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THE ECCENTRIC CONNECTIVITY POLYNOMIAL OF SOME GRAPH OPERATIONS

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Abstract. The eccentric connectivity index of a graph G, ξ^C, was proposed by Sharma, Goswami and Madan. It is defined as $\xi^C(G) = \sum_{u \in V(G)} deg_G(u)\varepsilon_G(u)$, where $deg_G(u)$ denotes the degree of the vertex x in G and $\varepsilon_G(u) = Max\{d(u,x) \mid x \in V(G)\}$. The eccentric connectivity polynomial is a polynomial version of this topological index. In this paper, exact formulas for the eccentric connectivity polynomial of Cartesian product, symmetric difference, disjunction and join of graphs are presented.

1. Introduction. Throughout this paper all graphs are assumed to be simple, finite and connected. A function Top from the class of connected graphs into real numbers with the property that $Top(G) = Top(H)$ whenever G and H are isomorphic is known as a **topological index** in the chemical literature; see [11]. There are many examples of such functions, especially those based on distances, which are applicable in chemistry. The **Wiener index** [19], defined as

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the sum of all distances between pairs of vertices in a graph, is probably the first and most studied such graph invariant, both from a theoretical and a practical point of view. If we define the distance $d(x, y)$ between vertices x and y of a graph G as the length of a minimum path connecting them then it is possible to redefine the Wiener index as $W(G) = \sum_{\{x,y\}\subseteq V(G)} d(x, y)$. The topological indices which are definable by distance function $d(-,-)$ are called a **distance-based topological index**.

The eccentric connectivity index of a graph G, ξ^C, is a new distance-based topological index which was proposed by Sharma, Goswami and Madan [13]. We encourage to interested readers to consult papers $[1-4, 6-8, 14, 16, 17]$ for chemical meaning and [12, 21] for mathematical properties of this new topological index.

A graph can be characterized by a number, by a matrix or by a polynomial. The characterization of graphs by a single topological index is usually impossible. For example, it is possible to find infinite pairs of graphs with the same Wiener index. It is an open question to find a topological index characterizing graphs. On the other hand, it is possible to characterize graphs by matrices. A well-known example of such matrices is an adjacency matrix. But the characterization of graphs by polynomials is a new branch of research in modern graph theory. This paper is an attempt in this line. We refer the interested readers to [5, 15, 20] for more information on this topic.

Throughout this paper our notation is standard and taken mainly from [13, 18].

2. Definitions. In this section we present the main concepts of the paper. We begin by **eccentric connectivity index**. It is defined as $\xi^C(G) = \sum_{u\in V(G)} deg_G(u)\varepsilon_G(u)$, where $deg_G(u)$ denotes the degree of the vertex u in G and $\varepsilon_G(u)$ is the largest distance between u and any other vertex v of G. The quantity $\varepsilon_G(u)$ is usually named the **eccentricity** of vertex u in G. The minimum and maximum of eccentricity among vertices of G are called the **radius** and **diameter** of G, respectively.

The **eccentric connectivity** and **total eccentricity polynomials** of G are defined as $\Xi(G, x) = \sum_{u\in V(G)} deg_G(u)x^{\varepsilon_G(u)}$ and $\Theta(G, x) = \sum_{u\in V(G)} x^{\varepsilon_G(u)}$, respectively. It is easy to see that the eccentric connectivity index and the total eccentricity of a graph can be obtained from the corresponding polynomials by evaluating their first derivatives at $x = 1$.

A vertex $u \in V(G)$ is called **well-connected** if $\varepsilon_G(u) = 1$, i.e., if it is adjacent to all other vertices in G. We denote the number of well-connected vertices in G by $w(G)$. Define $N_G(u) = \{v \in V(G) : uv \in E(G)\}$, where u is a fixed vertex of G. Then the **modified eccentric connectivity polynomial** of G is defined as $\xi_C(G, x) = \sum_{u \in V(G)} \delta(u) x^{\varepsilon_G(u)}$, where $\delta(v) = \sum_{u \in N_G(v)} deg(u)$. Then the **modified eccentric connectivity index** $\xi_C(G) = \sum_{u \in V(G)} \delta(u)\varepsilon_G(u)$ is the first derivative of $\xi_C(G, x)$ evaluated at $x = 1$.

3. Main results.

In this section we first compute the eccentric and modified eccentric connectivity polynomials of some well known graphs that will serve as basic building blocks in the considered composite graphs. We begin by introducing the **first Zagreb index**. The first Zagreb index was introduced more than forty years ago by Gutman and Trinajestić [9]. We recommend [10] for the success history of this topological index. It is defined as $M_1(G) = \sum_{u \in V(G)} deg(u)^2$.

