Fractional Calculus & Applied Analysis

An International Journal for Theory and Applications

VOLUME 13, NUMBER 3 (2010) ISSN 1311-0454

BOCHNER-HECKE THEOREMS FOR THE WEINSTEIN TRANSFORM AND APPLICATION

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Abstract

In this paper we prove Bochner-Hecke theorems for the Weinstein transform and we give an application to homogeneous distributions.

MSC 2010: 42B10, 44A15

Key Words and Phrases: Weinstein transform, Bochner-Hecke theorems, homogeneous distributions

1. Introduction

We consider the Weinstein operator $\Delta_{d,\alpha}$ defined on $\mathbb{R}^{d-1}\times]0,+\infty[$ by

$$\Delta_{d,\alpha} = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_d}\frac{\partial}{\partial x_d}\,, \quad \alpha \in \mathbb{R},\ \alpha > -\frac{1}{2}.$$

Then

$$\Delta_{d,\alpha} = \Delta_{d-1} + \ell_\alpha,$$

where Δ_{d-1} is the Laplacian operator in $\mathbb{R}^{d-1}$ and ℓ_α the Bessel operator with respect to the variable x_d defined by

$$\ell_\alpha = \frac{d^2}{dx_d^2} + \frac{2\alpha+1}{x_d}\,\frac{d}{dx_d}\,, \quad \alpha > -\frac{1}{2}\,.$$

The Weinstein operator $\Delta_{d,\alpha}$ has several applications in Pure and Applied Mathematics, especially in Fluid Mechanics, [3].

In this paper we consider the spherical harmonics associated with the Weinstein operator, and the Weinstein transform studied in [1], [2], [8], [9], [10].

The principles of constructing of multidimensional Fourier transforms associated with integral transforms, of the type considered in the paper are also well discussed in [4].

With the help of the Weinstein transform, the mean value property of the W-harmonic functions and the translation operator associated with the Weinstein operator, we prove a Hecke formula, a Funk-Hecke formula and Bochner-Hecke theorems for the Weinstein transform.

The analogues of these formulas and theorems have been proved in [6], [7], [12] for the classical Fourier transform on $\mathbb{R}^d$ and the Dunkl transform on $\mathbb{R}^d$.

As application of the Bochner-Heck theorems for the Weinstein transform, we determine the Weinstein transform of some homogeneous distributions on $\mathbb{R}^d$. An analogous application has been studied in the cases of the classical Fourier transform on $\mathbb{R}^d$ and the Dunkl transform on $\mathbb{R}^d$ (see [6], [10], [12]).

The contents of the paper is as follows:

- In Section 2 we give the main results concerning the Weinstein transform. - In Section 3 we study the translation operator associated with the Weinstein operator. - In Section 4 we define the mean value property of W-harmonic functions. - Section 5 is devoted to the Hecke formula associated with the Weinstein operator. - In Section 6 we give a proof of the Funk-Hecke formula for the Weinstein transform. - In Section 7 we give the Bochner-Hecke theorems for the Weinstein transform. - As an application of the results of the preceding sections, in Section 8 we determine the Weinstein transform of some homogeneous distributions on $\mathbb{R}^{d-1}\times]0,+\infty[$.

2. The eigenfunction of the operator $\Delta_{d,\alpha}$ and the Weinstein transform

2.1. The eigenfunction of the operator $\Delta_{d,\alpha}$

For all $\lambda = (\lambda_1, ..., \lambda_d) \in \mathbb{C}^d$, the system

$$\begin{cases} \dfrac{\partial^2 u(x)}{\partial x_i^2} &= -\lambda_i^2 u(x), \quad i = 1, ..., d-1, \\ \ell_\alpha u(x) &= -\lambda_d^2 u(x) \\ u(0) &= 1, \dfrac{\partial u}{\partial x_d}(0) = 0, \ \dfrac{\partial u}{\partial x_j}(0) = -i\lambda_j, \quad j = 1, ..., d-1, \end{cases}$$

has a unique solution on $\mathbb{R}^d$, denoted by Ψ_λ, and given by

$$\Psi_\lambda(x) = e^{-i\langle x', \lambda' \rangle} j_\alpha(x_d \lambda_d). \tag{2.1}$$

Here $x' = (x_1, ..., x_{d-1}), \lambda' = (\lambda_1, ..., \lambda_{d-1})$ and j_α is the normalized Bessel function of index α defined by

$$\forall\, z \in \mathbb{C}, \quad j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n! \Gamma(n+\alpha+1)}, \tag{2.2}$$

satisfying the Laplace type integral representation

$$\forall\, z \in \mathbb{C}, \quad j_\alpha(z) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^\pi e^{iz\cos\theta} (\sin\theta)^{2\alpha} d\theta\,. \tag{2.3}$$

Remark 2.1. From the relation (2.3) we deduce by change of variables that the function $j_\alpha(t\mu)$ admits for $\alpha > -\frac{1}{2}$, the Laplace type integral representation

$$j_\alpha(t\mu) = \frac{2\Gamma(\alpha+1)t^{-2\alpha}}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^t (t^2 - y^2)^{\alpha-\frac{1}{2}} \cos(\mu y) dy,$$

$\forall\ \mu \in \mathbb{C}, \forall\, t \in [0, +\infty[$.

By using [13], p. 165, and the preceding relation, we deduce that the function $j_\alpha(t\mu)$ possesses for $\alpha \in]-\frac{1}{2}, -\frac{3}{2}[$ the following type Laplace representation

$$j_\alpha(t\mu) = \frac{2\Gamma(\alpha+2)t^{-2(\alpha+1)}}{\sqrt{\pi}\Gamma(\alpha+\frac{3}{2})} \int_0^t (t^2-y^2)^{\alpha+\frac{1}{2}} [1 - \frac{\mu^2(t^2-y^2)}{2(\alpha+1)(2\alpha+3)}] \cos(ty) dy,$$

$\forall\, \mu \in \mathbb{C}, \quad \forall\, t \in [0, +\infty[$. As the kernel of this representation contains the parameter μ, then we cannot built a harmonic analysis associated with the Bessel operator ℓ_α for $\alpha \in]-\frac{1}{2}, -\frac{3}{2}[$, as for the case $\alpha > -\frac{1}{2}$. For this reason, we suppose in this paper the requirement $\alpha > -\frac{1}{2}$ (see [11]).

