_2t^2+........+c_{n-1}t^{n-1},$$

where $c_i\in R, i=0,1,2,......,n-1(n=[q]+1)$.

In view of Lemma 2.1, it follows that

$$I^{qc}D^{q}x(t)=x(t)+c_0+c_1t+c_2t^2+........+c_{n-1}t^{n-1},$$

for some $c_i\in R, i=0,1,2,......,n-1(n=[q]+1)$.

Now, we state a known result due to Sadovskii [24].

THEOREM 2.2. *Let B be a closed, convex and bounded subset of a Banach space X. If $F:B\to B$ is a condensing map, then F has a fixed point in B.*

LEMMA 2.3. *Let $0<q\leq 1$ and let $h:I\times X\times X\to X$ be continuous. A function x is a solution of the fractional integral equation*

$$x(t)=\begin{cases}\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}h(s)ds-\frac{1}{2}\left[\frac{1}{\Gamma(q)}\int_0^T(T-s)^{q-1}h(s)ds\right] \text{ if } t\in[0,t_1],\\ \quad\frac{1}{\Gamma(q)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}h(s)ds\\ +\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}h(s)ds-\frac{1}{2}\left[\frac{1}{\Gamma(q)}\int_0^T(T-s)^{q-1}h(s)ds\right]\\ +\sum_{i=1}^{k}I_i(x(t_i^-)), \text{if } t\in(t_k,t_{k+1}],\end{cases}\tag{2.1}$$

where $k=1,\ldots,m$, if and only if y is a solution of the fractional impulsive BVP

$$^{c}D^{q}x(t)=h(t),\quad t\in J',\tag{2.2}$$

$$\Delta x|_{t=t_k}=I_k(x(t_k^-)),\quad k=1,\ldots,m,\tag{2.3}$$

$$x(0)=-x(T).\tag{2.4}$$

P r o o f. Assume that x satisfies (2.2)-(2.4). If $t\in[0,t_1]$, then

$$^{c}D^{q}x(t)=h(t).$$

Lemma 2.1 implies

$$x(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)ds - \frac{1}{2}\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds\right].$$

If $t \in (t_1, t_2]$, then

$$\begin{aligned} x(t) &= x(t_1^+) + \frac{1}{\Gamma(q)}\int_{t_1}^t (t-s)^{q-1}h(s)ds - \frac{1}{2}\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds\right] \\ &= \Delta x|_{t=t_1} + x(t_1^-) + \frac{1}{\Gamma(q)}\int_{t_1}^t (t-s)^{q-1}h(s)ds \\ &\quad - \frac{1}{2}\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds\right] \\ &= I_1(x(t_1^-)) + \frac{1}{\Gamma(q)}\int_0^{t_1} (t_1-s)^{q-1}h(s)ds + \frac{1}{\Gamma(q)}\int_{t_1}^t (t-s)^{q-1}h(s)ds \\ &\quad - \frac{1}{2}\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds\right]. \end{aligned} \tag{2.5}$$

If $t \in (t_2, t_3]$, then we get

$$\begin{aligned} x(t) &= x(t_2^+) + \frac{1}{\Gamma(q)}\int_{t_2}^t (t-s)^{q-1}h(s)ds - \frac{1}{2}\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds\right] \\ &= \Delta x|_{t=t_2} + x(t_2^-) + \frac{1}{\Gamma(q)}\int_{t_2}^t (t-s)^{q-1}h(s)ds \\ &\quad - \frac{1}{2}\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds\right] \\ &= I_2(x(t_2^-)) + I_1(x(t_1^-)) + \frac{1}{\Gamma(q)}\int_0^{t_1} (t_1-s)^{q-1}h(s)ds \\ &\quad - \frac{1}{2}\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds\right] \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_1}^{t_2} (t_2-s)^{q-1}h(s)ds + \frac{1}{\Gamma(q)}\int_{t_2}^t (t-s)^{q-1}h(s)ds \\ &\quad - \frac{1}{2}\left[\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}h(s)ds\right]. \end{aligned} \tag{2.6}$$

If $t \in (t_k, t_{k+1}]$, then again from Lemma 2.1 we get (2.1).

