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FRACTIONAL INTEGRATION OF THE PRODUCT OF BESSEL FUNCTIONS OF THE FIRST KIND

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Dedicated to 75th birthday of Prof. A.M. Mathai

Abstract

Two integral transforms involving the Gauss-hypergeometric function in the kernels are considered. They generalize the classical Riemann-Liouville and Erdélyi-Kober fractional integral operators. Formulas for compositions of such generalized fractional integrals with the product of Bessel functions of the first kind are proved. Special cases for the product of cosine and sine functions are given. The results are established in terms of generalized Lauricella function due to Srivastava and Daoust. Corresponding assertions for the Riemann-Liouville and Erdélyi-Kober fractional integrals are presented.

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1. Introduction

This paper deals with two integral transforms defined for $x > 0$ and complex $\alpha, \beta, \eta \in \mathbb{C}$ ($\Re(\alpha) > 0$) by

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$$(\mathcal{I}_{0+}^{\alpha,\beta,\eta}f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)\mathrm{d}t \tag{1.1}$$

and

$$(\mathcal{I}_{-}^{\alpha,\beta,\eta}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} \, {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)\mathrm{d}t. \tag{1.2}$$

Here $\Gamma(\alpha)$ is the Euler gamma function [1, Section 1], $\Re(\alpha)$ denotes the real part of α, and ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function defined for complex $a, b, c, \in \mathbb{C},\ c \neq 0, -1, -2, \cdots$ by the hypergeometric series [1, 2.1(2)]

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{1.3}$$

where $(z)_k$ is the Pochhammer symbol defined for $z \in \mathbb{C}$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, \ldots\}$ by

$$(z)_0 = 1, \quad (z)_k = z(z+1)...(z+k-1) \quad (k \in \mathbb{N}). \tag{1.4}$$

The series in (1.3) is absolutely convergent for

$$|z| < 1 \quad \text{and} \quad |z| = 1 (z \neq 1), \quad \Re(c-a-b) > 0. \tag{1.5}$$

Operators (1.1) and (1.2) were introduced by Saigo [5], and their properties were investigated by many authors; see bibliography and a short survey of results in [3, Section 7.12, For Sections 7.7 and 7.8]. When $\beta = -\alpha$, (1.1) and (1.2) coincide with the classical left and right-hand sided Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C},\ \Re(\alpha) > 0$, [6, Section 5.1]:

$$(\mathcal{I}_{0+}^{\alpha,-\alpha,\eta}f)(x) = (\mathcal{I}_{0+}^{\alpha}f)(x) \equiv \frac{1}{\Gamma(\alpha)} \int_o^x (x-t)^{\alpha-1} f(t)\mathrm{d}t \quad (x>0), \tag{1.6}$$

$$(\mathcal{I}_{-}^{\alpha,-\alpha,\eta}f)(x) = (\mathcal{I}_{-}^{\alpha}f)(x) \equiv \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)\mathrm{d}t \quad (x>0). \tag{1.7}$$

If $\beta = 0$, (1.1) and (1.2) are the so-called Erdélyi-Kober fractional integrals defined for complex $\alpha, \eta \in \mathbb{C}$ ($\Re(\alpha) > 0$) by [6, Section 18.1]:

$$(\mathcal{I}_{0+}^{\alpha,0,\eta}f)(x) = (\mathcal{I}_{\eta,\alpha}^{+}f)(x) \equiv \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_o^x (x-t)^{\alpha-1} t^{\eta} f(t)\mathrm{d}t \quad (x>0), \tag{1.8}$$

$$(\mathcal{I}_{-}^{\alpha,0,\eta}f)(x) = (\mathcal{K}_{\eta,\alpha}^{-}f)(x) \equiv \frac{x^{\eta}}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\eta}f(t)\mathrm{d}t \quad (x>0). \tag{1.9}$$

We investigate compositions of integral transforms (1.1) and (1.2) with the product of Bessel function of the first kind, $J_{\nu}(z)$, which is defined for complex $z \in \mathbb{C}$ ($z \neq 0$) and $\nu \in \mathbb{C}$ ($\Re(\nu) > -1$) by [2, 7.2(2)]

$$J_{\nu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{z}{2})^{\nu+2k}}{\Gamma(\nu+k+1)k!}. \tag{1.10}$$

We prove that such compositions are expressed in terms of the generalized Lauricella function due to Srivastava and Daoust [7], which is defined by

$$F_{C:\ D';\cdots;D^{(n)}}^{A:\ B';\cdots;B^{(n)}}\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

$$= F_{C:\ D';\cdots;D^{(n)}}^{A:\ B';\cdots;B^{(n)}}\begin{bmatrix} [(a):\ \theta',\cdots,\ \theta^{(n)}],\ [(b'):\ \phi'];\cdots;\ [(b)^{(n)}:\ \phi^{(n)}]; \\ [(c):\ \psi',\cdots,\ \psi^{(n)}],\ [(d'):\ \delta'];\cdots;\ [(d)^{(n)}:\ \delta^{(n)}]; \end{bmatrix} z_1,\cdots,z_n$$

$$= \sum_{k_1,\cdots,k_n=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_{k_1\theta_j'+\cdots+k_n\theta_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{k_1\psi_j'+\cdots+k_n\psi_j^{(n)}}} \frac{\prod_{j=1}^{B'}(b_j')_{k_1\phi_j'}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{k_n\phi_j^{(n)}}}{\prod_{j=1}^{D'}(d_j')_{k_1\delta_j'}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{k_n\delta_j^{(n)}}} \frac{z_1^{k_1}}{k_1!}\cdots\frac{z_n^{k_n}}{k_n!}, \tag{1.11}$$

the coefficients

$$\left\{ \begin{array}{ll} \theta_j^{(m)}\ \ (j=1,\ldots,A);\ \ \phi_j^{(m)}\ \ (j=1,\ldots,B^{(m)}) & \\ \psi_j^{(m)}\ \ (j=1,\ldots,C);\ \ \delta_j^{(m)}\ \ (j=1,\ldots,D^{(m)});\ \forall\, m\in\{1,\ldots,n\} \end{array} \right\} \tag{1.12}$$

are real and positive, and (a) abbreviates the array of A parameters $a_1,\ldots,a_A$, $(b^{(m)})$ abbreviates the array of $B^{(m)}$ parameters $b_j^{(m)}$ $(j=1,\ldots,B^{(m)})$; $\forall\, m \in \{1,\ldots,n\}$, with similar interpretations for (c) and $(d^{(m)})$ $(m=1,\ldots,n)$. $(z)_a$ is a generalization of the Pochhammer symbol (1.4):

$$(z)_a = \frac{\Gamma(z+a)}{\Gamma(a)} \quad (z,a\in\mathbb{C}). \tag{1.13}$$

The multiple series (1.11) converges absolutely either

$$(i)\ \ \Delta_i > 0\ \ (i=1,\ldots,n),\ \ \forall\ \ z_1,\ldots,z_n \in \mathbb{C},$$

or

$$(ii)\ \ \Delta_i = 0 \ \ (i = 1, \dots, n), \ \ \forall \ \ z_1, \dots, z_n \in \mathbb{C}, \ \ |z_i| < \varrho_i \ (i = 1, \dots, n),$$

