

NUMERICAL SOLUTION OF FRACTIONAL
DIFFUSION-WAVE EQUATION WITH
TWO SPACE VARIABLES BY MATRIX METHOD

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Abstract

In the present paper we solve space-time fractional diffusion-wave equation with two space variables, using the matrix method. Here, in particular, we give solutions to classical diffusion and wave equations and fractional diffusion and wave equations with different combinations of time and space fractional derivatives. We also plot some graphs for these problems with the help of *MATLAB* routines.

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1. Introduction

Fractional calculus is now considered as a practical technique in many branches of science including physics [6, 24]. A growing number of works in science and engineering deal with dynamical systems described by fractional order equations that involve derivatives and integrals of non-integer

order [1, 20, 35]. These new models are more adequate than the previously used integer order models, because fractional order derivatives and integrals describe the memory and hereditary properties of different substances [27]. This is the most significant advantage of the fractional order models in comparison with integer order models, in which such effects are neglected. In the context of flow in porous media, fractional space derivatives model large motions through highly conductive layers or fractures, while fractional time derivatives describe particles that remain motionless for extended period of time [19].

Recent applications of fractional equations to a number of systems such as those exhibiting enormously slow diffusion or sub-diffusion have given opportunity for physicists to study even more complicated systems. The generalized diffusion equation allows describing complex systems with anomalous behavior in much the same way as simpler systems.

R.R. Nigmatullin [22, 23] considered a fractional diffusion equation in the form

$${}_0D_t^\alpha u(x, t) = \frac{d^2 u(x, t)}{dx^2}, \quad (1)$$

obtained from the standard diffusion equation by replacing the first order time derivative with derivative of arbitrary real order $\alpha > 0$. It is called the fractional diffusion-wave equation. This name has been suggested by Mainardi [14, 17]. For $\alpha = 1$ equation (1) becomes the classical diffusion equation, for $\alpha = 2$ it becomes the classical wave equation and for $1 < \alpha < 2$, it may be seen as a sort of interpolation between the classical diffusion and classical wave equations. For $0 < \alpha < 1$ we have so-called ultraslow diffusion, and values $1 < \alpha < 2$ correspond to so-called intermediate processes [7].

The symmetric space-fractional diffusion equation is obtained from the classical diffusion equation by replacing the second-order space derivative by a symmetric space fractional derivative of order β with $0 < \beta \leq 2$. This equation is written as [11, p.15, Eq. 6.1]

$$\frac{\partial u}{\partial t} = \chi \frac{\partial^\beta u}{\partial |x|^\beta}. \quad (2)$$

A space-time fractional diffusion equation, obtained from the standard diffusion equation by replacing the first order time derivative by a fractional derivative of order α , $0 < \alpha \leq 2$ and the second order space derivative by a fractional derivative of order $\beta > 0$, has been treated by a number of authors notably, Saichev and Zaslavsky [30], Uchaikin and Zolotarev [33], Gorenflo,

Iskenderov and Luchko [6], Scalas, Gorenflo and Mainardi [31], Metzler and Klafter [20], Mainardi [12] and Yang [34]. A number of researchers, as Mainardi [13, 15, 16], Ray [28, 29], Chen et al. [3], Schot et al. [32], Erochenkova and Lima [5], McLean and Mustafa [18], Murio and Carlos [21], Kochubei [10], Sayed [4] have studied the space and/or time fractional diffusion equations.

Recently, Podlubny et al. [25] and Podlubny [26] have developed a matrix method for solving the space-time fractional diffusion equation in the following form

$${}_0^C D_t^\alpha u(x, t) = \chi \frac{\partial^\beta u(x, t)}{\partial |x|^\beta}, \tag{3}$$

where ${}_0^C D_t^\alpha u$ is the Caputo derivative of order α , $0 < \alpha \leq 1$, defined by equation (7) and $\frac{\partial^\beta}{\partial |x|^\beta}$ is the symmetric Riesz fractional derivative of order β , $1 < \beta \leq 2$, defined by equation (8).

In the present paper, we extend the matrix method to solve space-time fractional diffusion-wave equations with two space variables:

$${}_0^C D_t^\alpha u = \chi \left(\frac{\partial^\beta u}{\partial |x|^\beta} + \frac{\partial^\gamma u}{\partial |y|^\gamma} \right), \tag{4}$$

where ${}_0^C D_t^\alpha u$ is the Caputo derivative of order α , $0 < \alpha \leq 2$, defined by equation (7) and $\frac{\partial^\beta}{\partial |x|^\beta}$, $\frac{\partial^\gamma}{\partial |y|^\gamma}$ are symmetric Riesz derivatives of order β , γ respectively, $0 < \beta \leq 2$, $0 < \gamma \leq 2$, defined by equation (8). The equation (4) yields the following forms of diffusion-wave equations for various values of the parameters α , β and γ :