The function Ψ_λ has a unique extension to $\mathbb{C}^d \times \mathbb{C}^d$. It has the following properties:

i) $\forall\, \lambda, z \in \mathbb{C}^d, \quad \Psi_\lambda(z) = \Psi_z(\lambda),$ (2.4)

ii) $\forall\ \lambda, z \in \mathbb{C}^d, \quad \Psi_\lambda(-z) = \Psi_{-\lambda}(z),$ (2.5)

iii) $\forall\ \lambda, x \in \mathbb{R}^d, \quad |\Psi_\lambda(x)| \leq 1.$ (2.6)

2.2. The Weinstein transform

NOTATIONS: We denote by

- $C_*(\mathbb{R}^d)$ the space of continuous functions on $\mathbb{R}^d$, even with respect to the last variable, resp. $C_{*,c}(\mathbb{R}^d)$ denotes the subspace formed by functions with compact support.
- $\mathcal{D}_*(\mathbb{R}^d)$ the space of C^∞ functions on $\mathbb{R}^d$, even with respect to the last variable and with compact support.
- $S_*(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$, even with respect to the last variable, and rapidly decreasing together with their derivatives.

The topology of $S_*(\mathbb{R}^d)$ is defined by the seminorms $P_{\ell,m}, (\ell, m) \in \mathbb{N}^2$, given by

$$P_{\ell,m}(\varphi) = \sup_{\substack{|\mu| \leq m \\ x \in \mathbb{R}^d}} (1 + \|x\|^2)^\ell |D^\mu \varphi(x)|,$$

where $D^\mu = \dfrac{\partial^{|\mu|}}{\partial x_1^{\mu_1} ... \partial x_d^{\mu_d}}, \mu = (\mu_1, ..., \mu_d), |\mu| = \mu_1 + ... + \mu_d.$

- $L^p_\alpha(\mathbb{R}^{d-1} \times [0, +\infty[), 1 \leq p \leq +\infty$, the space of measurable functions f on $\mathbb{R}^{d-1} \times [0, +\infty[$ such that

$$\|f\|_{\alpha,p} = \left(\int_{\mathbb{R}^{d-1} \times [0,+\infty[} |f(x)|^p d\mu_\alpha(x) \right)^{1/p} < +\infty \quad \text{if } p \in [1, +\infty[,$$

$$\|f\|_{\alpha,\infty} = \operatorname{ess\,sup}_{x \in \mathbb{R}^{d-1} \times [0,+\infty[} |f(x)| < \infty, \quad \text{if } p = +\infty,$$

where μ_α is the measure defined by

$$d\mu_\alpha(x) = x_d^{2\alpha+1} dx = x_d^{2\alpha+1} dx_1 ... dx_d.$$

- $\mathcal{E}_*(\mathbb{R})$ the space of C^∞-functions on $\mathbb{R}^d$, even with respect to the last variable.

DEFINITION 2.1. The Weinstein transform $\mathcal{F}_W$ is defined on $L^1_\alpha(\mathbb{R}^{d-1} \times [0, +\infty[)$ by

$$\forall\ \lambda \in \mathbb{R}^d, \quad \mathcal{F}_W(f)(\lambda) = \int_{\mathbb{R}^{d-1} \times [0,+\infty[} f(x) \Psi_\lambda(x) d\mu_\alpha(x). \qquad (2.7)$$

Proposition 2.2.

i) For all $f \in L^1_\alpha(\mathbb{R}^{d-1} \times [0, +\infty[)$*, the function* $\mathcal{F}_W(f)$ *is continuous on* $\mathbb{R}^d$ *and we have*

$$\|\mathcal{F}_W(f)\|_{\alpha,\infty} \leq \|f\|_{\alpha,1}. \tag{2.8}$$

ii) For all $f \in S_*(\mathbb{R}^d)$ *and* $n \in \mathbb{N}$*, we have*

$$\forall\, \lambda \in \mathbb{R}^d, \quad \mathcal{F}_W(\Delta^n_{d,\alpha} f)(\lambda) = P_n(\lambda)\mathcal{F}_W(f)(\lambda). \tag{2.9}$$

and

$$\forall\, \lambda \in \mathbb{R}^d, \quad \Delta^n_{d,\alpha}(\mathcal{F}_W(f))(\lambda) = \mathcal{F}_W(P_n f)(\lambda), \tag{2.10}$$

where $P_n(\lambda) = (-1)^n \|\lambda\|^{2n} = (-1)^n(\lambda_1^2 + ... + \lambda_d^2)^n$.

Theorem 2.1. *The Weinstein transform is a topological isomorphism from* $S_*(\mathbb{R}^d)$ *onto itself. The inverse transform is given by*

$$\forall\, x \in \mathbb{R}^d, \quad \mathcal{F}_W^{-1}(f)(x) = C_\alpha \mathcal{F}_W(f)(-x_1, ..., -x_{d-1}, x_d), \tag{2.11}$$

where

$$C_\alpha = \frac{1}{(2\pi)^{d-1} 2^{2\alpha} (\Gamma(\alpha+1))^2}\ . \tag{2.12}$$

Theorem 2.2.

i) Plancherel formula: For all $f \in S_*(\mathbb{R}^d)$ *we have*

$$\int_{\mathbb{R}^{d-1}\times[0,+\infty[} |f(x)|^2 d\mu_\alpha(x) = \int_{\mathbb{R}^{d-1}\times[0,+\infty[} |\mathcal{F}_W(f)(\lambda)|^2 d\nu_\alpha(\lambda), \tag{2.13}$$

where $d\nu_\alpha(\lambda) = C_\alpha d\mu_\alpha(\lambda)$*, with* C_α *the constant given by (2.12).*

ii) Plancherel theorem: The Weinstein transform $\mathcal{F}_W$ *extends uniquely to an isomorphism isometric from* $L^2_\alpha[\mathbb{R}^{d-1} \times [0, +\infty[)$ *onto* $L^2_\alpha(\mathbb{R}^{d-1} \times [0, +\infty[, d\nu_\alpha(\lambda))$.

3. The translation operator associated with the Weinstein operator

Definition 3.2. The translation operator $T_x, x \in \mathbb{R}^{d-1} \times [0, +\infty[$, associated with the operator $\Delta_{d,\alpha}$ is defined for a continuous function f on $\mathbb{R}^d$ which is even with respect to the last variable and for all $y = (y', y_d) \in \mathbb{R}^{d-1} \times [0, +\infty[$ by

$$T_x f(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^\pi f(x'+y', \sqrt{x_d^2+y_d^2+2x_d y_d \cos\theta})(\sin\theta)^{2\alpha} d\theta. \tag{3.1}$$

Proposition 3.1. *The translation operator $T_x, x \in \mathbb{R}^{d-1} \times [0,+\infty[$, satisfies the following properties:*

i) For all continuous function f on $\mathbb{R}^d$ which is even with respect to the last variable and $x, y \in \mathbb{R}^{d-1} \times [0,+\infty[$, we have

$$T_x f(y) = T_y f(x)\ , \quad T_0 f = f.$$

ii) For all f in $\mathcal{E}_*(\mathbb{R}^d)$ and $y \in \mathbb{R}^{d-1} \times [0,+\infty[$, the function $x \to T_x f(y)$ belongs to $\mathcal{E}_*(\mathbb{R}^d)$.