and divergent when $\Delta_i < 0$ $(i = 1, \dots, n)$; except for the trivial case $z_1 = \cdots z_n = 0$, where

$$\Delta_i \equiv \ 1 + \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} \ \ (i = 1, \dots, n), \tag{1.14}$$

$$\varrho_i = \min_{\mu_1, \dots, \mu_n > 0} \{E_i\} \ \ (i = 1, \dots, n), \tag{1.15}$$

with

$$E_i = (\mu_i)^{1 + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)}} \frac{\left\{ \prod_{j=1}^{C} \left(\sum_{i=1}^{n} \mu_i \psi_j^{(i)} \right)^{\psi_j^{(i)}} \right\} \left\{ \prod_{j=1}^{D^{(i)}} (\delta_j^{(i)})^{\delta_j^{(i)}} \right\}}{\left\{ \prod_{j=1}^{A} \left(\sum_{i=1}^{n} \mu_i \theta_j^{(i)} \right)^{\theta_j^{(i)}} \right\} \left\{ \prod_{j=1}^{B^{(i)}} (\phi_j^{(i)})^{\phi_j^{(i)}} \right\}}. \tag{1.16}$$

For more details see [7]. Special cases of (1.11) are established in terms of generalized hypergeometric function of one and two variables respectively, for the sake of completeness we define these functions here. A generalized hypergeometric function ${}_pF_q(z)$ is defined for complex $a_i, b_j \in \mathbb{C}$, $b_j \neq 0, -1, \dots (i = 1, 2, \dots p; j = 1, 2, \dots q)$ by the generalized hypergeometric series [1, 4.1(1)]

$${}_pF_q(a_1, \dots, a_p;\ b_1, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!}. \tag{1.17}$$

This series is absolutely convergent for all values of $z \in \mathbb{C}$ if $p \leq q$; and it is an entire function of z. We define a generalization of the Kampé de Fériet function by means of the double hypergeometric series [7]

$$F^{p:q;k}_{l:m;n}\left[{}^{(a_p):(b_q);(c_k);}_{(\alpha_l):(\beta_m);(\gamma_n);} x, y\right] = \sum_{r,s=0}^{\infty} \frac{\{\prod_{j=1}^{p}(a_j)_{r+s}\}\{\prod_{j=1}^{q}(b_j)_r\}\{\prod_{j=1}^{k}(c_j)_s\}}{\{\prod_{j=1}^{l}(\alpha_j)_{r+s}\}\{\prod_{j=1}^{m}(\beta_j)_r\}\{\prod_{j=1}^{n}(\gamma_j)_s\}} \frac{x^r}{r!} \frac{y^s}{s!}. \tag{1.18}$$

The above double series is absolutely convergent for all values of x and y, if $p+q < l+m+1$ and $p+k < l+n+1$. Also, if $p+q = l+m+1$ and $p+k = l+n+1$, we must have any one of the following sets of conditions:

$$p \leq l, \quad \max\{|x|, |y|\} < 1;$$

$$p > l, \quad |x|^{1/(p-l)} + |y|^{1/(p-l)} < 1.$$

The paper is organized as follows. Formulas for compositions of integral transforms (1.1) and (1.2) with the product of Bessel functions (1.10) are proved in terms of generalized Lauricella function (1.11) in Section 2 and 3, respectively. The corresponding results for the Riemann-Liouville and Erdélyi-Kober fractional integrals (1.6), (1.7) and (1.8), (1.9) are also presented in Sections 2 and 3. Special cases giving compositions of fractional integrals with the product of cosine and sine functions are considered in Sections 4.

2. Left-sided fractional integration of Bessel functions

Our results in Sections 2 and 3 are based on the preliminary assertions giving composition formulas of generalized fractional integrals (1.1) and (1.2) with a power function.

Lemma 1. ([4, Lemmas 1-2]) *Let* $\alpha, \beta, \eta \in \mathbb{C}$.

(a) If $\Re(\alpha) > 0$ *and* $\Re(\sigma) > \max\,[0,\ \Re(\beta-\eta)]$, *then*

$$(\mathcal{I}_{0+}^{\alpha,\beta,\eta} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)} x^{\sigma-\beta-1}. \tag{2.1}$$

(b) If $\Re(\alpha) > 0$ *and* $\Re(\sigma) < 1 + \min\,[\Re(\beta), \Re(\eta)]$, *then*

$$(\mathcal{I}_{-}^{\alpha,\beta,\eta} t^{\sigma-1})(x) = \frac{\Gamma(\beta-\sigma+1)\Gamma(\eta-\sigma+1)}{\Gamma(1-\sigma)\Gamma(\alpha+\beta+\eta-\sigma+1)} x^{\sigma-\beta-1}. \tag{2.2}$$

The generalized left-sided fractional integration (1.1) of the product of Bessel functions(1.10) is given by the following result.

Theorem 1. *Let* $n \in \mathbb{N}$, $\alpha, \beta, \eta, \sigma, \nu_j \in \mathbb{C}$ *and* $a_j, \rho_j \in \mathcal{R}_+$ $(j = 1, \ldots, n)$ *be such that*

$$\Re(\alpha) > 0, \quad \Re(\nu_j) > -1, \quad \Re(\sigma + \sum_{j=1}^{n} \rho_j \nu_j) > \max[0, \Re(\beta-\eta)]. \tag{2.3}$$

Then there holds the formula

$$\left(\mathcal{I}_{0+}^{\alpha,\beta,\eta}\left[t^{\sigma-1}\prod_{j=1}^{n}J_{\nu_j}(a_jt^{\rho_j})\right]\right)(x)$$

$$=x^{\sigma-\beta-1}\left(\prod_{j=1}^{n}\frac{(\frac{a_jx^{\rho_j}}{2})^{\nu_j}}{\Gamma(\nu_j+1)}\right)\frac{\Gamma(u)\Gamma(v)}{\Gamma(w)\Gamma(z)}$$

$$\times F_{2:1,\ldots,1}^{2:0,\ldots,0}\left[\begin{smallmatrix}[u:2\rho_1,\ldots,2\rho_n],[v:2\rho_1,\ldots,2\rho_n]\\ [w:2\rho_1,\ldots,2\rho_n],[z:2\rho_1,\ldots,2\rho_n]:[\nu_1+1:1],\ldots,[\nu_n+1:1]:\end{smallmatrix};-\frac{a_1^2x^{2\rho_1}}{4},\ldots,-\frac{a_n^2x^{2\rho_n}}{4}\right], \tag{2.4}$$

where $u=\sigma+\sum_{j=1}^{n}\rho_j\nu_j, v=\sigma+\eta-\beta+\sum_{j=1}^{n}\rho_j\nu_j, w=\sigma-\beta+\sum_{j=1}^{n}\rho_j\nu_j, z=\sigma+\alpha+\eta+\sum_{j=1}^{n}\rho_j\nu_j$ *and* $F_{2:1,\ldots,1}^{2:0,\ldots,0}[\cdot]$ *is given by (1.11).*