- $\alpha = 1, \beta = 2, \gamma = 2$: Classical diffusion equation,
- $0 < \alpha < 1, \beta = 2, \gamma = 2$: Time-fractional diffusion equation,
- $\alpha = 1, 0 < \beta \leq 2, 0 < \gamma < 2$ (or $\alpha = 1, 0 < \beta < 2, 0 < \gamma \leq 2$): Space-fractional diffusion equation,
- $0 < \alpha < 1, 0 < \beta < 2, 0 < \gamma < 2$: Space-time fractional diffusion equation,
- $\alpha = 2, \beta = 2, \gamma = 2$: Classical wave equation,
- $1 < \alpha < 2, \beta = 2, \gamma = 2$: Time-fractional wave equation,
- $\alpha = 2, 0 < \beta \leq 2, 0 < \gamma < 2$ (or $\alpha = 2, 0 < \beta < 2, 0 < \gamma \leq 2$): Space-fractional wave equation,
- $1 < \alpha < 2, 0 < \beta < 2, 0 < \gamma < 2$: Space-time fractional wave equation.

The paper is organized as follows. In Section 2, we provide definitions which shall be used in the subsequent sections. In Section 3, we give discretization of space-time fractional diffusion-wave equations with two space variables by extending the matrix approach given by Podlubny et al. [25]. In Section 4, we consider example of fractional diffusion equation with different combinations of time and space fractional derivatives. In Section 5, we consider example of fractional wave equation on the lines of Section 4. At the end, we give some graphs for the problems considered in Sections 4 and 5 with the help of *MATLAB* routines.

2. Definitions

For $\alpha > 0$, the *left-sided Riemann-Liouville fractional derivative of order α* , for a real valued function $\varphi(x)$ defined on $R_+ = (0, \infty)$, is defined as [9, p.70, Eq. 2.1.5]:

$${}_a D_x^\alpha \varphi(x) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx} \right)^m \int_a^x \frac{\varphi(\xi) d\xi}{(x-\xi)^{\alpha-m+1}}, \quad (m-1 < \alpha \leq m, m \in N). \quad (5)$$

The *right-sided Riemann-Liouville fractional derivative of order α* , is given by [9, p.70, Eq. 2.1.6]:

$${}_x D_b^\alpha \varphi(x) = \frac{1}{\Gamma(m-\alpha)} \left(-\frac{d}{dx} \right)^m \int_x^b \frac{\varphi(\xi) d\xi}{(\xi-x)^{\alpha-m+1}}, \quad (m-1 < \alpha \leq m), \quad (6)$$

Caputo fractional derivative of order α , is defined as [2]:

$${}^C D_t^\alpha \varphi(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{\varphi^{(m)}(\xi) d\xi}{(t-\xi)^{\alpha-m+1}}, \quad (m-1 < \alpha \leq m), \quad (7)$$

Symmetric Riesz derivative of order β is defined as [30]:

$$\frac{d^\beta \varphi(x)}{d|x|^\beta} = D_R^\beta \varphi(x) = \frac{1}{2} \left[{}_a D_x^\beta \varphi(x) + {}_x D_b^\beta \varphi(x) \right] \quad (m-1 < \beta \leq m). \quad (8)$$

The Kronecker matrix product is defined as below:

Let A and B be $n \times m$ and $p \times q$ matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pq} \end{bmatrix}, \quad (9)$$

then the Kronecker matrix product $A \otimes B$ is the $np \times mq$ matrix having the following block structure:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{bmatrix}. \tag{10}$$

3. Discretization of derivatives in time and space

We discretize the general space-time fractional diffusion-wave equation

$${}_0^C D_t^\alpha u - \chi \left(\frac{\partial^\beta u}{\partial |x|^\beta} + \frac{\partial^\gamma u}{\partial |y|^\gamma} \right) = f(x, y, t) \tag{11}$$

in the following form:

$$\left\{ B_n^{(\alpha)} \otimes E_{mp} - \chi \left(E_n \otimes R_m^{(\beta)} \otimes E_p + E_{nm} \otimes R_p^{(\gamma)} \right) \right\} u_{nmp} = f_{nmp}. \tag{12}$$

For this, we consider the nodes $(ih, jk, l\tau)$, $l = 1, 2, \dots, n$, corresponding to all time layers at $(i, j)^{th}$ spatial discretization node, where $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, p$.