iii) For all $x \in \mathbb{R}^{d-1} \times [0,+\infty[$, we have

$$\Delta_{d,\alpha} \circ T_x = T_x \circ \Delta_{d,\alpha}. \tag{3.2}$$

Proposition 3.2. *The space $S_*(\mathbb{R})$ is invariant under the operators $T_x, x \in \mathbb{R}^{d-1} \times [0,+\infty[$.*

Proposition 3.3. *For all f in $\mathcal{E}_*(\mathbb{R}^d)$ and $g \in S_*(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}^{d-1}\times[0,+\infty[} T_x f(y) g(y) d\mu_\alpha(y) = \int_{\mathbb{R}^{d-1}\times[0,+\infty[} f(y) T_x g(y) d\mu_\alpha(y). \tag{3.3}$$

Proposition 3.4. *For all f in $L^p_\alpha(\mathbb{R}^{d-1} \times [0,+\infty[), p \in [1,+\infty]$, and $x \in \mathbb{R}^{d-1} \times [0,+\infty[$ we have*

$$\|T_x f\|_{p,\alpha} \leq \|f\|_{p,\alpha}\ . \tag{3.4}$$

4. Mean value property of the W-harmonic functions

Definition 4.1. Let u be a function of class C^2 on $\mathbb{R}^{d-1} \times [0,+\infty[$, even with respect to the last variable. We say that the function u is W-harmonic, if

$$\forall\, x \in \mathbb{R}^{d-1} \times [0,+\infty[, \quad \Delta_{d,\alpha} u(x) = 0.$$

Definition 4.2. The mean value associated with the Weinstein operator $\Delta_{d,\alpha}$ of a function u in $C_*(\mathbb{R}^d)$ is defined by

$$\mathcal{M}^{\alpha}_{x,r}(u) = \frac{1}{\Omega_{d,\alpha}} \int_{S^{d-1}_{+}} T_x u(rw) w_d^{2\alpha+1} d\sigma_d(w), \tag{4.1}$$

where

$$\Omega_{d,\alpha} = \int_{S^{d-1}_{+}} w_d^{2\alpha+1} d\sigma_d(w) = \frac{\pi^{\frac{d-1}{2}}\Gamma(\alpha+1)}{\Gamma(\frac{d+2\alpha+1}{2})}$$

and $S^{d-1}_{+} = \{(x_1,...,x_{d-1},x_d) \in \mathbb{R}^d,\ x_1^2+...+x_{d-1}^2+x_d^2 = 1,\ x_d \geq 0\}$ and $d\sigma_d$ is the normalized surface measure on S^{d-1}_{+}.

DEFINITION 4.3. Let u be a function in $C_*(\mathbb{R}^d)$. We say that u satisfies the mean value property associated with the Weinstein operator $\Delta_{d,\alpha}$ if for all $r > 0$ and $x \in \mathbb{R}^{d-1} \times [0,+\infty[$. We have

$$u(x) = \mathcal{M}^{\alpha}_{x,r}(u).$$

THEOREM 4.1. *Let u be a W-harmonic function on $\mathbb{R}^{d-1} \times [0,+\infty[$. Then u satisfies the mean value property associated with the Weinstein operator $\Delta_{d,\alpha}$ (see [2], p.40).*

5. Hecke formula for the Weinstein transform

NOTATIONS: We denote by
- $\mathcal{P}^d_n$ the space of homogeneous polynomials of degree n.
- H^{α}_n the space of W-harmonic homogeneous polynomials of degree n. It is defined by

$$H^{\alpha}_n = (ker\Delta_{d,\alpha}) \cap \mathcal{P}^d_n.$$

THEOREM 5.1. *Let H be in H^{α}_n. Then we have the following relation*

$$\int_{\mathbb{R}^{d-1}\times[0,+\infty[} e^{-\frac{\|y\|^2}{2}} H(y)\Psi_y(x) y_d^{2\alpha+1} dy = c_{\alpha,n} e^{-\frac{\|x\|^2}{2}} H(x), \tag{5.1}$$

where

$$c_{\alpha,n} = 2^{\frac{d-1}{2}+\alpha} \pi^{\frac{d-1}{2}} i^n \Gamma(\alpha+1). \tag{5.2}$$

To prove this theorem we need the following lemma.

LEMMA 5.1. *Let H be in H^{α}_n, and f be a radial function in $S_*(\mathbb{R}^d)$. Then we have*

$$\int_{\mathbb{R}^{d-1}\times[0,+\infty[} T_x f(y) H(y) y_d^{2\alpha+1} dy = H(x) \int_{\mathbb{R}^{d-1}\times[0,+\infty[} f(y) y_d^{2\alpha+1} dy. \tag{5.3}$$

P r o o f. Since H is in H_n^α, then from Theorem 4.1 we have

$$H(x) = \mathcal{M}_{x,r}^\alpha(H) = \frac{1}{\Omega_{d,\alpha}} \int_{S_+^{d-1}} T_x H(r\omega) w_d^{2\alpha+1} d\sigma_d(\omega). \quad (5.4)$$

Let F be the function on $[0, +\infty[$ given by $f(x) = F(\|x\|)$. From the relation (3.3) and using the spherical coordinates we obtain

$$\int_{\mathbb{R}^{d-1}\times[0,+\infty[} T_x f(y) H(y) y_d^{2\alpha+1} dy$$
$$= \int_0^{+\infty} \int_{S_+^{d-1}} F(r) T_x H(ru)(ru_d)^{2\alpha+1} r^{d-1} dr d\sigma_d(u).$$

Using (5.4) and Fubini's theorem, we obtain

$$\int_{\mathbb{R}^{d-1}\times[0,+\infty[} T_x f(y) H(y) y_d^{2\alpha+1} dy = H(x) \int_{\mathbb{R}^{d-1}\times[0,+\infty[} f(y) y_d^{2\alpha+1} dy.$$

∎

P r o o f o f T h e o r e m 5.1.