P r o o f. First of all we note that Δ_i in (1.14) is given by $\Delta_i=1+n>0$ $(i=1,\ldots,n\in\mathbb{N})$, and therefore $F_{2:1,\ldots,1}^{2:0,\ldots,0}[\cdot]$ in the right hand side of (2.4) is defined. Now we prove (2.4). Applying equation (1.10), Using (1.1) and (1.11) and changing the orders of integration and summation, we find

$$\left(\mathcal{I}_{0+}^{\alpha,\beta,\eta}\left[t^{\sigma-1}\prod_{j=1}^{n}J_{\nu_j}(a_jt^{\rho_j})\right]\right)(x)$$

$$=\left(\mathcal{I}_{0+}^{\alpha,\beta,\eta}\left[t^{\sigma-1}\left(\sum_{k_1=0}^{\infty}\frac{(-1)^{k_1}(\frac{a_1t^{\rho_1}}{2})^{\nu_1+2k_1}}{\Gamma(\nu_1+k_1+1)\ k_1!}\right)\cdots\left(\sum_{k_n=0}^{\infty}\frac{(-1)^{k_n}(\frac{a_nt^{\rho_n}}{2})^{\nu_n+2k_n}}{\Gamma(\nu_n+k_n+1)\ k_n!}\right)\right]\right)(x)$$

$$=\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{(-1)^{k_1}(\frac{a_1}{2})^{\nu_1+2k_1}}{\Gamma(\nu_1+1)(\nu_1+1)_{k_1}\ k_1!}\cdots\frac{(-1)^{k_n}(\frac{a_n}{2})^{\nu_n+2k_n}}{\Gamma(\nu_n+1)(\nu_n+1)_{k_n}\ k_n!}$$

$$\times(\mathcal{I}_{0+}^{\alpha,\beta,\eta}\{t^{\sigma+\nu_1\rho_1+\cdots+\nu_n\rho_n+2\rho_1k_1+\cdots+2\rho_nk_n-1}\})(x).$$

By (2.3), for any $k_j\in\mathbb{N}_0$ $(j=1,\ldots,n)$ $\Re(\sigma+\sum_{j=1}^{n}\rho_j\nu_j+2\sum_{j=1}^{n}\rho_jk_j)\geq\Re(\sigma+\sum_{j=1}^{n}\rho_j\nu_j)>\max[0,\Re(\beta-\eta)]$. Applying Lemma 1(a) and using (2.1) with σ replaced by $\sigma+\sum_{j=1}^{n}\rho_j\nu_j+2\sum_{j=1}^{n}\rho_jk_j$ $(j=1,\ldots,n)$, we obtain

$$\left(\mathcal{I}_{0+}^{\alpha,\beta,\eta}\left[t^{\sigma-1}\prod_{j=1}^{n}J_{\nu_j}(a_jt^{\rho_j})\right]\right)(x)$$

$$= \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{(-1)^{k_1}(\frac{a_1}{2})^{\nu_1+2k_1}}{\Gamma(\nu_1+1)(\nu_1+1)_{k_1}\ k_1!} \cdots \frac{(-1)^{k_n}(\frac{a_n}{2})^{\nu_n+2k_n}}{\Gamma(\nu_n+1)(\nu_n+1)_{k_n}\ k_n!}$$

$$\times \frac{\Gamma(\sigma+\sum_{j=1}^{n}(\nu_j\rho_j+2\rho_jk_j))\Gamma(\sigma+\eta-\beta+\sum_{j=1}^{n}(\nu_j\rho_j+2\rho_jk_j))}{\Gamma(\sigma-\beta+\sum_{j=1}^{n}(\nu_j\rho_j+2\rho_jk_j))\Gamma(\sigma+\alpha+\eta+\sum_{j=1}^{n}(\nu_j\rho_j+2\rho_jk_j))}$$

$$\times x^{\sigma-\beta-1+\sum_{j=1}^{n}(\nu_j\rho_j+2\rho_jk_j)}$$

$$= x^{\sigma-\beta-1}\left(\prod_{j=1}^{n} \frac{(\frac{a_jx^{\rho_j}}{2})^{\nu_j}}{\Gamma(\nu_j+1)}\right) \frac{\Gamma(\sigma+\sum_{j=1}^{n}\rho_j\nu_j)\Gamma(\sigma+\eta-\beta+\sum_{j=1}^{n}\rho_j\nu_j)}{\Gamma(\sigma-\beta+\sum_{j=1}^{n}\rho_j\nu_j)\Gamma(\sigma+\alpha+\eta+\sum_{j=1}^{n}\rho_j\nu_j)} \times$$

$$\sum_{k_1,\ldots,k_n=0}^{\infty} \frac{(\sigma+\sum_{j=1}^{n}\rho_j\nu_j)_{2\rho_1k_1+\cdots+2\rho_nk_n}(\sigma+\eta-\beta+\sum_{j=1}^{n}\rho_j\nu_j)_{2\rho_1k_1+\cdots+2\rho_nk_n}}{(\sigma-\beta+\sum_{j=1}^{n}\rho_j\nu_j)_{2\rho_1k_1+\cdots+2\rho_nk_n}(\sigma+\alpha+\eta+\sum_{j=1}^{n}\rho_j\nu_j)_{2\rho_1k_1+\cdots+2\rho_nk_n}}$$

$$\times \frac{1}{(\nu_1+1)_{k_1}\cdots(\nu_n+1)_{k_n}} \frac{(-\frac{a_1^2x^{2\rho_1}}{4})^{k_1}}{k_1!} \cdots \frac{(-\frac{a_n^2x^{2\rho_n}}{4})^{k_n}}{k_n!}.$$

This, in accordance with Equation (1.11), gives the result in (2.4). This complete the proof of the theorem. ■

Corollary 1.1. *Let $\alpha, \sigma, \nu_j \in \mathbb{C}$ and $a_j, \rho_j \in \mathcal{R}_+$ $(j = 1, \ldots, n)$ be such that $\Re(\alpha) > 0$, $\Re(\nu_j) > -1$ and $\Re(\sigma + \sum_{j=1}^{n}\rho_j\nu_j) > 0$. Then*

$$\left(\mathcal{I}_{0+}^{\alpha}\left[t^{\sigma-1}\prod_{j=1}^{n}J_{\nu_j}(a_jt^{\rho_j})\right]\right)(x)$$

$$= x^{\sigma+\alpha-1}\left(\prod_{j=1}^{n}\frac{(\frac{a_jx^{\rho_j}}{2})^{\nu_j}}{\Gamma(\nu_j+1)}\right)\frac{\Gamma(\sigma+\sum_{j=1}^{n}\rho_j\nu_j)}{\Gamma(\sigma+\alpha+\sum_{j=1}^{n}\rho_j\nu_j)}$$

$$\times F_{1:1,\ldots,1}^{1:0,\ldots,0}\left[\begin{matrix}[\sigma+\sum_{j=1}^{n}\rho_j\nu_j:2\rho_1,\ldots,2\rho_n]: \\ [\sigma+\alpha+\sum_{j=1}^{n}\rho_j\nu_j:2\rho_1,\ldots,2\rho_n]:[\nu_1+1:1],\ldots,[\nu_n+1:1]:\end{matrix}\ ;-\frac{a_1^2x^{2\rho_1}}{4},\ldots,-\frac{a_n^2x^{2\rho_n}}{4}\right]. \tag{2.5}$$