Following the lines of the paper by Podlubny et al. [25, Eq. 31], we can say that values of α^{th} order Caputo derivative of $u(x, y, t)$, with respect to time t , at these nodes can be approximated using the discrete analogue of differentiation of arbitrary order given by

$$\left[u_{i,j,n}^{(\alpha)} \ u_{i,j,n-1}^{(\alpha)} \ \cdots \ u_{i,j,2}^{(\alpha)} \ u_{i,j,1}^{(\alpha)} \right] = B_n^{(\alpha)} \left[u_{i,j,n} \ u_{i,j,n-1} \ \cdots \ u_{i,j,2} \ u_{i,j,1} \right]^T, \tag{13}$$

where

$$B_n^{(\alpha)} = \frac{1}{\tau^\alpha} \begin{bmatrix} w_1^{(\alpha)} & w_2^{(\alpha)} & \cdots & \cdots & w_{n-1}^{(\alpha)} & w_n^{(\alpha)} \\ 0 & w_1^{(\alpha)} & w_2^{(\alpha)} & \cdots & \cdots & w_{n-1}^{(\alpha)} \\ 0 & 0 & w_1^{(\alpha)} & w_2^{(\alpha)} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & w_1^{(\alpha)} & w_2^{(\alpha)} \\ 0 & 0 & \cdots & 0 & 0 & w_1^{(\alpha)} \end{bmatrix}, \tag{14}$$

$$w_j^\alpha = (-1)^j \binom{\alpha}{j}, \quad j = 1, 2, \dots, n. \tag{15}$$

The symmetric Riesz derivatives of $u(x, y, t)$, with respect to space variables x and y , at these nodes can be approximated as [25, Eq. 27]

$$\begin{bmatrix} u_{m,j,l}^{(\beta)} & u_{m-1,j,l}^{(\beta)} & \dots & u_{2,j,l}^{(\beta)} & u_{1,j,l}^{(\beta)} \end{bmatrix} = R_m^{(\beta)} [u_{m,j,l} \ u_{m-1,j,l} \ \dots \ u_{2,j,l} \ u_{1,j,l}]^T, \quad (16)$$

$$\begin{bmatrix} u_{i,p,l}^{(\gamma)} & u_{i,p-1,l}^{(\gamma)} & \dots & u_{i,2,l}^{(\gamma)} & u_{i,1,l}^{(\gamma)} \end{bmatrix} = R_p^{(\gamma)} [u_{i,p,l} \ u_{i,p-1,l} \ \dots \ u_{i,2,l} \ u_{i,1,l}]^T, \quad (17)$$

where

$$R_m^{(\beta)} = h^{-\beta} \begin{bmatrix} w_1^{(\beta)} & w_2^{(\beta)} & w_3^{(\beta)} & w_4^{(\beta)} & \dots & w_m^{(\beta)} \\ w_2^{(\beta)} & w_1^{(\beta)} & w_2^{(\beta)} & w_3^{(\beta)} & \dots & w_{m-1}^{(\beta)} \\ w_3^{(\beta)} & w_2^{(\beta)} & w_1^{(\beta)} & w_2^{(\beta)} & \dots & w_{m-2}^{(\beta)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ w_{m-1}^{(\beta)} & \dots & w_3^{(\beta)} & w_2^{(\beta)} & w_1^{(\beta)} & w_2^{(\beta)} \\ w_m^{(\beta)} & w_{m-1}^{(\beta)} & \dots & w_3^{(\beta)} & w_2^{(\beta)} & w_1^{(\beta)} \end{bmatrix}, \quad (18)$$

$$w_s^{(\beta)} = \frac{(-1)^s \Gamma(\beta + 1) \cos(\beta\pi/2)}{\Gamma(\beta/2 - s + 1) \Gamma(\beta/2 + s + 1)}, \quad s = 1, 2, \dots, m, \quad (19)$$

and

$$R_p^{(\gamma)} = k^{-\gamma} \begin{bmatrix} w_1^{(\gamma)} & w_2^{(\gamma)} & w_3^{(\gamma)} & w_4^{(\gamma)} & \dots & w_m^{(\gamma)} \\ w_2^{(\gamma)} & w_1^{(\gamma)} & w_2^{(\gamma)} & w_3^{(\gamma)} & \dots & w_{m-1}^{(\gamma)} \\ w_3^{(\gamma)} & w_2^{(\gamma)} & w_1^{(\gamma)} & w_2^{(\gamma)} & \dots & w_{m-2}^{(\gamma)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ w_{m-1}^{(\gamma)} & \dots & w_3^{(\gamma)} & w_2^{(\gamma)} & w_1^{(\gamma)} & w_2^{(\gamma)} \\ w_m^{(\gamma)} & w_{m-1}^{(\gamma)} & \dots & w_3^{(\gamma)} & w_2^{(\gamma)} & w_1^{(\gamma)} \end{bmatrix}, \quad (20)$$

$$w_s^{(\gamma)} = \frac{(-1)^s \Gamma(\gamma + 1) \cos(\gamma\pi/2)}{\Gamma(\gamma/2 - s + 1) \Gamma(\gamma/2 + s + 1)}, \quad s = 1, 2, \dots, p. \quad (21)$$

