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WELL-POSEDNESS OF DIFFUSION-WAVE PROBLEM WITH ARBITRARY FINITE NUMBER OF TIME FRACTIONAL DERIVATIVES IN SOBOLEV SPACES H^s

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Abstract

We give the proofs of the existence and regularity of the solutions in the space $C^\infty(t>0; H_2^{s+2}(\mathbf{R}^n)) \cap C^0(t \geq 0; H^s(\mathbf{R}^n))$, $s \in \mathbf{R}$, for the 1-term, 2-term,..., n-term time-fractional equation evaluated from the time fractional equation of distributed order with spatial Laplace operator $\triangle_x$

$$\int_0^2 p(\beta) D_*^\beta u(x,t) d\beta = \triangle_x u(x,t),\ t \geq 0,\ x \in \mathbf{R}^n,$$

subject to the Cauchy conditions $u(0,x) = \varphi(x)$, $u_t(0,x) = \psi(x)$, where D_*^β is the operator in the Caputo sense. For the weight function $p(\beta)$ we take a finite positive-linear combination of delta distributions concentrated at points of interval $(0,2)$, with special distribution of derivative orders, $p(\beta) = \sum_{i=1}^n b_i \delta(\beta - \beta_i)$, $0 < \beta_1 < ... < \beta_{m-1} < 1 < \beta_m < ... < \beta_n < 2$, $b_i > 0$, $m = 1, 2, ..., n$.

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1. Preliminaries

Throughout of the paper we use the following notation: PDE means partial differential equations, RL stands for Riemann-Louville fractional derivative and by C we denote the Caputo fractional derivative.

1.1. Applications of fractional PDEs

Fractional calculus has applications in engineering, physics, finance, chemistry, bioengineering. The equation studied, for the diffusion-wave phenomena, enjoys an increasing interest among researches in visco-elasticity and anomalous diffusion, and there are numerical analysts who find them highly challenging.

Anomalous diffusion processes in which the characteristic displacement scales are the power of time are described by fractional diffusion equations. In the equations describing non-scaling anomalous diffusion processes (cf. [39], [40], [11]) fractional derivatives of distributed order are employed.

The distributed order derivative is a linear operator defined as a weighted sum of different fractional derivatives or an integral of such over their order, $\int_a^b d\beta p(\beta)\frac{\partial^\beta}{\partial z^\beta}$, which acts on the function of the corresponding variable z (cf. [10]). For distributed order diffusion equation z means space or time.

The distributed order fractional calculus with application to distributed order PDEs was started by [11], [12], [13], [38], [14], [39] gaining more and more interest among physicists and mathematicians (cf. [30], [31] and recently [24]). These equations have an application in visco-elasticity and in anomalous diffusion processes. They can be perturbed with nonlinear force term $f(t, u(x,t))$ which satisfies some regularity conditions, singular initial data or generalized stochastic process. The theoretical background of the equations with fractional derivatives of distributed order is developed in [42], [44], [24].

The effects of transformation from anomalous diffusion to normal one are observed in experimental findings in biophysics, plasma physics and econophysics.

In [10], the processes become more anomalous (w.r.t.) the time are described with these equations. Anomalous relaxation and diffusion phenomena which are less anomalous (w.r.t.) the time described by differential equations with the fractional derivatives of distributed order are discussed in [10].

A generalization of the telegraph equation to equation with fractional derivatives with two-term time fractional derivative is explored in [4] by the maximum principle. Then, the classical heat conduction equation, the telegraph equation and the classical wave equation are special cases of these generalized equations.

It can be seen in [14], [31], [19], etc., that PDEs of fractional order have been successfully used for modeling some relevant physical processes. From the mathematical point of view, only initial value problems and boundary value problems with constant coefficients are considered. In the paper [26] the initial boundary value problem for the generalized time fractional diffusion equation over an open bounded domain is studied. The mentioned equation is obtained from the classical diffusion equation by setting for the first order time derivative a fractional derivative of order α $(0 < \alpha \leq 1)$ and the second order spatial derivative by the linear second order differential operator.

In this paper we follow the classical approach concerning the initial data and C derivative. Recently, there exists a growing interest among the researches to the revision of this approach considering the past of the initial data. One possibility is multiplication by the Heaviside function, as in [32]. Another possibility is using the initialization with the function $\psi(t)$ from [21], given with the function $y(t) = y_{t_0}(t) + \psi(t)$, $t > t_0$, where y_{t_0} is the response of the system to the input $u(t)H(t - t_0)$ and $y(t)$ is the response of the system at rest before $t = 0$, to the input $u(t)$ (cf. [35]). Both approaches require new initialization of the C derivative, new procedure and technique in applying the Laplace-Fourier transform in succession to the diffusion-wave phenomena, which is the base of the method developed here. New spaces are also required, as well as the physical consistency and new existence-uniqueness results. In few words, we need a development of a new theory concerning the fractional calculus and its applications different than the existing one.

1.2. Background and groundwork

In this paper we discuss questions of existence, uniqueness and the construction of the solutions to Cauchy problems for general diffusion-wave equations with temporal fractional derivatives of C-type with distributed orders. We find it desirable to find explicit solutions of them and put them into relevant functional spaces. Using as a weight function the finite linear combination of delta distributions, we generalize time fractional equations to the linear combinations of the fractional derivatives of the unknown solution at different time-instants. We condensed our result in theorems concerning the regularity of the solutions for each problem separately.

For the essentials of the fractional calculus, see e.g. [17], [28], [27]), and the recent book [22].