We take $f(x) = e^{-\frac{\|x\|^2}{2}}$ with $\|x\|^2 = \sum_{i=1}^{d} x_i^2$. From relation (5.3) and the fact that

$$\int_{\mathbb{R}^{d-1}\times[0,+\infty[} e^{-\frac{\|y\|^2}{2}} y_d^{2\alpha+1} dy = 2^\alpha (2\pi)^{\frac{d-1}{2}} \Gamma(\alpha+1),$$

we obtain

$$\int_{\mathbb{R}^{d-1}\times[0,+\infty[} T_x(e^{-\frac{\|\xi\|^2}{2}})(y) H(y) y_d^{2\alpha+1} dy = H(x)(2\pi)^{\frac{d-1}{2}} 2^\alpha \Gamma(\alpha+1).$$

The relations (2.3), (3.1) give

$$T_x(e^{(-\frac{\|\xi\|^2}{2}})(y) = e^{-\frac{\|x\|^2+\|y\|^2}{2}} \psi_{iy}(x),$$

and then

$$\int_{\mathbb{R}^{d-1}\times[0,+\infty[} e^{-\frac{\|x\|^2+\|y\|^2}{2}} H(y)\psi_{iy}(x)\, y_d^{2\alpha+1} dy = 2^{\frac{d-1}{2}+\alpha} \pi^{\frac{d-1}{2}} \Gamma(\alpha+1) H(x).$$

By changing in this relation x by ix, and using the fact that

$$\forall\, x, y \in \mathbb{R}^d,\ \psi_{iy}(ix) = \psi_y(x),$$

we obtain

$$\int_{\mathbb{R}^{d-1}\times[0,+\infty[} e^{-\frac{\|y\|^2}{2}} H(y)\psi_y(x) y_d^{2\alpha+1} dy = c_{\alpha,n} e^{-\frac{\|x\|^2}{2}} H(x) \quad (5.5)$$

with $c_{\alpha,n}$ given by (5.2). ∎

REMARK. By a change of variables in (5.5), we obtain

$$\int_{\mathbb{R}^{d-1}\times[0,+\infty[} e^{-\lambda\frac{\|y\|^2}{2}} H(y) j_\alpha(x_d y_d) e^{-i\langle x',y'\rangle} y_d^{2\alpha+1} dy$$
$$= c_{\alpha,n}\lambda^{-(n+\alpha+\frac{d}{2}+\frac{1}{2})} e^{-\frac{\|x\|^2}{2\lambda}} H(x). \quad (5.6)$$

6. Funk-Hecke formula for the Weinstein transform

In this subsection we give a proof of a Funk-Hecke formula for the Weinstein transform.

THEOREM 6.1. *Let H be in H_n^α. Then for all $y\in\mathbb{R}^{d-1}\times[0,+\infty[$, we have*

$$\int_{S_+^{d-1}} H(u)\Psi_y(iu) u_d^{2\alpha+1} d\sigma_d(u) = a_{\alpha,n}\, j_{n+\alpha+\frac{d}{2}-\frac{1}{2}}(\|y\|) H(y), \quad (6.1)$$

where

$$a_{\alpha,n} = \frac{c_{\alpha,n} 2^{-n-\alpha-\frac{d}{2}+\frac{1}{2}}}{\Gamma(n+\alpha+\frac{d}{2}+\frac{1}{2})}, \quad (6.2)$$

and $j_{n+\alpha+\frac{d}{2}-\frac{1}{2}}$ is the normalized Bessel function of first kind and order $n+\alpha+\frac{d}{2}-1$.

P r o o f. Using the relation (5.6) and spherical coordinates $y = ru$ with $r\in]0,+\infty[$ and $u\in S_+^{d-1}$, and by making the change of variables $r=\sqrt{2s}$, we obtain

$$I = \int_{\mathbb{R}^{d-1}\times[0,+\infty[} e^{-\frac{\lambda}{2}\|y\|^2} H(y) e^{-i\langle x',y'\rangle} j_\alpha(x_d y_d) y_d^{2\alpha+1} dy$$
$$= \int_0^{+\infty}\int_{S_+^{d-1}} e^{-\lambda s}(2s)^{\frac{n+2\alpha+d-1}{2}} H(u) e^{-i\langle\sqrt{2s}x',u'\rangle} j_\alpha(\sqrt{2s}x_d u_d) u_d^{2\alpha+1} d\sigma_d(u) ds. \quad (6.3)$$

From Fubini's Theorem,

$$I = c_{\alpha,n}\lambda^{-(n+\alpha+\frac{d+1}{2})} e^{-\|x\|^2/2\lambda} H(x).$$

But from formula of [13], p. 394, we have

$$\lambda^{-(n+\alpha+\frac{d+1}{2})} e^{-\frac{\|x\|^2}{2\lambda}}$$
$$= \frac{1}{\Gamma(n+\alpha+\frac{d+1}{2})}\int_0^{+\infty} e^{-\lambda s} j_{n+\alpha+\frac{d-1}{2}}(\sqrt{2s}\|x\|) s^{n+\alpha+\frac{d-1}{2}} ds.$$

By using this relation in (6.3), we obtain

$$\int_0^{+\infty} e^{-\lambda s} s^{-\frac{n}{2}} [\int_{S_+^{d-1}} H(u) e^{-i\langle\sqrt{2s}x',u'\rangle} j_\alpha(\sqrt{2s}x_d u_d) u_d^{2\alpha+1} d\sigma_d(u)] s^{n+\alpha+\frac{d-1}{2}} ds$$

$$= \frac{c_{\alpha,n} 2^{\frac{n+2\alpha+d-1}{2}} H(x)}{\Gamma(n+\alpha+\frac{d+1}{2})} \int_0^{+\infty} e^{-\lambda s} j_{n+\alpha+\frac{d-1}{2}}(\sqrt{2s}\|x\|) s^{n+\alpha+\frac{d-1}{2}} ds.$$

The injectivity of the Laplace transform implies

$$\forall\ s > 0, \quad \int_{S_+^{d-1}} H(u) e^{-i\langle\sqrt{2s}x',u'\rangle} j_\alpha(\sqrt{2s}x_d u_d) u_d^{2\alpha+1} d\sigma_d(u)$$

$$= c_{\alpha,n} \frac{S^{\frac{n}{2}} 2^{-\frac{n}{2}-\alpha-\frac{d-1}{2}}}{\Gamma(n+\alpha+\frac{d+1}{2})} j_{n+\alpha+\frac{d-1}{2}}(\sqrt{2s}\|x\|) H(x).$$

For $s = \frac{1}{2}$ we obtain

$$\int_{S_+^{d-1}} H(u) e^{-i\langle x',u'\rangle} j_\alpha(x_d u_d) u_d^{2\alpha+1} d\sigma_d(u) = a_{\alpha,n} H(x) j_{n+\alpha+\frac{d-1}{2}}(\|x\|),$$

where

$$a_{\alpha,n} = \frac{2^{-n-\alpha-\frac{d}{2}+\frac{1}{2}}}{\Gamma(n+\alpha+\frac{d+1}{2})} c_{\alpha,n}.$$

■

7. Bochner-Hecke theorems for the Weinstein transform

In this section we give for the Weinstein transform the analogue of the classical Bochner-Hecke theorem, studied in [6], p. 66-70 and [7], p. 30-31.