Conversely, assume that x satisfies the impulsive fractional integral equation (2.1). If $t \in [0, t_1]$, then $x(0) = -x(T)$ and using the fact that ${}^cD^q$ is the left inverse of I^q, we get

$$ {}^cD^qx(t) = h(t), \quad \text{for each} \quad t \in [0, t_1]. $$

If $t \in [t_k, t_{k+1})$, $k = 1, \dots, m$ and using the fact that ${}^cD^qC = 0$, where C is a constant, we get

$$ {}^cD^qx(t) = h(t), \quad \text{for each} \quad t \in [t_k, t_{k+1}). \tag{2.7} $$

Also, we can easily show that

$$ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, \dots, m. \tag{2.8} $$

This completes the proof. ∎

3. Main results

Now consider the first order impulsive boundary value problem for fractional differential equation

$$ \begin{cases} {}^cD^qx(t) = f(t, x(t), (\chi x)(t)), & t \in I = [0, T], t \neq t_k, \\ \Delta x\big|_{t=t_k} = I_k(x(t_k^-)), & \\ x(0) = -x(T), & \end{cases} \tag{3.1} $$

where $0 < q < 1$, $k = 1, \dots, m$, $0 < q \le 1$, ${}^cD^q$ is the Caputo fractional derivative, $f : I \times X \times X \to X$ is a given function and for $\gamma : I \times I \to [0, \infty)$,

$$ (\chi x)(t) = \int_0^t \gamma(t, s)x(s)ds, $$

with $\gamma_0 = \max\left\{\int_0^t \gamma(t, s)x(s)ds : (t, s) \in I \times I\right\}$. $I_k : X \to X$,
$0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$,
$x(t_k^+) = \lim_{h\to 0^+} x(t_k+h)$ and $x(t_k^-) = \lim_{h\to 0^-} x(t_k+h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

We need the following assumptions to prove the existence of solutions of equation (3.1).

(**HA**). $f : I \times X \times X \to X$ is continuous and there exist a constants $L > 0, M > 0$ such that

$$ \|f(t, x, (\chi x)) - f(t, y, (\chi y))\| \le L\|x - y\| + M\|(\chi x) - (\chi y)\| \quad \text{for all } x, y \in X $$

and for each $t \in I$. For brevity, let us take $\eta = \frac{T^q}{\Gamma(q+1)}$.

(**HB**). The functions $I_k : X \to X$ are continuous and there exists a constants $M^* > 0, L^* > 0$ such that

$$\|I_k(x) - I_k(\overline{x})\| \leq L^*\|x - \overline{x}\| \quad \text{with} \quad \|I_k(x)\| \leq M^*$$

for each $x, \overline{x} \in X$ and $k = 1, \ldots, m$.

(**HC**). $f : I \times X \times X \to X$ is continuous and there exist a function $\mu \in L^1(I, R^+)$ such that

$$\sup \|f(t, x, (\chi x))\| \leq \mu(t), \forall (t, x, (\chi x)) \in I \times X \times X.$$

THEOREM 3.1. *If the assumptions $(HA), (HB)$ are satisfied and if $(L + \gamma_0 M) \leq (m + \frac{3}{2})/\eta + mM^*$, then equation (3.1) has a unique solution.*

P r o o f. $F : PC(I, X) \to PC(I, X)$ defined by

$$F(x)(t) = \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, x(s), (\chi x)(s)) ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, x(s), (\chi x)(s)) ds \tag{3.2}$$
$$-\frac{1}{2}\left[\frac{1}{\Gamma(q)} \int_0^T (T - s)^{q-1} f(s, x(s), (\chi x)(s)) ds\right] + \sum_{0<t_k<t} I_k(x(t_k^-)),$$

and we have to show that F has a fixed point. This fixed point is then a solution of the equation (3.1).

Let $M_1 = \sup_{t\in[0,T]} \|f(t, 0)\|$. Then we can show that $FB_r \subset B_r$, where $B_r := \{x \in PC(I, X) : \|x\| \leq r\}$. From the assumptions, we have to choose $r \geq 3[M_1(m + \frac{3}{2})\eta + mM^*]$, then $\|(Fx)(t)\|$

$$\leq \left\|\frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, x(s), (\chi x)(s)) ds\right\|$$
$$+ \left\|\frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, x(s), (\chi x)(s)) ds\right\|$$
$$+ \left\|\frac{1}{2}\left[\frac{1}{\Gamma(q)} \int_0^T (T - s)^{q-1} f(s, x(s), (\chi x)(s)) ds\right] + \sum_{0<t_k<t} I_k(x(t_k^-))\right\|.$$