Before we proceed to find matrices corresponding to the fractional time and space derivatives of $u(x, y, t)$, we need to arrange all function values $u_{i,j,l}$ at the discretization nodes in the form of a column vector as follows:

$$\begin{aligned}
 u_{nmp} = & \\
 & \begin{bmatrix}
 u_{p,m,n} & u_{p-1,m,n} & u_{p-2,m,n} & \cdots & u_{1,m,n} \\
 u_{p,m-1,n} & u_{p-1,m-1,n} & u_{p-2,m-1,n} & \cdots & u_{1,m-1,n} \\
 u_{p,m-2,n} & u_{p-1,m-2,n} & u_{p-2,m-2,n} & \cdots & u_{1,m-2,n} \\
 \cdots & \cdots & \cdots & \cdots & \cdots \\
 u_{p,1,n} & u_{p-1,1,n} & u_{p-2,1,n} & \cdots & u_{1,1,n} \\
 u_{p,m,n-1} & u_{p-1,m,n-1} & u_{p-2,m,n-1} & \cdots & u_{1,m,n-1} \\
 u_{p,m-1,n-1} & u_{p-1,m-1,n-1} & u_{p-2,m-1,n-1} & \cdots & u_{1,m-1,n-1} \\
 u_{p,m-2,n-1} & u_{p-1,m-2,n-1} & u_{p-2,m-2,n-1} & \cdots & u_{1,m-2,n-1} \\
 \cdots & \cdots & \cdots & \cdots & \cdots \\
 u_{p,1,n-1} & u_{p-1,1,n-1} & u_{p-2,1,n-1} & \cdots & u_{1,1,n-1} \\
 \cdots & \cdots & \cdots & \cdots & \cdots \\
 \cdots & \cdots & \cdots & \cdots & \cdots \\
 \cdots & \cdots & \cdots & \cdots & \cdots \\
 u_{p,m,1} & u_{p-1,m,1} & u_{p-2,m,1} & \cdots & u_{1,m,1} \\
 u_{p,m-1,1} & u_{p-1,m-1,1} & u_{p-2,m-1,1} & \cdots & u_{1,m-1,1} \\
 u_{p,m-2,1} & u_{p-1,m-2,1} & u_{p-2,m-2,1} & \cdots & u_{1,m-2,1} \\
 \cdots & \cdots & \cdots & \cdots & \cdots \\
 u_{p,1,1} & u_{p-1,1,1} & u_{p-2,1,1} & \cdots & u_{1,1,1} \end{bmatrix}^T
 \end{aligned} \tag{22}$$

Here, we first take the nodes of n^{th} time layer, then the nodes of $(n-1)^{th}$ time layer and so forth and put them in a vertical column stack. The matrix $T_{nmp}^{(\alpha)}$, transforming the vector u_{nmp} to the vector of the partial fractional derivative of order α with respect to time variable is obtained by Kronecker product of the matrix $B_n^{(\alpha)}$, given by (14), and the unit matrix E_{mp} (of order mp) as follows:

$$T_{nmp}^{(\alpha)} = B_n^{(\alpha)} \otimes E_{mp}. \tag{23}$$

Similarly the matrices $S1_{nmp}^{(\beta)}$ and $S2_{nmp}^{(\gamma)}$ transforming the vector u_{nmp} to the vector of the partial fractional derivative of order β and γ with respect to spatial variables x and y respectively, can be obtained as Kronecker products of appropriate unit matrices with the matrices $R_m^{(\beta)}$ and $R_p^{(\gamma)}$, given by equation (18) and (20) respectively, as follows:

$$S1_{nmp}^{(\beta)} = E_n \otimes R_m^{(\beta)} \otimes E_p, \tag{24}$$

$$S2_{nmp}^{(\gamma)} = E_{nm} \otimes R_p^{(\gamma)}. \tag{25}$$

Using these approximations in (11), we arrive at the form (12).

4. Solution of fractional and classical diffusion equation

Here, we give the numerical solution of a general space-time fractional diffusion equation, with time fractional derivative as left-sided Riemann-Liouville derivative or Caputo fractional derivative (as the two are equal at zero initial conditions) and space fractional derivatives as the symmetric Riesz fractional derivative. As its particular cases, we first consider the classical diffusion equation and the results obtained are in agreement with the analytical solution obtained by the method of separation of variables. Next, we obtain the numerical solution of a time-fractional diffusion equation with Caputo fractional derivative. Finally, we consider space-fractional diffusion equation with symmetric Riesz fractional derivatives.