A recent generalization of the Cauchy problem for the space-time fractional diffusion equation for $\beta \in (0, 2]$,

$$D^{\alpha}_{x,\theta} u(x,t) = D^{\beta}_{*} u(x,t), \quad x \in \mathbf{R}, \quad t \in \mathbf{R}_{+}, \quad \alpha \in (0,2], \quad |\theta| \leq \min\{\alpha, 2-\alpha\},$$

is given in [29], where the second-order space derivative is replaced by a Riesz-Feller derivative of order $\alpha \in (0, 2]$ and skewness θ, $|\theta| \leq min\{\alpha, 2 - \alpha\}$. For the expression of the Green function, Mellin-Barnes integrals in the complex plane are employed.

In [42], it is given a mathematical basis for the question of existence and uniqueness and construction of a solution to the Cauchy problem and multi-point problem for a general linear evolution equation with temporal fractional derivatives with distributed orders,

$$\int_0^m B(\beta) D^{\beta}_{*} u(x,t) d\beta = A(D) u(x,t), \quad t > 0, \quad \beta \in [0, m],$$

where $B(\beta)$ and $A(D)$ are linear closed operators on locally convex spaces X, D^{β}_{*} is the operator of fractional differentiation of order β in the C-sense.

Recall, for example, the definition of the pseudo-differential operator from [18], $A(D)$ with a symbol $A(\kappa)$

$$A(D)\varphi(x) = \frac{1}{(2\pi)^n} \int_G A(\kappa) \hat{F}(\kappa) e^{ix\kappa} d\kappa,$$

where $\hat{\varphi}$ is the Fourier transform of φ. The operator is defined on a function φ which belongs to some of the functional spaces of entire functions of exponential type, which restrictions to $\mathbf{R}^n$, are in the space $L^2(\mathbf{R}^n)$, and in general, to $L^p(\mathbf{R}^n)$, and their duals (operators $A(-D)$). The strong and the weak solutions are considered.

The extension of these ideas to the fractional differential equations is given in [37], [18] and [42]. The results are extended to non-homogeneous problems in Sobolev spaces by a density lemma (cf. [18]).

In the papers by Umarov et al., appropriate spaces of generalized functions (or distributions) are introduced for treatment, that we have been motivated and influenced by the methods and far-reaching results, but that in our present paper we try just to work in suitable Sobolev-like spaces.

We give in this paper the generalization to nD of the explicit solutions for a class of the time fractional equations in $1D$ in terms of the Mittag-Leffler function from [41] obtained by applying the transforms of Laplace and Fourier in succession, and of their inverses.

The equation with m-times derivatives in the interval $(0, 1)$ and $(n-m)$-derivatives in $(1, 2)$ is employed. For the operator $A(D)$ we take the Laplace operator $\triangle_x$. As a framework, we employ the Sobolev space $H_2^s(\mathbf{R}^n)$, $s \in \mathbf{R}$.

We recall some problems that are covered by our method.

In [18] it is given the Cauchy problem for PDEs of fractional order β, $\beta \in (m-1, m)$, $m \in \mathbf{N}$,

$$D_*^\beta u(x,t) = A(D)u(x,t), \quad t > 0, \quad x \in \mathbf{R}^n,$$

$$\frac{\partial^k u(x,0)}{\partial t^k} = \varphi_k(x), \quad x \in \mathbf{R}^n, \quad k = 0, 1, ..., m-1$$

φ_k, $k = 0, ..., m-1$, are given functions, $A(D)$ is a PDO with the symbol $A(\kappa)$, which is a continuous function defined on an open domain, subset of $\mathbf{R}^n$, D_*^β is the operator of Caputo fractional differentiation of the order $\beta > 0$.

Generalization of the multi-point value problem is obtained in (cf. [42], [18]) for the PDEs of fractional order

$$D_*^{\beta_m} u(x,t) + \sum_{k=1}^{m-1} A_k(D) D_*^{\beta_k} u(x,t) + A_0(D)u(x,t) = 0, \quad t \in (0,T), \quad x \in \mathbf{R}^n.$$

In particular, with $A(D) = \triangle_x$ is exhibited the time-fractional differential equation ([18])

$$D_*^\beta u(x,t) = \triangle_x u(x,t), \quad t \geq 0, \quad x \in \mathbf{R}^n, \quad \beta \in (0,2), \tag{1}$$

where $\triangle_x$ is the Laplace operator with the symbol $A(\kappa) = -|\kappa|^2$. This is the fractional model of diffusion when $\beta \in (0, 1)$, and the process between diffusion and wave propagation when $\beta \in (0, 2)$. When $t \to \infty$, the solution has the relaxation property, i.e. tends to zero.

When $\beta = 1$, we have the space-fractional equation

$$\frac{\partial u(x,t)}{\partial t} = D_0^\alpha u(x,t), \quad t > 0, \quad x \in \mathbf{R}^n, \quad \alpha > 0, \tag{2}$$

where $D_0^\alpha(\kappa) = -|\kappa|^\alpha$ has a connection with the inversion of the fractional Riesz potential. When $n = 1$, and $\alpha \in (0, 2)$, this is the model of Levy-Feller diffusion processes (cf. [18], and references therein). Relaxation holds for a solution of this problem, i.e. $\lim_{t\to\infty} u(x,t) = 0$.

As a generalization of (1) and (2), the space-time fractional differential equation

$$D_*^\beta u(x,t) = D_0^\alpha u(x,t), \;\; t > 0, \;\; x \in \mathbf{R}^n, \;\; \alpha, \beta > 0,$$

is considered. For $\alpha = 2$, (1) follows, and for $\beta = 1$, we have (2). Smoothness properties of the solution in the Sobolev space $H_2^s(\mathbf{R}^n)$, $s \in \mathbf{R}$, $\forall \beta$ from $(0, 2)$, are described in [18]. These results match with the classical ones for the Cauchy problem for the wave equation without $\beta = 1$.