THEOREM 7.1. *Let H be in H_n^α and f a measurable function on $[0, +\infty[$ such that*

$$\int_0^{+\infty} |f(x)|^{2n+2\alpha+d} < +\infty. \qquad (7.1)$$

Then the function $F(x) = f(\|x\|)H(x)$ belongs to $L^1_\alpha(\mathbb{R}^{d-1} \times [0, +\infty[)$ and its Weinstein transform is given by

$$\forall\ y \in \mathbb{R}^d, \quad \mathcal{F}_W(F)(\lambda) = c_{\alpha,n} H(\lambda) \mathcal{F}_B^{n+\alpha+\frac{d}{2}-1}(f)(\|\lambda\|), \qquad (7.2)$$

where $\mathcal{F}_B^\gamma$ is the Fourier-Bessel transform of order $\gamma, \gamma > -\frac{1}{2}$, given by

$$\mathcal{F}_B^\gamma(h)(\lambda) = \frac{1}{2^\gamma \Gamma(\gamma+1)} \int_0^{+\infty} h(r) j_\gamma(\lambda r) r^{2\gamma+1} dr.$$

P r o o f. The spherical coordinates and Fubini-Tonelli's Theorem imply that the function F belongs to $L^1_\alpha(\mathbb{R}^{d-1}\times[0,+\infty[)$,

$$\forall\,\lambda\in\mathbb{R}^d\,,\mathcal{F}_W(F)(\lambda)=\int_{\mathbb{R}^{d-1}\times[0,+\infty[}f(\|x\|)H(x)e^{-i\langle x',\lambda'\rangle}j_\alpha(x_d\lambda_d)x_d^{2\alpha+1}dx.$$

By using spherical coordinates, Fubini's Theorem and Theorem 6.1, we obtain

$$\begin{aligned}\forall\,\lambda\in\mathbb{R}^d,\ \ \mathcal{F}_W(F)(\lambda) &= a_{\alpha,n}H(\lambda)\int_0^{+\infty}j_{n+\alpha+\frac{d}{2}-\frac{1}{2}}(r\|\lambda\|)f(r)r^{2n+2\alpha+d}dr\\ &= c_{\alpha,n}H(\lambda)\mathcal{F}_B^{n+\alpha+\frac{d}{2}-\frac{1}{2}}(f)(\|\lambda\|),\end{aligned}$$

where $c_{\alpha,n}$ and $a_{\alpha,n}$ are the constants given by (5.2) and (6.2). ∎

To state and prove the second generalized Bochner-Hecke theorem, we need the following notations and lemmas.

NOTATIONS: Let for $n\in\mathbb{N}$ and $H\in H_n^\alpha$ we denote by

- $L^p_{(n,\alpha)}([0,+\infty[),p=1,2$, the space of measurable function f on $[0,+\infty[$ such that

$$\|f\|_{(n,\alpha),p}=\left(\int_0^{+\infty}|f(r)|^p r^{2n+2\alpha+d}dr\right)^{1/p}<+\infty.$$

- $L^2_{(\alpha,H)}(\mathbb{R}^d)=\{f(\|x\|)H(x)\in L^2_\alpha(\mathbb{R}^{d-1}\times[0,+\infty[)$ with f defined a.e. in $[0,+\infty[\}$.

LEMMA 7.1. *The operator* τ_H *from* $L^2_{(n,\alpha)}([0,+\infty[)$ *into* $L^2_{(\alpha,H)}(\mathbb{R}^d)$ *defined by*

$$\tau_H(f)(x)=f(\|x\|)H(x),$$

satisfies

$$\|\tau_H(f)\|_{\alpha,2}=k\|f\|_{(n,\alpha),2},$$

with

$$k=\left[\int_{S^{d-1}_+}|H(u)|^2d\sigma_d(u)\right]^{1/2}.$$

P r o o f. Using the spherical coordinates and Fubini's theorem, we obtain

$$\|\tau_H(f)\|^2_{\alpha,2}=\int_{\mathbb{R}^{d-1}\times[0,+\infty[}|\tau_Hf(x)|^2d\mu_\alpha(x)$$

$$=\Big(\int_{S^{d-1}_+}|H(u)|^2u_d^{2\alpha+1}d\sigma_d(u)\Big)\Big(\int_0^{+\infty}f(r)r^{2n+2\alpha+d}dr\Big)=k^2\|f\|^2_{(n,\alpha),2}.$$

∎

Lemma 7.2. *The set of linear combination of the functions* $r \to e^{-\lambda \frac{r^2}{2}}$, $\lambda > 0$, *is dense in* $L^2_{(n,\alpha)}([0,+\infty[)$.

P r o o f. We have to prove that each function φ in $L^2_{(n,\alpha)}([0,+\infty[)$ that satisfies

$$\int_0^{+\infty} \varphi(r) e^{-\mu r^2} r^{2n+2\alpha+d} dr = 0, \quad \text{for all } \mu > 0, \tag{7.3}$$

is the function equal to zero.

We consider the function

$$\psi(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ \varphi(\sqrt{x}) x^{n+\alpha+\frac{d}{2}-\frac{1}{2}} e^{-\frac{x}{2}}, & \text{if } x > 0 \end{cases}$$

By using the substitution $x = r^2$ and the Schwartz inequality, we obtain

$$\begin{aligned} \int_0^{+\infty} |\psi(x)| dx &= \int_0^{+\infty} |\varphi(\sqrt{x})| x^{n+\alpha+\frac{d}{2}-\frac{1}{2}} e^{-\frac{x}{2}} dx \\ &= 2\int_0^{+\infty} |\varphi(r)| r^{2n+2\alpha+d-1} e^{-\frac{r^2}{2}} r dr \end{aligned}$$

$$\leq 2\Big(\int_0^{+\infty} |\varphi(r)|^2 r^{2n+2\alpha+d} dr\Big)\Big(\int_0^{+\infty} e^{-r^2} r^{2n+2\alpha+d} dr\Big) < +\infty.$$

As supp ψ is contained in $[0,+\infty[$, then the function ψ is integrable on $\mathbb{R}$ with respect to the Lebesgue measure.