We use the described method to solve diffusion equation in most general situation, when both time and spatial derivatives are of fractional order. Consider the problem

$$\begin{aligned} {}_0^C D_t^\alpha u &= \frac{\partial^\beta u}{\partial |x|^\beta} + \frac{\partial^\gamma u}{\partial |y|^\gamma}, \quad 0 < \alpha \leq 1, \quad 0 < \beta \leq 2 \quad \text{and} \quad 0 < \gamma \leq 2, \\ u(x, 0, t) &= u(x, 1, t) = u(0, y, t) = u(1, y, t) = 0, \\ u(x, y, 0) &= x(1-x)y(1-y), \end{aligned} \tag{26}$$

where the time fractional derivative is the Caputo fractional derivative, the space fractional derivatives are considered as symmetric Riesz fractional derivatives.

For using Podlubny's matrix method (see [25, p.16]), the initial and boundary conditions must be equal to zero. If it is not so, then an auxiliary unknown function must be introduced, which satisfies the zero initial and boundary conditions. To reduce this problem to a problem with zero initial conditions (boundary conditions are already zero), we introduce an auxiliary function

$$v(x, y, t) = u(x, y, t) - u(x, y, 0). \tag{27}$$

As the Caputo fractional derivative of constant is zero, for the auxiliary function $v(x, y, t)$ defined by equation (27), we obtain

$$\begin{aligned} {}_0^C D_t^\alpha v - \frac{\partial^\beta v}{\partial |x|^\beta} + \frac{\partial^\gamma v}{\partial |y|^\gamma} &= f(x, y, t), \\ v(x, 0, t) &= v(x, 1, t) = v(0, y, t) = v(1, y, t) = 0; \quad v(x, y, 0) = 0, \end{aligned} \tag{28}$$

where

$$f(x, y, t) = -2[x(1-x) + y(1-y)]. \tag{29}$$

Problem (28)-(29) can be discretized as follows:

$$\left\{ B_n^{(\alpha)} \otimes E_{mp} - E_n \otimes R_m^{(\beta)} \otimes E_p - E_{nm} \otimes R_p^{(\gamma)} \right\} v_{nmp} = f_{nmp}, \quad (30)$$

where n is the number of time steps and p, m are the numbers of discretization intervals for spatial variables x and y , respectively.

To obtain the system for finding the unknown values of v_{nmp} for the inner nodes, we have to use the initial and boundary conditions. Since they all are zero, it is sufficient to delete the corresponding rows and columns in the system (30). The value of u_{nmp} follows from equation (27).

The results of computations, for different fractional values of α, β and γ are shown in **Figures 1** and **5**. In **Figure 1**, we fix up the values of y and draw graphs in x, t and u . In **Figure 5**, we fix up the values of t and draw graphs in x, y and u .

In particular, for $\alpha = 1, \beta = \gamma = 2$, problem (26) reduces to **Classical Diffusion Equation**. The results of computation of u_{nmp} are shown in **Figure 2** and **Figure 6**. In **Figure 2**, we fix up the values of y and draw graphs in x, t and u . In **Figure 6**, we fix up the values of t and draw graphs in x, y and u . The values of $u(x, y, t)$ obtained here, are practically in agreement with the analytical solution:

$$u(x, y, t) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{64}{l^3 m^3 \pi^6} \sin l\pi x \sin m\pi y e^{-(l^2+m^2)\pi^2 t}, \quad (l, m \text{ are odd}), \quad (31)$$

of problem (26), for $\alpha = 1, \beta = \gamma = 2$, obtained by the method of separation of variables.

For $\beta = \gamma = 2$, problem (26) reduces to **Time-Fractional Diffusion Equation**. The results of computation of u_{nmp} , for different fractional values of α , are shown in **Figure 3** and **Figure 7**. In **Figure 3**, we fix up the values of y and draw graphs in x, t and u . In **Figure 7**, we fix up the values of t and draw graphs in x, y and u .

For $\alpha = 1$, problem (26) reduces to **Space-Fractional Diffusion Equation**. The results of computations of u_{nmp} , for different fractional values of β and γ are shown in **Figure 4** and **Figure 8**. In **Figure 4**, we fix up the values of y and draw graphs in x, t and u . In **Figure 8**, we fix up the values of t and draw graphs in x, y and u .

In all numerical calculations, we have taken spatial step size $h = 0.1$ (for variable y), $k = 0.1$ (for variable x), time step size $\tau = \frac{hk}{12}$ and we plot graphs for $n = 37$ time steps.