$$\leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} [\|f(s, x(s), (\chi x)(s) - f(s,0,0) + f(s,0,0)\|] ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} [\|f(s, x(s), (\chi x)(s)) - f(s,0,0) + f(s,0,0)\|] ds$$
$$+ \frac{1}{2} \left[\frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} [\|f(s, x(s), (\chi x)(s)) - f(s,0,0) + f(s,0,0)\|] ds \right]$$
$$+ \sum_{0<t_k<t} \|I_k(x(t_k^-))\|$$
$$\leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} [\|f(s, x(s), (\chi x)(s))) - f(s,0,0)\| + \|f(s,0,0)\|] ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} [\|f(s, x(s), (\chi x)(s))) - f(s,0,0)\| + \|f(s,0,0)\|] ds$$
$$+ \frac{1}{2} \left[\frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} [\|f(s, x(s), (\chi x)(s))) - f(s,0,0)\| + \|f(s,0,0)\|] ds \right]$$
$$+ \sum_{0<t_k<t} \|I_k(x(t_k^-))\|$$
$$\leq \frac{m((L+\gamma_0 M)r + M_1)T^q}{\Gamma(q+1)} + \frac{((L+\gamma_0 M)r + M_1)T^q}{\Gamma(q+1)}$$
$$+ \frac{1}{2} \frac{(L+\gamma_0 M)r + M_1)T^q}{\Gamma(q+1)} + mM^*$$
$$\leq \frac{((L+\gamma_0 M)r + M_1)T^q}{\Gamma(q+1)} (m + \frac{3}{2}) + mM^* \leq r, \tag{3.3}$$

by the choice of L, M, M^* and r. Clearly, the fixed point of the operator F are the solution of the problem (3.1). Thus, F maps B_r into itself. Now, for $x, y \in PC(I, X)$, we have $\|(Fx)(t) - (Fy)(t)\|$

$$\leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|f(s, x(s), (\chi x)(s)) - f(x, y(s), (\chi y)(s))\| ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|f(s, x(s), (\chi x)(s)) - f(x, y(s), (\chi y)(s))\| ds$$
$$+ \frac{1}{2} \left[\frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} \|f(s, x(s), (\chi x)) - f(x, y(s), (\chi x))\| ds \right]$$
$$+ \sum_{0<t_k<t} \|I_k(x(t_k^-)) - I_k(y(t_k^-))\|$$

$$\leq \frac{(L+\gamma_0 M)}{\Gamma(q)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|x(s) - y(s)\| ds$$
$$+ \frac{(L+\gamma_0 M)}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|x(s) - y(s)\| ds$$
$$+ \frac{(L+\gamma_0 M)}{2} \left[\frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} \|x(s) - y(s)\| ds \right] + \sum_{k=1}^{m} L^* \|x(t_k^-) - y(t_k^-)\|$$
$$\leq \frac{mLT^q}{\Gamma(q+1)} \|x-y\| + \frac{(L+\gamma_0 M)T^q}{\Gamma(q+1)} \|x-y\| + \frac{1}{2} \frac{(L+\gamma_0 M)T^q}{\Gamma(q+1)} \|x-y\|$$
$$+ mL^* \|x-y\| \leq \left[\frac{(L+\gamma_0 M)T^q}{\Gamma(q+1)} (m + \frac{3}{2}) + mL^* \right] \|x-y\|. \tag{3.4}$$

Thus,

$$\|Fx - Fy\|_C \leq \Lambda_{L,m,T,q} \|x-y\|,$$

where $\Lambda_{L,m,T,q} = \left[(L+\gamma_0 M)\eta \left(m + \frac{3}{2}\right) + mL^* \right]$. And since $\Lambda_{L,m,T,q} < 1$, F is a contraction mapping and therefore there exists a unique fixed point $x \in B_r$ such that $Fx(t) = x(t)$. Any fixed point of F is the solution of the problem (3.1). ∎