On the other hand, for all $s > 0$, the substitution $x = r^2$ implies

$$\int_0^{+\infty} \psi(x) e^{-sx} dx = 2\int_0^{+\infty} \varphi(r) r^{2n+2\alpha+d} e^{-r^2(\frac{1}{2}+s)} dr.$$

From this relation and (7.1) we deduce that

$$\int_0^{+\infty} \psi(x) e^{-sx} dx = 0.$$

The injectivity of the Laplace transform implies that $\psi = 0$ and then $\varphi = 0$. ∎

Theorem 7.2. *Let f be in $L^2_{(n,\alpha)}([0,+\infty[)$. Then:*

i) The function $F(x) = f(\|x\|)H(x)$ belongs to $L^2_{(\alpha,H)}(\mathbb{R}^d)$, and its Weinstein transform is of the form

$$\mathcal{F}_W(F)(y) = g(\|y\|)H(y), \quad y \in \mathbb{R}^d, \tag{7.4}$$

with g in $L^2_{(n,\alpha)}([0,+\infty[)$.

ii) *If moreover, f belongs to $L^1_{(n,\alpha)}([0,+\infty[)$, then we have*

$$\forall\, r \geq 0, \quad g(r) = c_{\alpha,n}\mathcal{F}_B^{n+\alpha+\frac{d}{2}-\frac{1}{2}}(f)(r), \tag{7.5}$$

with $c_{\alpha,n}$ the constant given in (5.2).

P r o o f.

i) From Lemma 7.1 it is clear that the function $F(x) = f(\|x\|)H(x)$ belongs to $L^2_{(\alpha,H)}([0,+\infty[)$.

Also from this lemma, up to a constant of normalization, the application $\mathcal{F}_W \circ \tau_H$ is an isometry from $L^2_{(n,\alpha)}([0,+\infty[)$ into $L^2_\alpha(\mathbb{R}^{d-1} \times [0,+\infty[)$. From relation (5.1) this isometry applies to all functions of the type $e^{-\lambda\frac{\|x\|^2}{2}}, \lambda > 0$, in the space $L^2_{(\alpha,H)}(\mathbb{R}^d)$. Then by using Lemma 7.2, we deduce that the space $L^2_{(\alpha,H)}(\mathbb{R}^d)$ is invariant under the Weinstein transform. Thus

$$\mathcal{F}_W(F)(y) = g(\|y\|)H(y), \quad y \in \mathbb{R}^d,$$

with g in $L^2_{(n,\alpha)}([0,+\infty[)$.

ii) Moreover, if f belongs to $L^1_{(n,\alpha)}([0,+\infty[)$, then we have

$$\int_0^{+\infty} |f(r)|r^{n+2\alpha+d}dr = \int_0^1 (|f(r)|r^n)r^{2\alpha+d}dr + \int_1^{+\infty} |f(r)|r^{n+2\alpha+d}dr.$$

By applying the Schwartz inequality to the first integral and by replacing r^n by r^{2n} in the second integral, we obtain

$$\int_0^{+\infty} |f(r)|r^{n+2\alpha+d}dr \leq \Big(\int_0^1 |f(r)r^n|^2 r^{2\alpha+d}dr\Big)^{1/2}\Big(\int_0^1 r^{2\alpha+d}dr\Big)^{1/2}$$
$$+ \int_1^{+\infty} |f(r)|r^{2n+2\alpha+d}dr \leq \frac{1}{2\alpha+d+1}\|f\|_{(n,\alpha),2} + \|f\|_{(n,\alpha),1} < +\infty.$$

Thus the function f satisfies the condition (7.1), Theorem 7.1, implies (7.4). We obtain (7.5) from (7.4) and (7.2).

■

8. Application

In this section we use the results of the preceding section to obtain the Weinstein transform of some homogeneous distributions on $\mathbb{R}^d$.

8.1. The Weinstein transform of distributions

NOTATIONS: We denote by

- $D'_*(\mathbb{R}^d)$ the space of distributions on $\mathbb{R}^d$ which are even with respect to the last variable. It is the topological dual of $\mathcal{D}_*(\mathbb{R}^d)$.
- $S'_*(\mathbb{R}^d)$ the space of tempered distributions on $\mathbb{R}^d$ which are even with respect to the last variable. It is the topological dual of $S_*(\mathbb{R}^d)$.

DEFINITION 8.1. The Weinstein transform of a distribution S in $S'_*(\mathbb{R}^d)$ is defined by

$$\langle \mathcal{F}_W(S), \varphi\rangle = \langle S, \mathcal{F}_W(\varphi)\rangle \ , \quad \varphi \in S_*(\mathbb{R}^d). \tag{8.1}$$

THEOREM 8.1. *The Weinstein transform is a topological isomorphism from $S'_*(\mathbb{R}^d)$ onto itself. The inverse transform is given by*

$$\langle \mathcal{F}_W^{-1}(S), \varphi\rangle = \langle S, \mathcal{F}_W^{-1}(\varphi)\rangle \ , \quad \varphi \in S_*(\mathbb{R}).$$

8.2. The Weinstein transform of homogeneous distributions

Let $\beta \in \mathbb{R}$. A function f defined on $\mathbb{R}^d$ is homogeneous of degree β, if for all $\lambda > 0$, we have

$$f(\lambda x) = \lambda^\beta f(x). \tag{8.2}$$

Let f be a locally integrable function on $\mathbb{R}^d$ with respect to the Lebesgue measure, and homogeneous of degree β. We consider the distribution $T_{fx_d^{2\alpha+1}}$ of $D'_*(\mathbb{R}^d)$ given by the function $fx_d^{2\alpha+1}$.

For all φ in $D_*(\mathbb{R}^d)$ and $\lambda > 0$ we have

$$\langle T_{fx_d^{2\alpha+1}}, \varphi_\lambda\rangle = \lambda^{-(d+2\alpha+\beta+1)}\langle T_{fx_d^{2\alpha+1}}, \varphi\rangle, \tag{8.3}$$

where $\varphi_\lambda(x) = \varphi(\lambda x)$ for all $x \in \mathbb{R}^d$. Since

$$\begin{aligned}\langle T_{fx_d^{2\alpha+1}}, \varphi_\lambda\rangle &= \int_{\mathbb{R}^{d-1}\times[0,+\infty[} f(x)\varphi_\lambda(x)d\mu_\alpha(x)\\ &= \int_{\mathbb{R}^{d-1}\times[0,+\infty[} f(x)\varphi(\lambda x)x_d^{2\alpha+1}dx_1...dx_d,\end{aligned}$$

then by the substitution $u = \lambda x$ we obtain

$$\begin{aligned}\langle T_{f_{x_d}^{2\alpha+1}},\varphi_\lambda\rangle &= \lambda^{-(d+2\alpha+\beta+1)}\int_{\mathbb{R}^{d-1}\times[0,+\infty[} f(u)\varphi(u)u_d^{2\alpha+1}du_1...du_d\\ &= \lambda^{-(d+2\alpha+\beta+1)}\langle T_{fx_d^{2\alpha+1}},\varphi\rangle.\end{aligned}$$

The relation (8.2) implies that in the Weinstein theory's, we say that a distribution S in $D'_*(\mathbb{R}^d)$ is homogeneous of degree β, if for all φ in $D_*(\mathbb{R}^d)$ and $\lambda > 0$, we have

$$\langle S,\varphi_\lambda\rangle = \lambda^{-(d+2\alpha+\beta+1)}\langle S,\varphi\rangle. \tag{8.4}$$

Remark. All homogeneous distributions in $D'_*(\mathbb{R}^d)$ belong to $S'_*(\mathbb{R}^d)$ (see [5], p. 154).