Corollary 1.2. *Let $\alpha, \eta, \sigma, \nu_j \in \mathbb{C}$ and $a_j, \rho_j \in \mathcal{R}_+$ $(j = 1, \ldots, n)$ be such that $\Re(\alpha) > 0$, $\Re(\nu_j) > -1$ and $\Re(\sigma + \sum_{j=1}^{n}\rho_j\nu_j) > -\Re(\eta)$. Then*

$$\left(\mathcal{I}_{\eta,\alpha}^{+}\left[t^{\sigma-1}\prod_{j=1}^{n}J_{\nu_j}(a_jt^{\rho_j})\right]\right)(x)$$

$$= x^{\sigma-1}\left(\prod_{j=1}^{n}\frac{(\frac{a_j x^{\rho_j}}{2})^{\nu_j}}{\Gamma(\nu_j+1)}\right)\frac{\Gamma(\sigma+\eta+\sum_{j=1}^{n}\rho_j\nu_j)}{\Gamma(\sigma+\alpha+\eta+\sum_{j=1}^{n}\rho_j\nu_j)}$$

$$\times F^{1:0,\dots,0}_{1:1,\dots,1}\left[\begin{matrix}[\sigma+\eta+\sum_{j=1}^{n}\rho_j\nu_j:2\rho_1,\dots,2\rho_n]: \\ [\sigma+\alpha+\eta+\sum_{j=1}^{n}\rho_j\nu_j:2\rho_1,\dots,2\rho_n]:[\nu_1+1:1],\dots,[\nu_n+1:1]:\end{matrix};-\frac{a_1^2x^{2\rho_1}}{4},\dots,-\frac{a_n^2x^{2\rho_n}}{4}\right]. \tag{2.6}$$

Corollary 1.3. *Let $\alpha,\beta,\sigma,\nu_1,\nu_2\in\mathbb{C}$ and $a_1,a_2,\rho_1,\rho_2\in\mathcal{R}_+$ be such that $\Re(\alpha)>0$, $\Re(\nu_1)>-1$, $\Re(\nu_2)>-1$ and $\Re(\sigma+\rho_1\nu_1+\rho_2\nu_2)>\max[0,\Re(\beta-\eta)]$. Then*

$$\left(\mathcal{I}^{\alpha,\beta,\eta}_{0+}\left[t^{\sigma-1}J_{\nu_1}(t)J_{\nu_2}(t)\right]\right)(x)$$

$$=\frac{x^{c-1}}{2^{\nu_1+\nu_2}}\,\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)\Gamma(\nu_1+1)\Gamma(\nu_2+1)}$$

$$\times F^{4:0,0}_{4:1,1}\left[\begin{matrix}[\frac{a}{2}:1,1],[\frac{a+1}{2}:1,1],[\frac{b}{2}:1,1],[\frac{b+1}{2}:1,1]: \\ [\frac{c}{2}:1,1],[\frac{c+1}{2}:1,1],[\frac{d}{2}:1,1],[\frac{d+1}{2}:1,1]:[\nu_1+1:1],\dots,[\nu_n+1:1]:\end{matrix};-\frac{x^2}{4},-\frac{x^2}{4}\right], \tag{2.7}$$

where $a=\sigma+\nu_1+\nu_2$, $b=\sigma+\eta-\beta+\nu_1+\nu_2$, $c=\sigma-\beta+\nu_1+\nu_2$, $d=\sigma+\alpha+\eta+\nu_1+\nu_2$ and $F^{4:0,0}_{4:1,1}[\cdot]$ is defined in (1.18).

Corollaries 1.1 and 1.2 follow from Theorem 1 in respective cases $\beta=-\alpha$ and $\beta=0$, if we take (1.6) and (1.8) into account. Corollary 1.3 follows from Theorem 1, if we put $n=2, a_1=1, a_2=1, \rho_1=1, \rho_2=1$, use (1.11) and take into account the relation

$$(z)_{2k}=2^{2k}\left(\frac{z}{2}\right)_k\left(\frac{z+1}{2}\right)_k \quad (z\in\mathbb{C},\ k\in\mathbb{N}_0), \tag{2.8}$$

where $(z)_k$ is the Pochhammer symbol (1.4).

Remark 1. When $n=1, a_1=1, \rho_1=1, \nu_1=\nu$ equation (2.4) is reduced to

$$\left(\mathcal{I}^{\alpha,\beta,\eta}_{0+}\left[t^{\sigma-1}J_{\nu}(t)\right]\right)(x)$$

$$=\frac{x^{\sigma+\nu-\beta-1}}{2^{\nu}}\,\frac{\Gamma(\sigma+\nu)\Gamma(\sigma+\nu+\eta-\beta)}{\Gamma(\sigma+\nu-\beta)\Gamma(\sigma+\nu+\alpha+\eta)\Gamma(\nu+1)}$$

$$\times {}_4F_5\left[\begin{matrix}\frac{\sigma+\nu}{2},\frac{\sigma+\nu+1}{2},\frac{\sigma+\nu+\eta-\beta}{2},\frac{\sigma+\nu+\eta-\beta+1}{2} \\ \nu+1,\frac{\sigma+\nu-\beta}{2},\frac{\sigma+\nu-\beta+1}{2},\frac{\sigma+\nu+\alpha+\eta}{2},\frac{\sigma+\nu+\alpha+\eta+1}{2}\end{matrix};-\frac{x^2}{4}\right]. \tag{2.9}$$

This result was proved in [4, Theorem 3].

3. Right-sided fractional integration of Bessel functions

The following result yields generalized right-hand sided fractional integration (1.2) of the product of Bessel functions.

THEOREM 2. *Let $\alpha, \beta, \eta, \sigma, \nu_j \in \mathbb{C}$ and $a_j, \rho_j \in \mathcal{R}_+$ $(j = 1, \ldots, n)$ be such that*

$$\Re(\alpha) > 0,\ \Re(\nu_j) > -1,\ \ \Re(\sigma - \sum_{j=1}^{n} \rho_j \nu_j) < 1 + \min[\Re(\beta), \Re(\eta)]. \qquad (3.1)$$

Then there holds the formula

$$\left(\mathcal{I}_{-}^{\alpha,\beta,\eta}\left[t^{\sigma-1}\prod_{j=1}^{n} J_{\nu_j}\left(\frac{a_j}{t^{\rho_j}}\right)\right]\right)(x)$$

$$= x^{\sigma-\beta-1}\left(\prod_{j=1}^{n}\frac{(\frac{a_j}{2x^{\rho_j}})^{\nu_j}}{\Gamma(\nu_j+1)}\right)\frac{\Gamma(p)\Gamma(q)}{\Gamma(r)\Gamma(s)}$$

$$\times F_{2:1,\ldots,1}^{2:0,\ldots,0}\left[\begin{matrix}[p{:}2\rho_1,\ldots,2\rho_n],[q{:}2\rho_1,\ldots,2\rho_n]{:} \\ [r{:}2\rho_1,\ldots,2\rho_n],[s{:}2\rho_1,\ldots,2\rho_n]{:}[\nu_1+1{:}1],\ldots,[\nu_n+1{:}1]{:}\end{matrix};-\frac{a_1^2}{4x^{2\rho_1}},\ldots,-\frac{a_n^2}{4x^{2\rho_n}}\right], \qquad (3.2)$$

where $p = 1 + \beta - \sigma + \sum_{j=1}^{n} \rho_j \nu_j$, $q = 1 + \eta - \sigma + \sum_{j=1}^{n} \rho_j \nu_j$, $r = 1 - \sigma + \sum_{j=1}^{n} \rho_j \nu_j$, $s = \alpha + \beta + \eta - \sigma + \sum_{j=1}^{n} \rho_j \nu_j + 1$ *and* $F_{2:1,\ldots,1}^{2:0,\ldots,0}[\cdot]$ *is given by (1.11).*