The paper [37], based on the research [18], is devoted to the time-fractional inhomogeneous pseudo-differential equations of fractional order

$$D_*^\beta u(x,t) = A(D_x)u(x,t) + f(x,t), \;\; u(x,0) = \varphi(x), \;\; t > 0, \;\; x \in \mathbf{R}^n,$$

where $f(x,t)$ and $\varphi(x)$ are given functions, the symbol $A(\kappa)$ is a real analytic function on an open domain $G \subset \mathbf{R}$, D_*^β, $\beta \in (0,1)$, is the operator of fractional differentiation of order β in Caputo sense.

The Duhamel principle for Cauchy problem for linear inhomogeneous the time-fractional PDEs is established in [43].

In [23], the authors consider the diffusion-wave problem involving Riemann-Louiville approach. For $\beta \in (0, 2)$ the explicit solution is given in a form of H functions and for $n = 1$ the solution is established via the Wright function and Mittag-Leffler function, including asymptotic behaviour. The initial data are fractional derivatives of $C^\infty(\mathbf{R}^n)$ functions. Relaxation properties of the solutions are considered.

Solvability of linear fractional differential equations in Banach spaces, Roumieu, Gevrey and Beurling type, are proved in [16]. Solving fractional evolution equations which use the theory of semigroups and the Schauder fixed point theorem and their generators are the subject of [5]. The thesis of E. Bazhlekova, and her papers ([6], [7], [8], [9]) are concerned with abstract fractional evolution equations. There, methods from the functional analysis, theory of semigroups and Schauder fixed point theorem are used.

In [45], continuous and discrete techniques for several approximations for some models of fractional-order systems are proposed.

In [24], Kochubei develops a mathematical theory for the equation

$$\int_0^1 D_*^\beta u(x,t)\mu(\beta)d\beta = Bu(x,t) + f(x,t), \;\; t > 0, \;\; x \in \mathbf{R}^n,$$

with the C fractional derivative $D_*^\beta u$, where B is an elliptic operator in the spatial variable, $\mu(\beta)$ is a weight function. This method is used in physics to model anomalous diffusion in fractal media. Moreover, the ultra slow-diffusion with a logarithmic growth appearing in polymer physics and in models of a particle's motion in a quenched random force field, iterated map models, etc. can be solved. Fractional differentiation and integrations are employed with the classical theory of the Cauchy problem for the heat equation. This approach is different from the approach in [42], which does not differ between $B = +\triangle$ and $B = -\triangle$. Due to that, in the paper [24] the classical theory for the heat equation is obtained for $B = \triangle$.

1.3. Notation

The transforms of Laplace and Fourier, applied in succession to a generic function $u(x,t)$, are defined as:

$$(\mathcal{L}_t u)(x,s) = \int_0^\infty u(x,t)e^{-st}dt, \quad s \in \mathbf{C}, \ t > 0;$$

$$(\mathcal{F}_t u)(\kappa,t) = \int_{\mathbf{R}^n} u(x,t)e^{ix\cdot\kappa}dx, \ \ x, \ \kappa \in \mathbf{R}^n.$$

The Riemann-Liouville fractional integral of order $\beta > 0$ is

$$J^\beta f(t) := \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}f(\tau)d\tau, \ \ \beta > 0.$$

The Riemann-Liouville fractional derivative of order $\beta > 0$ is defined as

$$D^\beta f(t) = \begin{cases} \dfrac{d^m}{dt^m}[\dfrac{1}{\Gamma(m-\beta)}\displaystyle\int_0^t \dfrac{f(\tau)}{(t-\tau)^{\beta+1-m}}d\tau], & \beta \in (m-1,m), \\ \dfrac{d^m}{dt^m}f(t), & \beta = m. \end{cases} \quad m \in \mathbf{N},$$

and the fractional derivative in the so-called Caputo sense is given by

$$D_*^\beta f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\beta)}\displaystyle\int_0^t \dfrac{f^{(m)}(\tau)}{(t-\tau)^{\beta+1-m}}d\tau, & \beta \in (m-1,m), \\ \dfrac{d^m}{dt^m}f(t), & \beta = m. \end{cases} \quad m \in \mathbf{N}.$$

The Laplace transform of a fractional derivative reads

$$\mathcal{L}(D_*^\beta f(t);s) = s^\beta \tilde{f}(s) - \sum_{k=0}^{m-1} s^{\beta-1-k}f^{(k)}(0^+), \ \ \beta \in (m-1,m),$$

where $f^{(k)}(0^+) := \lim_{t\to 0^+} f(t)$.

For more details concerning the fractional calculus operators, see e.g. [17], [27], [22].

As a framework, we use the Sobolev spaces (cf. [3]).

Let P^n be the subset of $\mathbf{R}^n$ whose elements have as coordinates non-negative integers. If $k = (k_1, ..., k_n) \in P^n$, then $|k| = k_1 + ... + k_n$. Instead of P^1, we usually use only P.

Let Ω be an open set. Let p be a real number such that $1 \leq p \leq \infty$ and let $m \in P$. Denote by $H_p^m(\Omega)$ the subset of the Schwartz distributions $\mathcal{D}'(\Omega)$ such that for its elements the following holds:

1. $u \in H_p^m(\Omega)$ iff $\forall \alpha \in P^n$, $|\alpha| \leq m$,
2. $\partial^\alpha u \in L^p(\Omega)$ where we consider α as the derivative in the distributional sense.