Proposition 8.1. *Let S be in $D'_*(\mathbb{R}^d)$, homogeneous of degree β. Then its Weinstein transform is homogeneous of degree $-(d+2\alpha+\beta+1)$.*

P r o o f. By the substitution $t = \lambda x$, we obtain, for all $y \in \mathbb{R}^d$:

$$\begin{aligned}\mathcal{F}_W(\varphi_\lambda)(y) &= \int_{\mathbb{R}^{d-1}\times[0,+\infty[}\varphi(t)e^{-i\langle\frac{t'}{\lambda},y'\rangle}j_\alpha(\frac{t_dy_d}{\lambda})\lambda^{-d-2\alpha-1}t_d^{2\alpha+1}dt\\ &= \lambda^{-d-2\alpha-1}\mathcal{F}_W(\varphi)(\frac{y}{\lambda}).\end{aligned}$$

From this relation and (8.4), we obtain

$$\langle\mathcal{F}_W(S),\varphi_\lambda\rangle = \langle S,\mathcal{F}_W(\varphi_\lambda)\rangle = \lambda^{-d-2\alpha-1}\langle S_y,\mathcal{F}_W(\varphi)(\frac{y}{\lambda})\rangle.$$

Thus,

$$\langle\mathcal{F}_W(S),\varphi_\lambda\rangle = \lambda^{\beta}\langle S,\mathcal{F}_W(\varphi)\rangle.$$

This completes the proof. ■

Proposition 8.2. *Let H be in H_n^α and $s \in \mathbb{C}$. Then the function $G_s(x) = \dfrac{H(x)}{\|x\|^s}$ is homogeneous of degree $n-s$.*

Proposition 8.2. *The Weinstein transform of the function G_S with $n < Res < n+2\alpha+d+1$ is given by*

$$\mathcal{F}_W(G_s)(y) = M_{\alpha,n,s}\,\frac{H(y)}{\|y\|^{2n+2\alpha+1+d-s}}\,, \quad y\in\mathbb{R}^d,$$

where

$$M_{\alpha,n,s} = \frac{c_{\alpha,n}}{\Gamma(s/2)}\,2^{n+\alpha+\frac{d}{2}-s+\frac{1}{2}}\Gamma(n+\alpha+\frac{d}{2}-s+\frac{1}{2}), \tag{8.5}$$

and $c_{\alpha,n}$ is given by (5.2).

P r o o f. We suppose first that
$$n+\alpha+\frac{1}{2}+\frac{d}{2}<Res<n+2\alpha+1+d.$$
We write G_s in the form
$$G_s(x)=G_s(x)\mathbf{1}_{B(0,1)}(x)+G_s(x)\mathbf{1}_{B^c(0,1)}(x),$$
where $B(0,1)$ is the closed unit ball of $\mathbb{R}^d$ and $B^c(0,1)$ its complementary domain, and $\mathbf{1}_{B(0,1)}, \mathbf{1}_{B^c(0,1)}$ are their characteristic functions.

It is clear that $G_s(x)\mathbf{1}_{B(0,1)}(x)$ is in $L^1_\alpha(\mathbb{R}^{d-1}\times[0,+\infty[)$ and $G_s(x)\mathbf{1}_{B^c(0,1)}$ is in $L^2_\alpha(\mathbb{R}^{d-1}\times[0,+\infty[)$.

By applying to these functions Theorems 7.1 and 7.2, we deduce that
$$\mathcal{F}_W(G_s)(y)=\mathcal{F}_W\Big(\frac{H(x)}{\|x\|^s}\Big)(y)=g(\|y\|)H(y),\quad y\in\mathbb{R}^d, \tag{8.6}$$
with a function g defined (a.e) on $[0,+\infty[$. As from Propositions 8.1, 8.2, the function $\mathcal{F}_W(G_s)$ is homogeneous of degree $-d-2\alpha-n+s-1$, then the function g is homogeneous of degree $-d-2\alpha-1-2n+s$. Thus it is necessarily of the form
$$g(\|y\|)=\frac{M_{n,\alpha,s}}{\|y\|^{2n+2\alpha+d+1-s}}, \tag{8.7}$$
where $M_{n,\alpha,s}$ is a constant. On the other hand, from (8.6), (8.7), for all φ in $S_*(\mathbb{R}^d)$ we have
$$\begin{aligned}\langle G_S,\mathcal{F}_W(\varphi)\rangle &= \langle\mathcal{F}_W(G_s),\varphi\rangle\\ &= \int_{\mathbb{R}^{d-1}\times[0,+\infty[}\frac{H(x)}{\|x\|^s}\mathcal{F}_W(\varphi)(x)d\mu_\alpha(x)\\ &= \int_{\mathbb{R}^{d-1}\times[0,+\infty[}\frac{M_{n,\alpha,s}}{\|x\|^{2n+2\alpha+d-s}}H(x)\varphi(x)d\mu_\alpha(x).\end{aligned} \tag{8.8}$$
To obtain the value of $M_{n,\alpha,s}$ we consider the function $\varphi(x)=e^{-\frac{\|x\|^2}{2}}H(x)$.

Then from (5.1) the relation (8.8) takes the form
$$\begin{aligned}c_{\alpha,n}&\int_{\mathbb{R}^{d-1}\times[0,+\infty[}\frac{H(x)}{\|x\|^s}\,e^{-\frac{\|x\|^2}{2}}H(x)d\mu_\alpha(x)\\ &= M_{n,\alpha,s}\int_{\mathbb{R}^{d-1}\times[0,+\infty[}\frac{H^2(x)e^{-\frac{\|x\|^2}{2}}}{\|x\|^{2n+2\alpha+d+1-s}}d\mu_\alpha(x).\end{aligned}$$
By using spherical coordinates and Fubini's theorem, we deduce that
$$c_{\alpha,n}\int_0^{+\infty}e^{-\frac{r^2}{2}}r^{2n+2\alpha+d-s}dr=M_{n,\alpha,s}\int_0^{+\infty}e^{-\frac{r^2}{2}}r^{s-1}dr.$$
The definition of the function gamma implies the relation (8.5).

We have proved the relation (8.8) in the case $n+\alpha+\frac{1}{2}+\frac{d}{2} < Res < n+2\alpha+1+d$. But the two members of this relation are analytic functions of the complex variable s in the strip $n < Res < n+2\alpha+1+d$.