THEOREM 3.2. *Assume that* $(HA) - (HC)$ *hold with* $\mu(t)\eta < 1$. *Then the fractional impulsive boundary value problem with antiperiodic condition of equation (3.1) has at least one solution on* I, *provided that*

$$\mu(t)\eta + m(\mu(t)\eta + M^*) < 1. \tag{3.5}$$

P r o o f. For each positive integer r, let

$$B_r : \{x \in PC(I,X) : \|x\| \leq r, 0 \leq t \leq T\},$$

then B_r, for each r, is a bounded, closed, convex set in $PC(I,X)$. So F is well defined on B_r. We claim that there exists a positive number r such that $FB_r \subseteq B_r$. If it is not true, then for each positive number r, there is a function $x_r \in B_r$ but $Fx_r \notin B_r$, that is, $\|Fx_r(t)\| > r$ for some $t \in [0,T]$. However, on the other hand, we have

$$r \leq \|(Fx_r)(t)\| = \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|f(s, x_r(s))\| ds$$
$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|f(s, x_r(s))\| ds - \frac{1}{2} \left[\frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} \|f(s, x_r(s))\| ds \right]$$
$$+ \sum_{0<t_k<t} \|I_k(x_r(t_k^-))\| \leq m\mu(t)\gamma + \mu(t)r\gamma + \frac{1}{2}[\mu(t)r\gamma] + mrM^*.$$

Dividing both sides by r, we get

$$\frac{3}{2}\mu(t)\eta + m(\mu(t)\eta + M^*) \geq 1. \tag{3.6}$$

This contradicts (3.5). Hence $FB_r \subseteq B_r$, for some positive number r.

Now define the operators F_1 and F_2 on B_r as

$$\begin{aligned} F_1(x)(t) := & \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, x(s), (\chi x)))ds \\ & + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s, x(s), (\chi x)))ds \end{aligned} \tag{3.7}$$

and

$$F_2(x)(t) := \sum_{0<t_k<t} I_k(x(t_k^-)) - \frac{1}{2}\left[\frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1} f(s, x(s), (\chi x))ds\right] \tag{3.8}$$

We will show that F_2 is a contraction mapping and F_1 is a compact operator. It follows from the assumption (H2), F_2 is a contraction mapping if $mL^* + \frac{1}{2}M\gamma < 1$. The continuity of f implies that the operator $F_1$is continuous. Also F_1 is uniformly bounded on B_r as

$$\|F_1(x)\| \leq \frac{(m+1)T^q}{\Gamma(q+1)} \|\mu\|_{L^1}. \tag{3.9}$$

Now let us prove that $(F_1 x)(t)$ is equicontinuous. Let $t_1, t_2 \in I,\ \ t_1 < t_2$ and $x \in B_r$. Using the fact that f is bounded on the compact set $I \times B_r \times \chi(B_r)$. We define (thus $\sup_{(s,x,y)\in I\times B_r\times\chi(B_r)} \|f(t, s, x(s), (\chi x(s))\| := c_0 < \infty$), we will get

$$\begin{aligned} & \|F_1 x(t_2) - F_1 x(t_1)\| \\ & = \frac{1}{\Gamma(q)}\left[\sum_{0<t_k<t_2-t_1} (t_2 - t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, x(s), (\chi x)))ds\right] \\ & \quad + \frac{1}{\Gamma(q)}\left[\sum_{0<t_k<t_1} (t_2 - t_1) \int_{t_{k-1}}^{t_k} (t_2 - s)^{q-1} f(s, x(s), (\chi x)))ds\right] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s),(\chi x)))ds\\
&+\frac{1}{\Gamma(q)}\int_{t_k}^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]f(s,x(s),(\chi x)))ds\\
\le&\frac{c_0}{\Gamma(q)}\left[\sum_{0<t_k<t_2-t_1}|(t_k-t_{k-1})^{q-1}(t_2-t_k)|+\sum_{0<t_k<t_1}|(t_2-t_1)(t_k-t_{k-1})^{q-1}|\right]\\
&+\frac{c_0}{\Gamma(q+1)}[2(t_2-t_1)^q+|(t_2-t_k)^q-(t_1-t_k)^q|], \qquad (3.10)
\end{aligned}
$$

which does not depend on x. So $F_1(B_r)$ is relatively compact. As $t_2 \to t_1$, the righthand side of the above inequality tends to zero. By the Arzela-Ascoli theorem, F_1 is compact operator. These arguments show that $F = F_1 + F_2$ which is a condensing mapping on B_r, and by the Sadovskii fixed point theorem there exists a fixed point for F on B_r, which is a solution of the problem (3.1). The proof is complete. ∎

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