The spaces $H_p^m(\Omega)$ are called the Sobolev spaces.

2. Well-posedness of diffusion-wave problem with arbitrary finite number of time fractional derivatives in Sobolev spaces H^s, $s \in \mathbf{R}$

We transfer the exact solutions from [41] given for $1D$ till the nD case and prove that if the order of the time-fractional derivative is any real number from interval $(0,2)$ or finite set of real numbers distributed in intervals $(0,1)$ and $(1,2)$ respectively, i.e. $0 < \beta_1 < \beta_2 < ... < \beta_{m-1} < 1 < \beta_m < \beta_{m+1} < .... < \beta_n < 2$, the solution belongs to the Sobolev space $H_2^{s+2}(\mathbf{R}^n)$ with respect to the spatial variable, if the initial data belong to $H_2^s(\mathbf{R}^n)$. We use the results of [42], the properties of Dubinski's (cf. [15]) pseudo-differential operators and the density lemma (cf. [18]) which allow us to prove that the solutions belong to the Sobolev space $H_2^{s+2}(\mathbf{R}^n)$, $(s \in \mathbf{R})$, n is arbitrary natural number, if the initial data are in $H_2^s(\mathbf{R}^n)$.

For $\varphi(x) \in H_2^s(\mathbf{R}^n)$, the operator $A(D)$ of Dubinskii type is defined as

$$A(D)\varphi(x) = \frac{1}{2\pi}\int_{\mathbf{R}^n} A(\kappa)\mathcal{F}\varphi(\kappa)e^{ix\kappa}d\kappa = \frac{1}{2\pi}\int_G A(\kappa)\mathcal{F}\varphi(\kappa)e^{ix\kappa}d\kappa.$$

It is pseudo-differential operator with a symbol $A(\kappa) = (1 + |\kappa|^2)$ which is a real analytic function in an open set G. Outside G or on its boundary $A(\kappa)$ it may have singularities of arbitrary order. The operator is defined in [18] on a function φ which belongs to some case of the functional spaces which are entire functions of exponential type, which restrictions to $\mathbf{R}^n$ are in the space $L^2(\mathbf{R}^n)$ and in general $L^p(\mathbf{R}^n)$ and their duals, operators $A(-D)$. The strong and the weak solutions are considered. If

$$|A(\kappa)| \leq C(1+|\kappa|^2)^{l/2}, \ \ C>0, \ \ \kappa \in \mathbf{R}^n,$$

there exists a unique continuous closure $\bar{A}(D): H_2^s(\mathbf{R}^n) \to H_2^{s-l}(\mathbf{R}^n), \ \ s \in \mathbf{R}, \ \ l \in \mathbf{R}$, (cf. [18]). Since the Dubinski operator does not make difference between $A(D) = \pm\triangle_x$, (cf. [18], [46]) in nonhomogeneous case is considered the operator $A(D) = I - \triangle_x$, where I is identity operator.

We give a proof for a homogeneous operator $A(D) = \triangle_x$ with symbol $A(\kappa) = -|\kappa|^2$, $\kappa \in \mathbf{R}^n$.

2.1. Existence-uniqueness results for the diffusion-wave problem of distributed order

Explicit solutions in terms of the Mittag-Leffler function are given in [41] by the technique of Gorenflo-Mainardi which consists of the application of the Laplace and Fourier transformations and their inverses in succession. The precision of the corresponding solutions to the diffusion-wave problem with one, two, three,...., finite n number of fractional derivatives is verified by numerical experiments in *Mathematica*.

In this section, we recall some basic formulas for the exact solutions generated to the n-dimensional case (w.r.t.) the spatial variable x, to the diffusion-wave problem and put them into corresponding functional spaces. For calculation and main asymptotical properties, as well as the plots of the solution, see [41] in one-dimensional case.

We deal with the problem for $p(\beta) > 0:$

$$\int_0^2 p(\beta) D_*^{\beta} u(x,t) d\beta = A(D)u(x,t), \ \ x \in \mathbf{R}^n, \ \ t \geq 0, \tag{3}$$

where $A(D) = \triangle_x$. We examine the first, a one-term Green function on intervals $(0,1)$, and $(1,2)$ separately, $x \in \mathbf{R}^n$. Explicit solution of this case also is given in [17] and [23] but we repeat it here, due to generality of the method. For analogous representation formulas in $1D$ for one-term, two-term,..., n-term equations, cf. [41].

In what follows, we consider separately the fundamental solution or the Green function $u_{1fund}(x,t)$ (resp. $u_{2fund}(x,t)$), corresponding to the initial data

$$u(x,0) = \varphi(x) \quad (\text{resp. } u_t(x,0)) = \psi(x)). \tag{4}$$

Calling them $u_{jfund}(x,t)$, $j=1,2$, we obtain for arbitrary initial conditions $u(x,0)$ and $u_t(x,0)$, the solution $u(x,t)$ by the convolution

$$u(x,t)=\int_{\mathbf{R}^n} u(s,0)u_{1fund}(t,x-s)ds+\int_{\mathbf{R}^n} u_t(s,0)u_{2fund}(t,x-s)ds. \quad (5)$$

2.1.1. Case $\beta_1 \in (0,1)$, $p(\beta)=b_1\delta(\beta-\beta_1)$

For the operator $A(D)=\triangle_x$, the equation (3) becomes

$$b_1 D_*^{\beta_1} u(x,t)=\triangle_x u(x,t),\ \ x\in\mathbf{R}^n. \quad (6)$$

The explicit solution reads

$$u(x,t)=E_{\beta_1}(1/b_1 t^{\beta_1}\triangle_x)u(x,0)),\ \ x\in\mathbf{R}^n. \quad (7)$$