The identity (8.8) is then true in this strip.

This completes the proof of the theorem. ■

We consider now the function

$$G(x) = \frac{H(x)}{\|x\|^{n+2\alpha+1+d}}, \tag{8.9}$$

where H is in H_n^α, with $n \geq 1$.

LEMMA 8.1. *We denote also by G, the distribution defined by the relation*

$$\begin{aligned} \langle G, \varphi \rangle &= vp \int_{\mathbb{R}^d} G(x)\varphi(x) d\mu_\alpha(x) \\ &= \lim_{\varepsilon\to 0} \int_{\|x\|\geq\varepsilon>0} G(x)\varphi(x) d\mu_\alpha(x), \quad \varphi \in S_*(\mathbb{R}^d). \end{aligned} \tag{8.10}$$

Then this distribution belongs to $S'_*(\mathbb{R}^d)$.

P r o o f. We have

$$\int_{\mathbb{R}^d} G(x)\varphi(x)d\mu_\alpha(x) = \int_{B(0,1)} G(x)\varphi(x)d\mu_\alpha(x) + \int_{B^c(0,1)} G(x)\varphi(x)d\mu_\alpha(x), \tag{8.11}$$

where $B(0,1)$ is the unit closed ball of $\mathbb{R}^d$ and $B^c(0,1)$ its complementary domain. As the function $G(x)\mathbf{1}_{B^c(0,1)}(x)$ with $\mathbf{1}_{B^c(0,1)}$ the characteristic function of $B^c(0,1)$, belongs to $L^2_\alpha(\mathbb{R}^{d-1}\times[0,+\infty[)$, then we deduce that there exist $\ell \in \mathbb{N}\backslash\{0\}$ and a positive constant c_1 such that

$$\left| \int_{B^c(0,1)} G(x)\varphi(x)d\mu_\alpha(x) \right| \leq c_1 P_{\ell,0}(\varphi). \tag{8.12}$$

On the other hand, as the degree of H is greater than one, then by using spherical coordinates and Fubini's theorem and the orthogonality of the polynomials H, we obtain

$$\int_{\varepsilon\leq\|x\|\leq 1} G(x)d\mu_\alpha(x) = \int_\varepsilon^1 \frac{1}{r}\left(\int_{S_+^{d-1}} H(u)d\sigma_d(u)\right)dr = 0.$$

Thus

$$\int_{\varepsilon\le\|x\|\le 1} G(x)\varphi(x)d\mu_\alpha(x) = \int_{\varepsilon\le\|x\|\le 1} G(x)[\varphi(x)-\varphi(0)]d\mu_\alpha(x).$$

From Taylor's formula we deduce that

$$|\varphi(x)-\varphi(0)| \le \|x\| \sup_{x\in\mathbb{R}^d} |\frac{\partial}{\partial x_1}\varphi(x)+...+\frac{\partial}{\partial x_d}\ \varphi(x)|. \tag{8.13}$$

As the function $\|x\|G(x)\mathbf{1}_{B(0,1)}(x)$ belongs to $L^1_\alpha(\mathbb{R}^{d-1}\times[0,+\infty[)$, then

$$\int_{\varepsilon\le\|x\|\le 1} |G(x)||\varphi(x)-\varphi(0)|d\mu_\alpha(x) \le c_2 \sup_{x\in\mathbb{R}^d} |\frac{\partial}{\partial x_1}\varphi(x)+...+\frac{\partial}{\partial x_d}\varphi(x)|,$$

with

$$c_2 = \int_{B(0,1)} \|x\|G(x)d\mu_\alpha(x).$$

Using (8.10), (8.11), (8.12), (8.13), we deduce that there exists a positive constant C such that

$$|\langle G,\varphi\rangle| \le CP_{\ell,1}(\varphi).$$

Thus the distribution G belongs to $S'_*(\mathbb{R}^d)$. ■

THEOREM 8.3. *The Weinstein transform of the distribution G given by (8.10) is the distribution T_F in $S'_*(\mathbb{R}^d)$ given by the function F, with*

$$F(y) = M^0_{n,\alpha}\ \frac{H(y)}{\|y\|^n}\ , \quad y\in\mathbb{R}^d, \tag{8.14}$$

where

$$M^0_{n,\alpha} = C_\alpha 2^{-\alpha-\frac{1}{2}-\frac{d}{2}}\ \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+2\alpha+1+d}{2})}, \tag{8.15}$$

where C_α is the constant given by (2.12).

P r o o f. We shall see that to obtain (8.14) it suffices to take $s = n+2\alpha+1+d$ in Theorem 8.2.

In the proof of Theorem 8.2 we have shown that for $n < Res < n+2\alpha+1+d$, we have

$$\forall\ \varphi\in S(\mathbb{R}^d)\ ,\ M_{n,\alpha,s}\int_{\mathbb{R}^d} \frac{H(y)\varphi(y)}{\|y\|^{2n+2\alpha+1+d-s}}d\mu_\alpha(y) = \int_{\mathbb{R}^d} \frac{H(y)}{\|y\|^s}\mathcal{F}_W(\varphi)(y)d\mu_\alpha(y). \tag{8.16}$$

It is clear that in the left handside, when s tends to $n+2\alpha+1+d$, we obtain $M^0_{n,\alpha}\int_{\mathbb{R}^d}\frac{H(y)}{\|y\|^n}d\mu_\alpha(y)$ with $M^0_{n,\alpha}$ given by (8.15). On the other hand, by using the fact that $\int_{S_+^{d-1}} H(u)d\sigma_d(u) = 0$, and by considering the function $\psi = \mathcal{F}_W(\varphi)$ in the right handside of (8.16), we obtain

$$\lim_{s\to n+2\alpha+1+\alpha} \int_{\mathbb{R}^d} \frac{H(y)}{\|y\|^s} \mathcal{F}_W(\varphi)(y)d\mu_\alpha(y)$$

$$\begin{aligned}
&= \lim_{s\to n+2\alpha+1+d} \Big[\int_{B(0,1)} \frac{H(y)}{\|y\|^s} [\psi(y)-\psi(0)]d\mu_\alpha(y) + \int_{B^c_{(0,1)}} \frac{H(y)}{\|y\|} \psi(y)d\mu_\alpha(u) \Big] \\
&= \int_{B(0,1)} G(y)[\psi(y) - \psi(0)]d\mu_\alpha(y) + \int_{B^c(0,1)} G(y)\psi(y)d\mu_\alpha(y) \\
&= vp \int_{\mathbb{R}^d} G(y)\psi(y)d\mu_\alpha(y) = \langle G, \psi \rangle.
\end{aligned}$$

Thus we obtain (8.14). ∎

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