P r o o f. First of all we note that Δ_i in (1.14) is given by $\Delta_i = 1+n > 0$ $(i = 1, \ldots, n \in \mathbb{N})$, and therefore $F_{2:1,\ldots,1}^{2:0,\ldots,0}[\cdot]$ in the right hand side of (3.2) is defined. Now we prove (3.2). Using Equations (1.2) and (1.10) and changing the orders of integration and summation, we have

$$\left(\mathcal{I}_{-}^{\alpha,\beta,\eta}\left[t^{\sigma-1}\prod_{j=1}^{n} J_{\nu_j}\left(\frac{a_j}{t^{\rho_j}}\right)\right]\right)(x)$$

$$= (\mathcal{I}_{-}^{\alpha,\beta,\eta}\left\{t^{\sigma-1}\sum_{k_1=0}^{\infty}\frac{(-1)^{k_1}(\frac{a_1}{2t^{\rho_1}})^{\nu_1+2k_1}}{\Gamma(\nu_1+k_1+1)\ k_1!}\cdots\sum_{k_n=0}^{\infty}\frac{(-1)^{k_n}(\frac{a_n}{2t^{\rho_n}})^{\nu_n+2k_n}}{\Gamma(\nu_n+k_n+1)\ k_n!}\right\}(x)$$

$$= \sum_{k_1,\ldots,k_n=0}^{\infty}\frac{(-1)^{k_1}(\frac{a_1}{2})^{\nu_1+2k_1}}{\Gamma(\nu_1+1)(\nu_1+1)_{k_1}\ k_1!}\cdots\frac{(-1)^{k_n}(\frac{a_n}{2})^{\nu_n+2k_n}}{\Gamma(\nu_n+1)(\nu_n+1)_{k_n}\ k_n!}$$

$$\times(\mathcal{I}_{-}^{\alpha,\beta,\eta}\{t^{\sigma-\nu_1\rho_1-\cdots-\nu_n\rho_n-2\rho_1k_1-\cdots-2\rho_nk_n-1}\})(x).$$

By (3.1), for any $k_j \in\in \mathbb{N}_0$ $(j=1,\ldots,n)$ $\Re(\sigma-\sum_{j=1}^n \rho_j\nu_j - 2\sum_{j=1}^n \rho_j k_j) \leq \Re(\sigma - \sum_{j=1}^n \rho_j\nu_j) < 1+\min[\Re(\beta),\Re(\eta)]$. Applying Lemma 1(b) and using (2.2) with σ replaced by $\sigma-\sum_{j=1}^n \rho_j\nu_j - 2\sum_{j=1}^n \rho_j k_j$ $(j=1,\ldots,n)$, we obtain

$$\left(\mathcal{I}_{-}^{\alpha,\beta,\eta}\left[t^{\sigma-1}\prod_{j=1}^n J_{\nu_j}\left(\frac{a_j}{t^{\rho_j}}\right)\right]\right)(x)$$

$$= \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{(-1)^{k_1}(\frac{a_1}{2})^{\nu_1+2k_1}}{\Gamma(\nu_1+1)(\nu_1+1)_{k_1}\ k_1!}\cdots\frac{(-1)^{k_n}(\frac{a_n}{2})^{\nu_n+2k_n}}{\Gamma(\nu_n+1)(\nu_n+1)_{k_n}\ k_n!}$$

$$\times\frac{\Gamma(\beta-\sigma+1+\sum_{j=1}^n(\nu_j\rho_j+2\rho_jk_j))\Gamma(\eta-\sigma+1+\sum_{j=1}^n(\nu_j\rho_j+2\rho_jk_j))}{\Gamma(1-\sigma+\sum_{j=1}^n(\nu_j\rho_j+2\rho_jk_j))\Gamma(1+\alpha+\beta+\eta-\sigma+\sum_{j=1}^n(\nu_j\rho_j+2\rho_jk_j))}$$

$$\times\ x^{\sigma-\beta-1-\sum_{j=1}^n(\nu_j\rho_j+2\rho_jk_j)}$$

$$= x^{\sigma-\beta-1}\left(\prod_{j=1}^n \frac{(\frac{a_j}{2x^{\rho_j}})^{\nu_j}}{\Gamma(\nu_j+1)}\right)\frac{\Gamma(p)\Gamma(q)}{\Gamma(r)\Gamma(s)}$$

$$\times\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{(p)_{2\rho_1k_1+\cdots+2\rho_nk_n}(q)_{2\rho_1k_1+\cdots+2\rho_nk_n}}{(r)_{2\rho_1k_1+\cdots+2\rho_nk_n}(s)_{2\rho_1k_1+\cdots+2\rho_nk_n}}$$

$$\times\ \frac{1}{(\nu_1+1)_{k_1}\cdots(\nu_n+1)_{k_n}}\frac{(-\frac{a_1^2}{4x^{2\rho_1}})^{k_1}}{k_1!}\cdots\frac{(-\frac{a_n^2}{4x^{2\rho_n}})^{k_n}}{k_n!}.$$

By equation (1.11), this yields the result in (3.2). ■

Corollary 2.1. *Let $\alpha,\sigma,\nu_j \in \mathbb{C}$ and $a_j,\rho_j \in \mathcal{R}_+$ $(j=1,\ldots,n)$ be such that $\Re(\nu_j)>-1$, and $0<\Re(\alpha)<1-\Re(\sigma-\sum_{j=1}^n \rho_j\nu_j)$. Then*

$$\left(\mathcal{I}_{-}^{\alpha}\left[t^{\sigma-1}\prod_{j=1}^n J_{\nu_j}\left(\frac{a_j}{t^{\rho_j}}\right)\right]\right)(x)$$

$$= x^{\sigma+\alpha-1}\left(\prod_{j=1}^n \frac{\left(\frac{a_j}{2x^{\rho_j}}\right)^{\nu_j}}{\Gamma(\nu_j+1)}\right)\frac{\Gamma(1-\sigma-\alpha+\sum_{j=1}^n\rho_j\nu_j)}{\Gamma(1-\sigma+\sum_{j=1}^n\rho_j\nu_j)}$$

$$\times F_{1:1,\ldots,1}^{1:0,\ldots,0}\left[\begin{smallmatrix}[1-\sigma-\alpha+\sum_{j=1}^n\rho_j\nu_j:2\rho_1,\ldots,2\rho_n]:\\ [1-\sigma+\sum_{j=1}^n\rho_j\nu_j:2\rho_1,\ldots,2\rho_n]:[\nu_1+1:1],\ldots,[\nu_n+1:1]:\end{smallmatrix};-\frac{a_1^2}{4x^{2\rho_1}},\ldots,-\frac{a_n^2}{4x^{2\rho_n}}\right]. \tag{3.3}$$