In general, for $x\in\mathbf{R}^n$, for unbounded pseudo-differential operator $A(D)=\triangle_x$ with symbol $A(\kappa)=-|\kappa|^2$, we have

$$(A_x^j f)(x)=\frac{1}{(2\pi)}\int_{\mathbf{R}}(-|\kappa|^2)^j(\mathcal{F}_x f)(\kappa)e^{-i\kappa x}d\kappa,\ \ (j=1,2,...),$$

the solution operator is given

$$u(x,t)=\sum_{j=0}^{\infty}\frac{(1/b_1 t^{\beta_1})^j}{\Gamma(\beta_1 j+1)}A_x^j(u(x,0)),$$

and

$$u(x,t)=\sum_{j=0}^{\infty}\frac{(1/b_1 t^{\beta_1})^j}{\Gamma(\beta_1 j+1)}(\triangle_x^j)(u(x,0)), \quad (8)$$

where $\triangle_x^1=\triangle_x$, $\triangle_x^j=\triangle_x^1\triangle_x^{j-1}$, $j=2,3,...$ (cf. [23]). The solution operators are also pseudo-differential operators. Because of that we study the properties of their symbols and use them to prove the solvability theorems.

Following the approach from [37], [18], we prove the following existence uniqueness result to the equation (6).

THEOREM 1. *Let* $\beta_1\in(0,1)$. *If* $A(D)=\triangle_x$, $p(\beta)==b_1\delta(\beta-\beta_1)$ *for diffusion-wave problem (3), we obtain the existence-uniqueness result in the space* C^∞ $(t\in(0,\infty);\ H^{s+2}(\mathbf{R}^n))\cap C^0(t\in[0,\infty),H^s(\mathbf{R}^n))$, $s\in\mathbf{R}$. *The representation of the exact solution is given with (8).*

P r o o f. Since the solution operator for the Laplace operator, $A(D)=\triangle_x$, is given by $S(\beta_1,x,t)=E_{\beta_1}(1/b_1 t^{\beta_1}\triangle_x)$, then according to ([18])

(where the asymptotic properties of the Mittag-Leffler function are used), $|S(\beta_1,t,\kappa)| \le C(1+|\kappa|^2)^{-2}$, as $|\kappa| \to \infty$. Then, from (7) we have the following estimate for the solution

$$||u(x,t)||_{H_2^{s+2}(\mathbf{R}^n)} = ||E_{\beta_1}(1/b_1 t^{\beta_1}\triangle_x)u(x,0)||_{H_2^{s+2}(\mathbf{R}^n)} = (\int_{\mathbf{R}^n} e^{ix\kappa}|\hat{u}(\kappa,0)|^2$$

$$|(1+|\kappa|)^{-2}|^2|(1+|\kappa|)^{(s+2)}|^2 d\kappa)^{1/2} \le C(\int_{\mathbf{R}^n} e^{ix\kappa}|\hat{u}(\kappa,0)|^2(1+|\kappa|^2)^{2s}d\kappa)^{1/2}$$

$$\le C||u(x,0)||_{H_2^s(\mathbf{R}^n)}.$$

■

Limiting behavior reads $||u(x,t)||_{H_2^{s+2}(\mathbf{R}^n)} \to C||u(x,0)||_{H_2^s(\mathbf{R}^n)}$, when, $t \to 0$. When $t \to \infty$, then $||u(x,t)||_{H_2^{s+2}(\mathbf{R}^n)} \to 0$. Relaxation property holds uniformly (w.r.t.) $x \in \mathbf{R}^n$ for $s+2 > n/2$, (cf. [18]) due to the Sobolev imbedding.

2.1.2. Case $p(\beta) = b_2\delta(\beta-\beta_2)$, $\beta_2 \in (1,2)$

The equation is

$$b_2 D_*^{\beta_2}u(x,t) = \triangle_x u(x,t), \quad x \in \mathbf{R}^n.$$

The Laplace and Fourier transformations, in succession, and their inverses give the solution

$$u(x,t) = E_{\beta_2}(1/b_2 t^{\beta_2}\triangle_x)u(x,0) + tE_{\beta_2,2}(1/b_2 t^{\beta_2}\triangle_x)u_t(x,0)).$$

Then, fot $t \ge T$,

$$||u(x,t)||_{H_2^{s+2}} \le \frac{1}{1+T^{\beta_2}}||u(x,0)||_{H^s} + \frac{T}{1+T^{\beta_2}}||u_t(x,0)||_{H^s},$$

and when $t \to 0^+$, then $||u(x,t)||_{H_2^{s+2}} \le C||u(x,0)||_{H^s}$. When $t \to \infty$, then, $||u(x,t)||_{H_2^{s+2}} \to 0$. The relaxation property $\lim_{t\to\infty}||u(x,t)||_{H_2^{s+2}} = 0$ holds uniformly (w.r.t.) $x \in \mathbf{R}^n$ when $s+2 > n/2$, i.e. $s > n/2-2$.