5. Solution of fractional and classical wave equation

We use the described method to solve wave equation in most general situation, when both time and space derivatives are of fractional order. Consider the problem

$$\begin{aligned} {}_0^C D_t^\alpha u &= \frac{\partial^\beta u}{\partial |x|^\beta} + \frac{\partial^\gamma u}{\partial |y|^\gamma}, \quad 1 < \alpha \leq 2, \quad 0 < \beta \leq 2 \quad \text{and} \quad 0 < \gamma \leq 2, \\ u(x, 0, t) &= u(x, 1, t) = u(0, y, t) = u(1, y, t) = 0, \\ u(x, y, 0) &= \sin 3\pi x \sin \pi y, \quad \left(\frac{\partial u}{\partial t}\right)_{t=0} = 0, \end{aligned} \tag{32}$$

where time fractional derivative is the Caputo fractional derivative, the spatial fractional derivatives are considered as symmetric Riesz fractional derivatives. To reduce this problem to a problem with zero initial conditions (boundary conditions are already zero), we introduce an auxiliary function

$$v(x, y, t) = u(x, y, t) - u(x, y, 0). \tag{33}$$

As the Caputo fractional derivative of constant is zero, for the auxiliary function $v(x, y, t)$ defined by equation (33), we obtain

$$\begin{aligned} {}_0^C D_t^\alpha v - \frac{\partial^\beta v}{\partial |x|^\beta} + \frac{\partial^\gamma v}{\partial |y|^\gamma} &= f(x, y, t) \\ v(x, 0, t) &= v(x, 1, t) = v(0, y, t) = v(1, y, t) = 0; \quad v(x, y, 0) = 0, \\ \left(\frac{\partial v}{\partial t}\right)_{t=0} &= 0, \end{aligned} \tag{34}$$

where

$$f(x, y, t) = -10\pi^2 \sin 3\pi x \sin \pi y. \tag{35}$$

Problem (34)-(35) can be discretized as

$$\left\{ B_n^{(\alpha)} \otimes E_{mp} - E_n \otimes R_m^{(\beta)} \otimes E_p - E_{nm} \otimes R_p^{(\gamma)} \right\} v_{nmp} = f_{nmp}, \tag{36}$$

where n is the number of time steps and p, m are the numbers of discretization intervals for spatial variables x and y , respectively.

To obtain the system for finding the unknown values of v_{nmp} for the inner nodes, we have to use the initial and boundary conditions. Since they all are zero, it is sufficient to delete the corresponding rows and columns in the system (36). The value of u_{nmp} follows from equation (33).

The results of computations, for different fractional values of α, β and γ are shown in **Figure 9** and **Figure 13**. In **Figure 9**, we fix up the values

of y and draw graphs in x, t and u . In **Figure 13**, we fix up the values of t and draw graphs in x, y and u .

Particularly for $\alpha = \beta = \gamma = 2$, problem (32) reduces to **Classical Wave Equation**. The results of computation of u_{nmp} are shown in **Figure 10** and **Figure 14**. In **Figure 10**, we fix up the values of y and draw graphs in x, t and u . In **Figure 14**, we fix up the values of t and draw graphs in x, y and u . The values of $u(x, y, t)$ obtained here, are practically in agreement with the analytical solution

$$u(x, y, t) = \sin 3\pi x \sin \pi y \cos \sqrt{10}\pi t \tag{37}$$

of problem (32), for $\alpha = \beta = \gamma = 2$, obtained by the method of separation of variables.

For $\beta = \gamma = 2$, problem (32) reduces to **Time-Fractional Wave Equation**. The results of computation of u_{nmp} , for different fractional values of α , are shown in **Figure 11** and **Figure 15**. In **Figure 11**, we fix up the values of y and draw graphs in x, t and u . In **Figure 15**, we fix up the values of t and draw graphs in x, y and u .

For $\alpha = 2$, problem (32) reduces to **Space Fractional Wave Equation**. The results of computations of u_{nmp} , for different fractional values of β and γ are shown in **Figure 12** and **Figure 16**. In **Figure 12**, we fix up the values of y and draw graphs in x, t and u . In **Figure 16**, we fix up the values of t and draw graphs in x, y and u .

In all numerical calculations, we have taken spatial step size $h = 0.2$ (for variable y), $k = 0.03125$ (for variable x), time step size $\tau = \frac{hk}{12}$ and we plot graphs for $n = 25$ time steps.