COROLLARY 2.2. *Let* $\alpha, \eta, \sigma, \nu_j \in \mathbb{C}$ *and* $a_j, \rho_j \in \mathcal{R}_+$ $(j = 1, \ldots, n)$ *be such that* $\Re(\alpha) > 0$, $\Re(\nu_j) > -1$, *and* $\Re(\sigma - \sum_{j=1}^n \rho_j \nu_j) < 1 + \Re(\eta)$. *Then*

$$\left(\mathcal{K}_{\eta,\alpha}^{-}\left[t^{\sigma-1}\prod_{j=1}^{n} J_{\nu_j}\left(\frac{a_j}{t^{\rho_j}}\right)\right]\right)(x)$$

$$= x^{\sigma-1}\left(\prod_{j=1}^{n}\frac{(\frac{a_j}{2x^{\rho_j}})^{\nu_j}}{\Gamma(\nu_j+1)}\right)\frac{\Gamma(1+\eta-\sigma+\sum_{j=1}^n \rho_j\nu_j)}{\Gamma(1+\eta+\alpha-\sigma+\sum_{j=1}^n \rho_j\nu_j)}$$

$$\times F_{1:1,\ldots,1}^{1:0,\ldots,0}\left[\begin{matrix}[1+\eta-\sigma+\sum_{j=1}^n \rho_j\nu_j:2\rho_1,\ldots,2\rho_n]: \\ [1+\alpha+\eta-\sigma+\sum_{j=1}^n \rho_j\nu_j:2\rho_1,\ldots,2\rho_n]:[\nu_1+1:1],\ldots,[\nu_n+1:1]:\end{matrix}; -\frac{a_1^2}{4x^{2\rho_1}}, \ldots, -\frac{a_n^2}{4x^{2\rho_n}}\right]. \tag{3.4}$$

COROLLARY 2.3. *Let* $\alpha, \beta, \eta, \sigma, \nu_1, \nu_2 \in \mathbb{C}, a_1, a_2$ *and* $\rho_1, \rho_2 \in \mathcal{R}_+$ *be such that* $\Re(\alpha) > 0$, $\Re(\nu_1) > -1$, $\Re(\nu_2) > -1$, $\Re(\sigma - \rho_1\nu_1 - \rho_2\nu_2) < 1 + min[\Re(\beta), \Re(\eta)]$ *and* $\Re(\beta - \sigma + \nu_1 + \nu_2 + 1) > 0, \Re(\eta - \sigma + \nu_1 + \nu_2 + 1) > 0$. *Then*

$$\left(\mathcal{I}_{-}^{\alpha,\beta,\eta}\left[t^{\sigma-1}J_{\nu_1}\left(\frac{1}{t}\right)J_{\nu_2}\left(\frac{1}{t}\right)\right]\right)(x)$$

$$= \frac{x^{\sigma-\nu_1-\nu_2-\beta-1}}{2^{\nu_1+\nu_2}}\;\frac{\Gamma(c)\Gamma(f)}{\Gamma(g)\Gamma(h)\Gamma(\nu_1+1)\Gamma(\nu_2+1)}$$

$$\times F_{4:1,1}^{4:0,0}\left[\begin{matrix}[\frac{e}{2}:1,1],[\frac{e+1}{2}:1,1],[\frac{f}{2}:1,1],[\frac{f+1}{2}:1,1]: \\ [\frac{g}{2}:1,1],[\frac{g+1}{2}:1,1],[\frac{h}{2}:1,1],[\frac{h+1}{2}:1,1]:[\nu_1+1:1],\ldots,[\nu_n+1:1]:\end{matrix}; -\frac{1}{4x^2}, -\frac{1}{4x^2}\right], \tag{3.5}$$

where $e = \beta - \sigma + \nu_1 + \nu_2 + 1$, $f = \eta - \sigma + \nu_1 + \nu_2 + 1$, $g = 1 - \sigma + \nu_1 + \nu_2$, $h = \alpha + \beta + \eta - \sigma + \nu_1 + \nu_2 + 1$ *and* $F_{4:1,1}^{4:0,0}[\cdot]$ *is defined in (1.18).*

According to (1.7) and (1.9), Corollaries 2.1 and 2.2 follow from Theorem 2 in respective cases $\beta = -\alpha$ and $\beta = 0$. Corollary 1.3 follows from Theorem 1, if we put $n = 2, a_1 = 1, a_2 = 1, \rho_1 = 1, \rho_2 = 1$ and take (2.8) into account.

REMARK 2. When $n = 1, a_1 = 1, \rho_1 = 1, \nu_1 = \nu$, equation (3.9) is reduced to

$$\left(\mathcal{I}_{-}^{\alpha,\beta,\eta}\left[t^{\sigma-1}J_{\nu}\left(\frac{1}{t}\right)\right]\right)(x)$$

$$= \frac{x^{\sigma-\nu-\beta-1}}{2^{\nu}}\;\frac{\Gamma(\beta-\sigma+\nu+1)\Gamma(\eta-\sigma+\nu+1)}{\Gamma(1-\sigma+\nu)\Gamma(\alpha+\beta+\eta-\sigma+\nu+1)\Gamma(\nu+1)}$$

$$\times\ {}_4F_5\left[\begin{matrix}\frac{\beta-\sigma+\nu+1}{2},\frac{\beta-\sigma+\nu+2}{2},\frac{\eta-\sigma+\nu+1}{2},\frac{\eta-\sigma+\nu+2}{2}\\ \nu+1,\frac{1-\sigma+\nu}{2},\frac{2-\sigma+\nu}{2},\frac{\alpha+\beta+\eta-\sigma+\nu+1}{2},\frac{\alpha+\beta+\eta-\sigma+\nu+2}{2}\end{matrix};-\frac{1}{4x^2}\right]. \qquad (3.6)$$

This formula was proved in [4, Theorem 4].

4. Fractional integration of cosine and sine functions

For $\nu=-\frac{1}{2}$ and $\nu=\frac{1}{2}$, the Bessel function $J_\nu(z)$ in (1.10) coincides with cosine- and sine-functions, apart from the multiplier $\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}$:

$$J_{-\frac{1}{2}}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cos(z),\quad J_{\frac{1}{2}}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\sin(z). \qquad (4.1)$$

Setting $\nu_1=\cdots=\nu_n=-\frac{1}{2}$ and $\rho_1=\cdots=\rho_n=1$, from Theorem 1 and Corollaries 1.1 and 1.2 we deduce the following results:

THEOREM 3. *Let $\alpha,\beta,\eta,\sigma\in\mathbb{C}, a_j\in\mathcal{R}_+, j=1,\ldots,n$ be such that*

$$\Re(\alpha)>0,\quad \Re(\sigma)>0,\quad \Re(\sigma+\eta-\beta)>0,\quad \Re(\sigma)>\max[0,\Re(\beta-\eta)]$$