2.1.3. Case $p(\beta) = b_1\delta(\beta-\beta_1) + b_2\delta(\beta-\beta_2)$, $\beta_1 \in (0,1)$, $\beta_2 \in (1,2)$

We have the equation with two-fractional derivatives

$$b_1 D_*^{\beta_1}u(x,t) + b_2 D_*^{\beta_2}u(x,t) = \triangle_x u(x,t), \quad x \in \mathbf{R}^n, \tag{9}$$

subject to the initial data $u(x,0) = \varphi(x)$, $u_t(x,0) = \psi(x)$. The transforms of Laplace and Fourier in succession and their inverses give the solution which consists of the three parts:

$$u(x,t) = I_1(x,t) + I_2(x,t) + I_3(x,t), \tag{10}$$

where

$$I_1(x,t) = b_1/b_2 \sum_{k=0}^{\infty} \frac{1}{k! b_2^k} \triangle_x^{(k)} u(x,0) t^{\beta_2(k+1)-\beta_1}$$
$$E^{(k)}_{\beta_2-\beta_1,\beta_2+\beta_1(k-1)+1}(-b_1/b_2 t^{\beta_2-\beta_1}),$$

$$I_2(x,t) = \sum_{k=0}^{\infty} \frac{1}{k! b_2^k} \triangle_x^{(k)} u(x,0) t^{\beta_2 k} E^{(k)}_{\beta_2-\beta_1,\beta_1 k+1}(-b_1/b_2 t^{\beta_2-\beta_1}).$$

$$I_3(x,t) = \sum_{k=0}^{\infty} \frac{1}{k! b_2^k} \triangle_x^{(k)} u_t(x,0) t^{\beta_2 k+1} E^{(k)}_{\beta_2-\beta_1,\beta_1 k+2}(-b_1/b_2 t^{\beta_2-\beta_1}).$$

According to [34], p.21, formula (1.82), the following structural formula holds:

$$I_1(x,t) = \sum_{k=0}^{\infty} \frac{1}{k! b_2^k} \triangle_x^{(k)} u(x,0) D^{-(\beta_2-\beta_1)} S(t),$$

$$I_2(x,t) = \sum_{k=0}^{\infty} \frac{1}{k! b_2^k} \triangle_x^{(k)} u(x,0) S(t),$$

$$I_3(x,t) = \sum_{k=0}^{\infty} \frac{1}{k! b_2^k} \triangle_x^{(k)} u_t(x,0) D^{-1}(S(t)),$$

where

$$S(t) = t^{\beta_2 k} E^{(k)}_{\beta_2-\beta_1,\beta_1 k+1}(-\frac{b_1}{b_2} t^{\beta_2-\beta_1}), \tag{11}$$

and D is the Riemann-Louville fractional derivative. Then it follows

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k! b_2^k} [(D^{-(\beta_2-\beta_1)}(S(t)) + S(t)) u(x,0) + D^{-1}(t) u_t(x,0)],$$

where $S(t)$ is given with (11). In particular, for $u_t(x,0) = 0$, we have

$$u(x,t) = I_1(x,t) + I_2(x,t) = \sum_{k=0}^{\infty} \frac{1}{k! b_2^k} \triangle_x^{(k)} u(x,0)(D^{-(\beta_2-\beta_1)} S(t) + S(t)).$$

We can calculate the solution (10) such that the derivative is in terms of the Mittag-Leffler function:

$$J_1(x,t) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} (\frac{b_1}{b_2})^{i+1} t^{(\beta_2-\beta_1)(i+1)} E^{(i)}_{\beta_2,\beta_2-\beta_1(i+1)+1}(1/b_2 t^{\beta_2} \triangle_x) u(x,0).$$

$$J_2(x,t) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} (\frac{b_2}{b_1})^{i+1} t^{(\beta_1-\beta_2)(i+1)} E^{(i)}_{\beta_1,\beta_1-\beta_2(i+1)+1}(1/b_1 t^{\beta_1} \triangle_x) u(x,0).$$

$$J_3(x,t) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} (\frac{b_2}{b_1})^{i+1} t^{(\beta_1-\beta_2)(i+1)+1} E^{(i)}_{\beta_1,\beta_1-\beta_2(i+1)+2}(1/b_1 t^{\beta_1} \triangle_x) u_t(x,0).$$

The first type of formulas are calculated for $t > 1$ and the second ones for $t < 1$. Then, we can calculate the explicit solution

$$u(x,t) = K_1(x,t) + K_2(x,t) + K_3(x,t), \tag{12}$$

where

$$K_1(x,t) = I_1(x,t)H(t-1) + J_1(x,t)H(1-t),$$

$$K_2(x,t) = I_2(x,t)H(t-1) + J_2(x,t)H(1-t)$$

$$K_3(x,t) = I_3(x,t)H(t-1) + J_3(x,t)H(1-t),$$

$H(\cdot)$ is the Heaviside function.

Concerning the asymptotic behavior, the relaxation property holds when $t \to \infty$. Namely, $\lim_{t\to\infty} ||u(x,t)||_{H_2^{s+2}} = 0$ uniformly for $T \geq t$, and $s > n/2 - 2$. When $t \to 0$, then $||u(x,t)||_{H_2^{s+2}} \leq C||u(x,0)||_{H_2^s}$. This holds from the asymptotic behavior of the functions $J_i(x,t)$, $i = 1,2,3$.

The asymptotic behavior for the equations with one and two-term Green function in sup-norm, such as for the uniformly distributed fractional relaxation (when weight function is $p(\beta) = 1$) is given in [27] for $\beta \in (0,1)$. Tauberian theorems are employed.