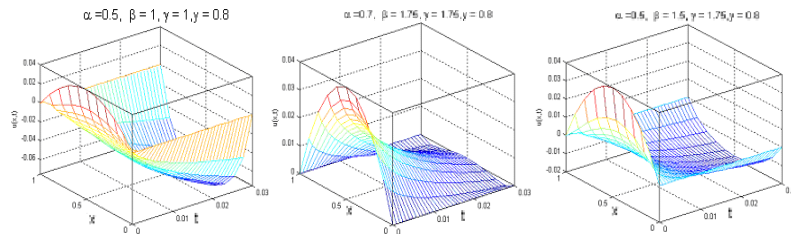


Figure 1

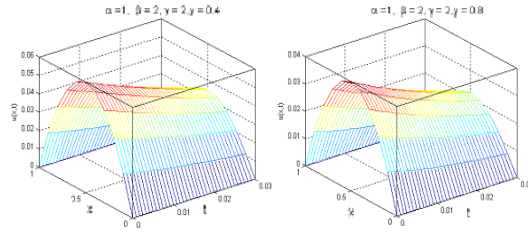


Figure 2

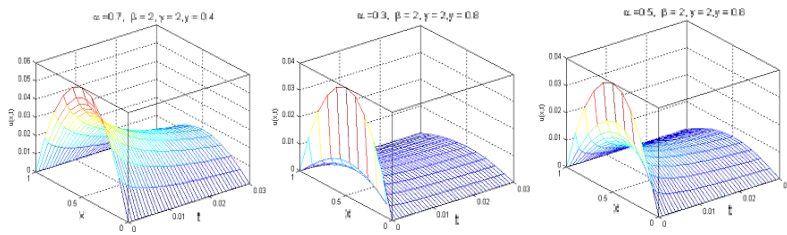


Figure 3

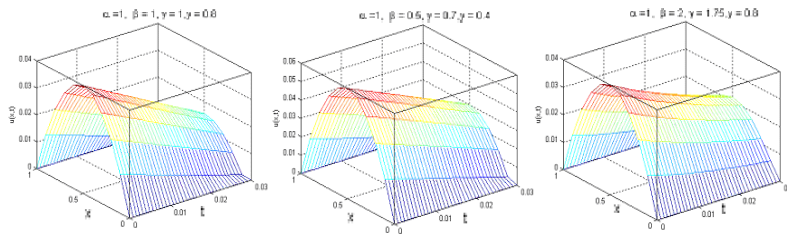


Figure 4

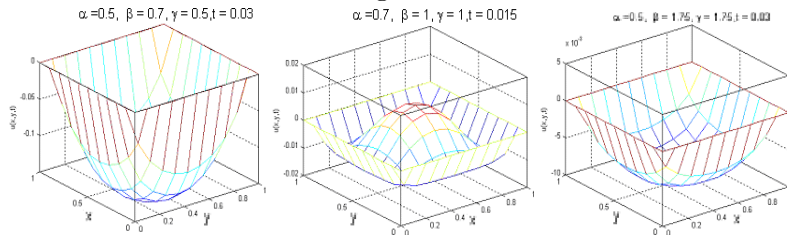


Figure 5

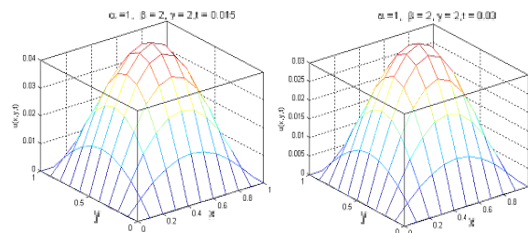


Figure 6

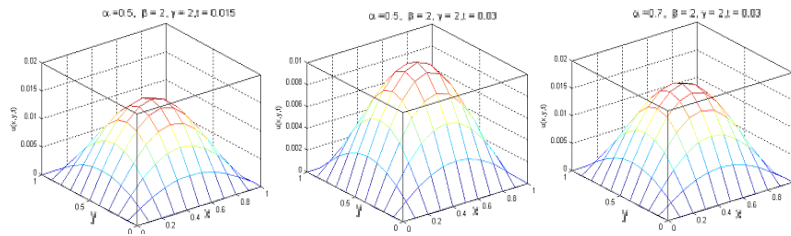


Figure 7

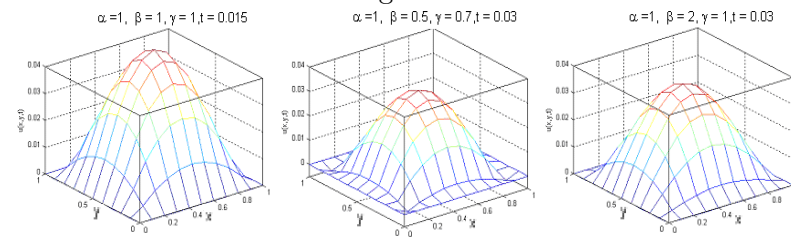


Figure 8

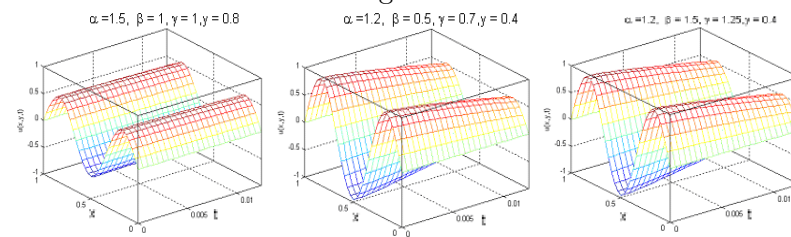