Then there holds the formula

$$\left(\mathcal{I}_{0+}^{\alpha,\beta,\eta}\left[t^{\sigma-1}\prod_{j=1}^{n}\cos(a_jt)\right]\right)(x)$$
$$=x^{\sigma-\beta-1}\frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)}$$
$$\times F^{2:0,\ldots,0}_{2:1,\ldots,1}\left[\begin{matrix}[\sigma:2,\ldots,2],[\sigma+\eta-\beta:2,\ldots,2]:\\ [\sigma-\beta:2,\ldots,2],[\alpha+\eta+\sigma:2,\ldots,2]:[\frac{1}{2}:1],\ldots,[\frac{1}{2}:1]:\end{matrix};-\frac{a_1^2x^2}{4},\ldots,-\frac{a_n^2x^2}{4}\right]. \qquad (4.2)$$

COROLLARY 3.1. *Let $\alpha,\sigma\in\mathbb{C}$ and $a_j\in\mathcal{R}_+$ $(j=1,\ldots,n)$ be such that $\Re(\alpha)>0$ and $\Re(\sigma)>0$. Then*

$$\left(\mathcal{I}_{0+}^{\alpha}\left[t^{\sigma-1}\prod_{j=1}^{n}\cos(a_jt)\right]\right)(x)$$
$$=x^{\sigma+\alpha-1}\frac{\Gamma(\sigma)}{\Gamma(\sigma+\alpha)}F^{1:0,\ldots,0}_{1:1,\ldots,1}\left[\begin{matrix}[\sigma:2,\ldots,2]:\\ [\sigma+\alpha:2,\ldots,2]:[\frac{1}{2}:1],\ldots,[\frac{1}{2}:1]:\end{matrix};-\frac{a_1^2x^2}{4},\ldots,-\frac{a_n^2x^2}{4}\right]. \qquad (4.3)$$

Corollary 3.2. *Let $\alpha, \eta, \sigma \in \mathbb{C}$ and $a_j \in \mathcal{R}_+$ $(j = 1, \ldots, n)$ be such that $\Re(\alpha) > 0$ and $\Re(\sigma) > -\Re(\eta)$. Then*

$$\left(\mathcal{I}^{+}_{\eta,\alpha}\left[t^{\sigma-1}\prod_{j=1}^{n}\cos(a_j t)\right]\right)(x)$$

$$= x^{\sigma-1}\frac{\Gamma(\sigma+\eta)}{\Gamma(\sigma+\alpha+\eta)}F^{1:0,\ldots,0}_{1:1,\ldots,1}\left[{}^{[\sigma+\eta:2,\ldots,2]:}_{[\sigma+\alpha+\eta:2,\ldots,2]:[\frac{1}{2}:1],\ldots,[\frac{1}{2}:1]:};-\frac{a_1^2x^2}{4},\ldots,-\frac{a_n^2x^2}{4}\right]. \tag{4.4}$$

Theorem 4. *Let $\alpha, \beta, \eta, \sigma \in \mathbb{C}$ and $a_j \in \mathcal{R}_+$ $(j = 1, \ldots, n)$ be such that*

$$\Re(\alpha) > 0, \quad \Re(\sigma) > 0, \quad \Re(\sigma+\eta-\beta) > 0, \quad \Re(\sigma) > \max[0, \Re(\beta-\eta)].$$

Then

$$\left(\mathcal{I}^{\alpha,\beta,\eta}_{0+}\left[t^{\sigma-n-1}\prod_{j=1}^{n}\sin(a_j t)\right]\right)(x)$$

$$= \frac{\pi^{\frac{n}{2}}}{2^n}\left(\prod_{j=1}^{n}a_j\right)x^{\sigma-\beta-1}\frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)}$$

$$\times F^{2:0,\ldots,0}_{2:1,\ldots,1}\left[{}^{[\sigma:2,\ldots,2],[\sigma+\eta-\beta:2,\ldots,2]:}_{[\sigma-\beta:2,\ldots,2],[\alpha+\eta+\sigma:2,\ldots,2]:[\frac{3}{2}:1],\ldots,[\frac{3}{2}:1]:};-\frac{a_1^2x^2}{4},\ldots,-\frac{a_n^2x^2}{4}\right]. \tag{4.5}$$

Corollary 4.1. *Let $\alpha, \sigma \in \mathbb{C}$ and $a_j \in \mathcal{R}_+$ $(j = 1, \ldots, n)$ be such that $\Re(\alpha) > 0$ and $\Re(\sigma) > 0$. Then*

$$\left(\mathcal{I}^{\alpha}_{0+}\left[t^{\sigma-n-1}\prod_{j=1}^{n}\sin(a_j t)\right]\right)(x)$$

$$= \frac{\pi^{\frac{n}{2}}}{2^n}\left(\prod_{j=1}^{n}a_j\right)x^{\sigma-\beta-1}\frac{\Gamma(\sigma)}{\Gamma(\sigma+\alpha)}$$

$$\times F^{1:0,\ldots,0}_{1:1,\ldots,1}\left[{}^{[\sigma:2,\ldots,2]:}_{[\sigma+\alpha:2,\ldots,2],:[\frac{3}{2}:1],\ldots,[\frac{3}{2}:1]:};-\frac{a_1^2x^2}{4},\ldots,-\frac{a_n^2x^2}{4}\right]. \tag{4.6}$$

Corollary 4.2. *Let $\alpha, \eta, \sigma \in \mathbb{C}$ and $a_j \in \mathcal{R}_+$ $(j = 1, \ldots, n)$ be such that $\Re(\alpha) > 0$ and $\Re(\sigma) > -\Re(\eta)$. Then*

$$\left(\mathcal{I}^{+}_{\eta,\alpha}\left[t^{\sigma-n-1}\prod_{j=1}^{n}\sin(a_j t)\right]\right)(x)$$

$$= \frac{\pi^{\frac{n}{2}}}{2^n}\left(\prod_{j=1}^{n} a_j\right) x^{\sigma-1}\frac{\Gamma(\sigma+\eta)}{\Gamma(\sigma+\alpha+\eta)}$$

$$\times\, F^{1:0,\ldots,0}_{1:1,\ldots,1}\left[\begin{matrix}[\sigma+\eta:2,\ldots,2]: \\ [\alpha+\eta+\sigma:2,\ldots,2]:[\frac{3}{2}:1],\ldots,[\frac{3}{2}:1]:\end{matrix}\;;-\frac{a_1^2x^2}{4},\ldots,-\frac{a_n^2x^2}{4}\right]. \qquad (4.7)$$

Similarly, setting $\nu_1 = \cdots = \nu_n = -\frac{1}{2}$ and $\rho_1 = \cdots = \rho_n = 1$ and taking (4.1) into account, from Theorem 2 and Corollaries 2.1 and 2.2, we obtain the following results:

Theorem 5. *Let $\alpha, \beta, \eta, \sigma \in \mathbb{C}$ and $a_j \in \mathcal{R}_+$ $(j = 1, \ldots, n)$ be such that*

$$\Re(\alpha) > 0,\quad \Re(\beta-\sigma) > 0,\quad \Re(\eta-\sigma) > 0,\quad \Re(\sigma) < \min[\Re(\beta), \Re(\eta)].$$