THEOREM 2. *Let* $u(x,0) \in H_2^s(\mathbf{R}^n)$, $s \in \mathbf{R}$, *and* $0 < \beta_1 < 1 < \beta_2 < 2$, *in the diffusion-wave problem (9). Then, there exists a unique solution in the space* $C^\infty(t \in (0,\infty); H^{s+2}(\mathbf{R}^n)) \cap C^0(t \in [0,\infty), H^s(\mathbf{R}^n))$, $s \in \mathbf{R}$. *The exact solution is given by (10) resp. (12).*

P r o o f. Denoting the maximal term $I_1(x,t)$ (w.r.t.) $t \in [0,T]$ in (10), by $C(T)$, and having in mind that

$$E^{(i)}_{\alpha,\beta}(y) = \sum_{j=0}^{\infty} \frac{(j+i)!y^j}{j!\Gamma(\alpha j + \alpha i + \beta)}, i = 0,1,2,..., \quad (cf.[34]),$$

we obtain that asymptotic expansion (cf. [34])

$$|J_1(x,t)| \leq \frac{C(T)}{[1+|1/b_2 A(D)|]} u(x,0) \tag{13}$$

holds for $J_1(x,t)$ (resp. $I_1(x,t)$). For the operator $A(D) = \triangle_x$ we obtain the following estimate for $J_1(x,t)$, $J_2(x,t)$:

$$||u(x,t)||_{H_2^{s+2}} \leq C(T)(\int_{\mathbf{R}^n} e^{ix\kappa}||(1+|\kappa|^2)|^{-2}|^2|(1+|\kappa|^2)^{s+2}|^2|\hat{u}(\kappa,0)|^2 d\kappa)^{1/2}$$

$$\leq C(T)(\int_{\mathbf{R}^n} e^{ix\kappa}|(1+|\kappa|^2)^s|^2|\hat{u}(\kappa,0)|^2 d\kappa)^{1/2} \leq C(T)||u(x,0)||_{H_2^s}.$$

For $J_3(x,t)$ we employ as the initial data $u_t(x,0)$.

Thus, for the operator $A(D) = \triangle_x$ there exists continuous closure $A : H_2^{s+2}(\mathbf{R}^n) \to H_2^s(\mathbf{R}^s)$, (cf. [18]) in the two-term diffusion-wave problem. ■

2.1.4. Case $p(\beta) = b_1\delta(\beta-\beta_1) + b_2\delta(\beta-\beta_2) + b_3\delta(\beta-\beta_3)$, $\beta_1,\beta_2 \in (0,1)$, $\beta_3 \in (1,2)$

Repeating the same procedure (cf. [41]), we obtain the explicit solution by summing the four parts. The first part of $I_1(x,t)$ reads:

$$I_1(x,t) = \sum_{m=0}^{\infty}(-1)^m(\frac{b_1}{b_3})^{m+1}\sum_{k=0}^{m}\binom{m}{k}\frac{\triangle_x^k u(x,0)}{(-1)^k b_1^k} t^{(\beta_3-\beta_2)(k+1)+k\beta_1+(\beta_2-\beta_1)(m+1)}$$

$$\times E^{(k)}_{\beta_3-\beta_1,(\beta_3-\beta_1)+k\beta_1+(\beta_2-\beta_1)(m+1)+1}(-b_2/b_3 t^{\beta_3-\beta_1}).$$

Analogously, we obtain the other parts of the solution and their estimates.

2.1.5. Case $p(\beta) = \sum_{i=1}^m b_i\delta(\beta-\beta_i) + \sum_{i=m+1}^n b_i\delta(\beta-\beta_i)$

In this way, we have m jumps in interval $(0,1)$ and $(n-m)$ jumps in interval $(1,2)$, in the function $p(\beta)$. Denote the jump points from the interval $(0,1)$ by $\sum_{i=1}^m b_i\beta_i$, and the points from the interval $(1,2)$ by $\sum_{i=m+1}^n b_i\beta_i \in (1,2)$. We have to solve the equation

$$\sum_{i=1}^{m} b_i D_*^{\beta_i} u(x,t) + \sum_{i=m+1}^{n} b_i D_*^{\beta_i} u(x,t) = \triangle_x u(x,t),$$

or in general,

$$D_*^{\beta_n}u(x,t)+\sum_{i=1}^{n-1}b_iD_*^{\beta_i}u(x,t)+b_1D^{\beta_1}u(x,t)=\triangle_x u(x,t),\ \ t\in(0,T),\ \ x\in\mathbf{R}, \tag{14}$$

subject to the initial data $u(x,0)=\varphi(x)$, $u_t(x,0)=\psi(x)$.

We can calculate the explicit solution using the approach for the n-term equation from [34]. For details, see [41].

The following theorem gives the regularity properties for the solution to the Cauchy problem (3) with (4) when the weight function is the finite linear combination of the Dirac measures massed at the points of the interval $(0,2)$, for the following distribution of points: $0<\beta_1<...<\beta_m<1<\beta_{m+1}<...<\beta_n<2$, $m=1,2,...,n$.

THEOREM 3. *Suppose* $0<\beta_1<...<\beta_m<1<\beta_{m+1}<...<\beta_n<2$, $T>0$ *be fixed and* $u(x,0)\in H_2^s(\mathbf{R}^n)$. *Then, there exists a positive constant* C_β *such that the solution to the problem (14) has the following regularity properties in the corresponding Sobolev spaces:*

$$u\in C^\infty(t\in(0,\infty);H_2^{s+2}(\mathbf{R}^n))\cap C^0(t\in[0,\infty),H_2^s(\mathbf{R}^n)),\ \ s\in\mathbf{R}.$$