Figure 9

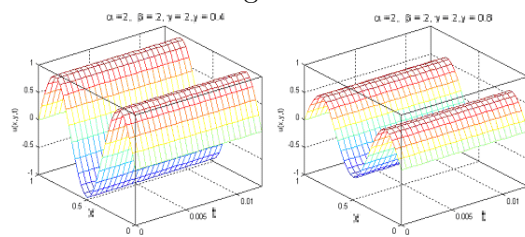


Figure 10

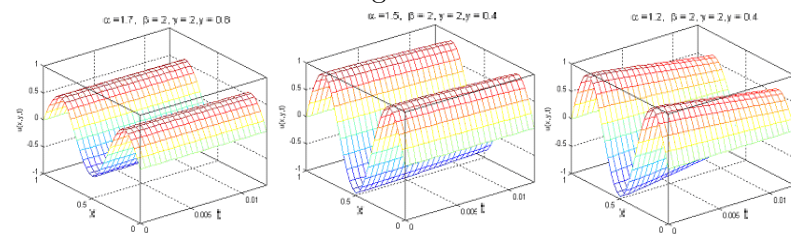


Figure 11

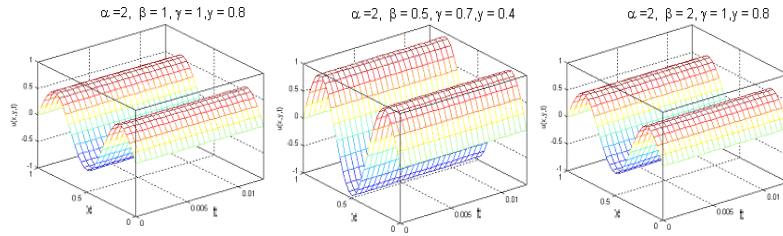


Figure 12

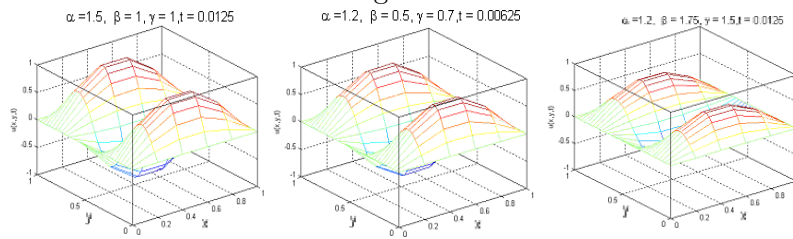


Figure 13

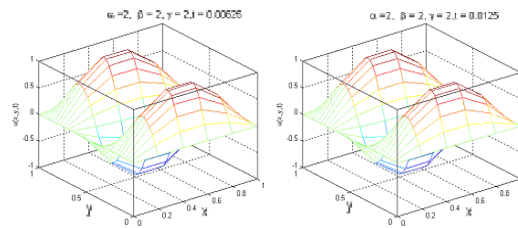


Figure 14

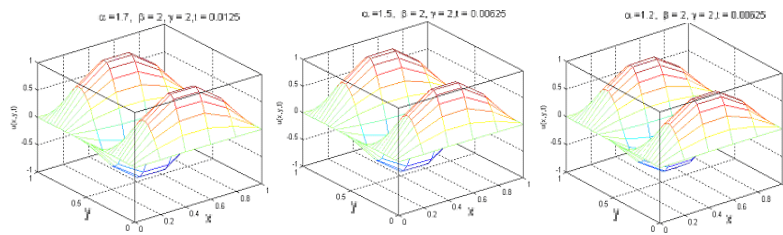


Figure 15

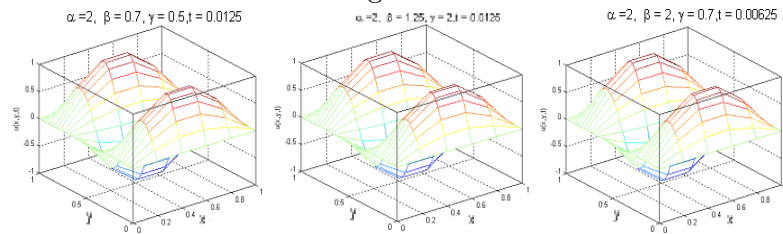


Figure 16

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