Then

$$\left(\mathcal{I}^{\alpha,\beta,\eta}_{-}\left[t^{\sigma}\prod_{j=1}^{n}\cos\left(\frac{a_j}{t}\right)\right]\right)(x)$$

$$= x^{\sigma-\beta}\frac{\Gamma(\beta-\sigma)\Gamma(\eta-\sigma)}{\Gamma(-\sigma)\Gamma(\alpha+\beta+\eta-\sigma)}$$

$$\times\, F^{2:0,\ldots,0}_{2:1,\ldots,1}\left[\begin{matrix}[\beta-\sigma:2,\ldots,2],[\eta-\sigma:2,\ldots,2]: \\ [-\sigma:2\ldots,2],[\alpha+\beta+\eta-\sigma:2,\ldots,2]:[\frac{1}{2}:1],\ldots,[\frac{1}{2}:1]:\end{matrix}\;;-\frac{a_1^2}{4x^2},\ldots,-\frac{a_n^2}{4x^2}\right]. \qquad (4.8)$$

Corollary 5.1. *Let $\alpha, \sigma \in \mathbb{C}$ and $a_j \in \mathcal{R}_+$ $(j = 1, \ldots, n)$ be such that $0 < \Re(\alpha) < -\Re(\sigma)$. Then*

$$\left(\mathcal{I}^{\alpha}_{-}\left[t^{\sigma}\prod_{j=1}^{n}\cos\left(\frac{a_j}{t}\right)\right]\right)(x)$$

$$= x^{\sigma+\alpha}\frac{\Gamma(-\alpha-\sigma)}{\Gamma(-\sigma)}F^{1:0,\ldots,0}_{1:1,\ldots,1}\left[\begin{matrix}[-\alpha-\sigma:2,\ldots,2]: \\ [-\sigma:2,\ldots,2]:[\frac{1}{2}:1],\ldots,[\frac{1}{2}:1]:\end{matrix}\;;-\frac{a_1^2}{4x^2},\ldots,-\frac{a_n^2}{4x^2}\right]. \qquad (4.9)$$

COROLLARY 5.2. *Let $\alpha, \eta, \sigma \in \mathbb{C}$ and $a_j \in \mathcal{R}_+$ $(j = 1, \dots, n)$ be such that $\Re(\alpha) > 0$ and $\Re(\sigma) < \Re(\eta)$. Then*

$$\left(\mathcal{K}_{\eta,\alpha}^{-}\left[t^{\sigma}\prod_{j=1}^{n}\cos\left(\frac{a_j}{t}\right)\right]\right)(x)$$

$$= x^{\sigma}\frac{\Gamma(\eta-\sigma)}{\Gamma(\alpha+\eta-\sigma)}F_{1:1,\dots,1}^{1:0,\dots,0}\left[\begin{matrix}[\eta-\sigma:2,\dots,2]: \\ [\alpha+\eta-\sigma:2,\dots,2]:[\frac{1}{2}:1],\dots,[\frac{1}{2}:1]:\end{matrix};-\frac{a_1^2}{4x^2},\dots,-\frac{a_n^2}{4x^2}\right]. \tag{4.10}$$

THEOREM 6. *Let $\alpha, \beta, \eta, \sigma \in \mathbb{C}$ and $a_j \in \mathcal{R}_+$ $(j = 1, \dots, n)$ be such that*

$$\Re(\alpha) > 0, \quad \Re(\beta - \sigma > -1, \quad \Re(\eta - \sigma) > -1, \quad \Re(\sigma) < 1 + \min[\Re(\beta), \Re(\eta)].$$

Then there holds the formula

$$\left(\mathcal{I}_{-}^{\alpha,\beta,\eta}\left[t^{\sigma+n-1}\prod_{j=1}^{n}\sin\left(\frac{a_j}{t}\right)\right]\right)(x)$$

$$= \frac{\pi^{\frac{n}{2}}}{2^n}\left(\prod_{j=1}^{n}a_j\right)x^{\sigma-\beta-1}\frac{\Gamma(\beta-\sigma+1)\Gamma(\eta-\sigma+1)}{\Gamma(1-\sigma)\Gamma(\alpha+\beta+\eta-\sigma+1)}$$

$$\times\ F_{2:1,\dots,1}^{2:0,\dots,0}\left[\begin{matrix}[\beta-\sigma:2,\dots,2],[\eta-\sigma+1:2,\dots,2]: \\ [1-\sigma:2,\dots,2],[\alpha+\beta+\eta-\sigma+1:2,\dots,2]:[\frac{3}{2}:1],\dots,[\frac{3}{2}:1]:\end{matrix};-\frac{a_1^2}{4x^2},\dots,-\frac{a_n^2}{4x^2}\right]. \tag{4.11}$$

COROLLARY 6.1. *Let $\alpha, \sigma \in \mathbb{C}$ and $a_j \in \mathcal{R}_+$ $(j = 1, \dots, n)$ be such that $0 < \Re(\alpha) < 1 - \Re(\sigma)$. Then*

$$\left(\mathcal{I}_{-}^{\alpha}\left[t^{\sigma+n-1}\prod_{j=1}^{n}\sin\left(\frac{a_j}{t}\right)\right]\right)(x)$$

$$= \frac{\pi^{\frac{n}{2}}}{2^n}\left(\prod_{j=1}^{n}a_j\right)x^{\sigma+\alpha-1}\frac{\Gamma(1-\alpha-\sigma)}{\Gamma(1-\sigma)}$$

$$\times\, F^{1:0,\dots,0}_{1:1,\dots,1}\left[{}^{[1-\sigma-\alpha:2,\dots,2]:}_{[1-\sigma:2,\dots,2]:[\frac{3}{2}:1],\dots,[\frac{3}{2}:1]:}\,;-\frac{a_1^2}{4x^2},\dots,-\frac{a_n^2}{4x^2}\right]. \tag{4.12}$$

Corollary 6.2. *Let $\alpha,\eta,\sigma\in\mathbb{C}$ and $a_j\in\mathcal{R}_+$ $(j=1,\dots,n)$ be such that $\Re(\alpha)>0$ and $\Re(\sigma)<1+\Re(\eta)$. Then*

$$\left(\mathcal{K}^{-}_{\eta,\alpha}\left[t^{\sigma+n-1}\prod_{j=1}^{n}\sin\left(\frac{a_j}{t}\right)\right]\right)(x)$$

$$=\frac{\pi^{\frac{n}{2}}}{2^n}\left(\prod_{j=1}^{n}a_j\right)x^{\sigma-1}\frac{\Gamma(\eta-\sigma+1)}{\Gamma(\alpha+\eta-\sigma+1)}$$

$$\times\, F^{1:0,\dots,0}_{1:1,\dots,1}\left[{}^{[\eta-\sigma+1:2,\dots,2]:}_{[\alpha+\eta-\sigma+1:2,\dots,2]:[\frac{3}{2}:1],\dots,[\frac{3}{2}:1]:}\,;-\frac{a_1^2}{4x^2},\dots,-\frac{a_n^2}{4x^2}\right]. \tag{4.13}$$

Remark 3. When $n=1, a_1=1$, then all the results in Section 4 coincide with that proved in [4, Sections 5 and 6].

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