P r o o f. It follows from the representation of the solution in a form of the derivatives of the Mittag-Leffler function and the choice of operator $A(D)=\triangle_x$, and $t\geq T$. We have as a generalization of the two-term time fractional diffusion-wave problem the finite sum of the estimates whose each part is estimated by

$$||u(x,t)||_{H_2^{s+2}(\mathbf{R}^n)}\leq C(\int_{\mathbf{R}^n}|(1+|\kappa|^2)^{-2}|^2(1+|\kappa|^2)^{2(s+2)}|\hat{u}(0,x)|^2e^{ix\kappa}d\kappa)^{1/2}$$

$$\leq C||u(x,0)||_{H_2^s(\mathbf{R}^n)}.$$

Thus, Theorem 3 holds for the equation (14). ∎

REMARK 1. For general operator $A(D)$ with symbol $A(\kappa)$ which satisfies the estimate

$$|A(\kappa)|\leq C(1+|\kappa|^2)^{l/2},\ \ C>0,\ \ l\in\mathbf{R},$$

we have

$$|u(x,t)||_{H_2^{s+l}(\mathbf{R}^n)} \leq C(\int_{\mathbf{R}} |(1+|\kappa|^2)^{-l/2}|^2|(1+|\kappa|^2)^{s+l/2}|^2|\hat{u}(0,x)|^2 e^{ix\kappa} d\kappa)^{1/2}$$
$$\leq C||u(x,0)||_{H_2^s(\mathbf{R}^n)},$$

since the corresponding operator has the following asymptotic behavior (w.r.t.) the spatial variable: $\dfrac{1}{(1+|\kappa|^2)^{l/2}}$, $l \in \mathbf{R}$, (cf. (13)). The same holds for the second fundamental solution which is in convolution with the second initial data in (5). Thus, for the operator $A(D)$ corresponding to $A(\kappa)$ there exists a continuous closure $A : H_2^{s+l}(\mathbf{R}^n) \to H_2^s(\mathbf{R}^n)$, $s, l \in \mathbf{R}$. For more details, see [18].

Remark 2. We can employ in the same way the Sobolev spaces $H_p^s(\mathbf{R}^n)$, Besov spaces $B_{pq}^s(\mathbf{R}^n)$ and Triebel-Lizorkin spaces $F_{pq}^s(\mathbf{R}^n)$, see [18].

3. Discussion

The representation formulas obtained in a form of derivatives of the Mittag-Leffler function which are the solutions of the diffusion-wave problems are useful from both, the mathematical and physical point of view. From them, we obtain the properties of the solutions. We see that it is not important how many different fractional orders are taken in the given equation between two consecutive integers, the regularity of the solution is always the same as in the two-term equation. When $0 < \beta_1 < ... < \beta_m < 1 < \beta_{m+1} < ...\beta_n < 2$, the solution is completely continuous (w.r.t.) the temporal variable t on the corresponding time interval. The same holds for the other cases. Concerning the behavior (w.r.t.) the spatial variable, the relaxation property holds when $t \to \infty$. When $t \to 0$, we have boundedness by the initial data in the corresponding norm.

The fractional diffusion-wave operator is a mixture of diffusion and wave operator but the diffusion operator have stronger smoothing properties when the distribution of the orders is $0 < \beta_1 < ... < \beta_m < 1 < \beta_{m+1} < ... < \beta_n < 2$, $T > 0$. Thus, the influence of the heat operator is greater than the influence of the wave operator. Due to that, the solution of the problem (3)-(4) in interval $\beta \in (0, 2)$, in this case inherits the regularity properties of the heat operator when $\beta = 1$, and the solution operator smoothes the singularities of the initial data. The diffusive effect remains for the derivatives orders till the point two. Then, it vanishes.

Thus, in this case the influence of the diffusion operator depends on how close to it we are and the regularity properties hold. The situation is completely different for the following distribution of the orders $0 < \beta_1 < ... < \beta_{m-1} < \beta_m < ... < \beta_n < 1$, and $1 < \beta_1 < ... < \beta_{m-1} < \beta_m < ... < \beta_n < 2$. For more details, see [41].

At the end, we give a remark on the new approach to the initial data in fractional differential equations, different than the presented one, due to the recent growing interest in that subject. Following the convenient approach given in [34], [17], [22] and many others, the RL derivative is associated with the fractional order initial data while the C derivative is suitable for the initial value problems in physics and engineering when the initial conditions are integer-order derivatives. Then, the problems are easier to solve. It was shown in recent papers [32], [33], [1], [2], [20], [21] that several definitions of fractional differentiation, including the C one, do not permit to take into account initial conditions in a convenient way and are not physically consistent in many cases. For the initial condition it is necessary to consider not only a point $t = 0^+$, but also a point $t = 0^-$, (cf. [25]), since the initial data contain the past of a system without relation to the future.

The classical approach is revised in [33] using the following aspects: the signals cover whole set of real numbers; for $t > t_0$, the Heaviside step function arises, $H(t - t_0)$; initial data depends on the past, while do not depend on the future; the system behavior is independent on the use of fractional derivatives.

The RL and C derivatives are considered in [32] in the general framework of the linear ordered system using the jump formula. The RL and C derivatives are marked as pseudo differential conditions, i.e. they do not correspond to true initial conditions of the system. They appear naturally and independently of the way of derivatives which are computed. The classes of the differential equations which can be solved by RL and C derivatives are presented, but they belong to the very restrictive set.

In [21], [20], it is distinguished the importance of the proper initialization of a system, evolving in time according to a differential equation of fractional order. There, the effect of the past is incorporated by an initialization function for the RL and the Grunwald fractional calculus. The same was done in the works [2], [1] concerning the Caputo fractional derivative.

In [35], it is proved that the RL and C fractional derivatives can not use the usual initial data in a convenient way from a physical point of view. They suggest a representation for fractional order system which give a physically compatible initialization consisting of classical linear integer system and parabolic equation system. Then, a fractional order system is between these two systems and in particular corresponds for modeling of diffusion. In this representation the time memory effect takes into account the system past on an infinite time interval. Thus, a physical meaning and appropriate initialization of the fractional derivatives have not been solved yet in a proper way.

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