Example 1. Let K_n be the complete graph on n vertices. Then for every $v \in V(K_n)$, $\delta(v) = (n-1)^2$ and $\varepsilon(v) = 1$. This implies that $\xi_C(K_n; x) = n(n-1)^2 x$, $\Xi(K_n; x) = n(n-1)x$ and $\Theta(K_n; x) = nx$.

Example 2. Let C_n denote the cycle of length n. Then for each vertex v of G, $\delta(v) = 4$ and $\varepsilon(v) = \lfloor \frac{n}{2} \rfloor$. Hence, $\xi_C(C_n; x) = 4nx^{\lfloor \frac{n}{2} \rfloor}$, $\Xi(C_n; x) = 2nx^{\lfloor \frac{n}{2} \rfloor}$ and $\Theta(C_n; x) = nx^{\lfloor \frac{n}{2} \rfloor}$.

Example 3. Let $S_n = K_{1,n}$ be the star graph with $n+1$ vertices. Then the central vertex v has degree n, $\delta(v) = n$ and eccentricity 1, while the remaining n vertices have degree 1 and eccentricity 2 and for all vertices $\delta(v) = n$. Hence $\xi_C(S_n; x) = n^2x^2 + nx$, $\Xi(S_n; x) = nx(x+1)$ and $\Theta(S_n; x) = x(x+n)$.

Example 4. A wheel W_n is a graph of order $n+1$ which contains a cycle on n vertices and a central vertex connected to each vertex of the cycle. Again, the central vertex v has degree n, $\delta(v) = 3n$ and eccentricity 1, while the peripheral vertices u have degree 3, $\delta(v) = n + 6$ and eccentricity 2. So, $\xi_C(W_n; x) = (n^2+6n)x^2 + 3nx$, $\Xi(S_n; x) = nx(x+1)$ and its eccentric connectivity polynomial is given by $\Xi(W_n; x) = nx(3x+1)$, while the total eccentricity polynomial is the same as for S_n.

Example 5. Consider a complete n-partite graph $M_{m_1m_2\cdots m_n}$ containing $v = |V(G)|$ vertices. By definition, the vertex set $V(G)$ of this graph can be partitioned into subsets $V_1, V_2, \cdots, V_n$ of V such that for every $1 \le i \le n$ there is no edge between the vertices of V_i. If none of V_i is a single vertex, then by direct calculation one can see that:

$$\Xi(M_{m_1m_2\cdots m_n}; x) = 2\left(\sum_{i\neq j} m_i m_j\right) x^2$$

and

$$\xi_C(M_{m_1m_2\cdots m_n}; x) = \left(\sum_{i=1}^{n} m_i \sum_{k=1,k\neq i}^{n} m_k |V(G) - m_k|\right) x^2.$$

If some of the classes are singletons, the above expression must be modified by adding an appropriate linear term. This illustrates the problems arising when a graph contains well-connected vertices.

Example 6. Let P_n be the path on n vertices ($n \ge 4$). Then

$$\begin{aligned}
\xi(P_n; x) &= 4x^{n-1} + 6x^{n-2} + 8x^{\lceil n/2\rceil} \frac{1 - x^{\lfloor n/2\rfloor - 1}}{1 - x} + 4x^{\lfloor n/2\rfloor} [n \quad \text{odd}],\\
\Xi(P_n; x) &= 2x^{n-1} + 4x^{\lceil n/2\rceil} \frac{1 - x^{\lfloor n/2\rfloor - 1}}{1 - x} + 2x^{\lfloor n/2\rfloor} [n \quad \text{odd}],\\
\Theta(P_n; x) &= 2x^{\lceil n/2\rceil} \frac{1 - x^{\lfloor n/2\rfloor}}{1 - x} + x^{\lfloor n/2\rfloor} [n \quad \text{odd}].
\end{aligned}$$

Here the square brackets on the right-hand side evaluate to 1 if the enclosed logical expression is true, and to 0 otherwise.

Lemma 1. *Let G be a graph then $M_1(G) = \sum_{u\in V(G)} \delta(u)$.*

Proof. By using the definition one can see that:

$$M_1(G) = \sum_{u\in V(G)} deg(u)^2 = \sum_{u\in V(G)} \sum_{v\in N_G(u)} deg(v) = \sum_{u\in V(G)} \delta(u). \qquad \square$$

3.1. Cartesian product. For given graphs G_1 and G_2 their **Cartesian product** $G_1 \Box G_2$ is defined as the graph on the vertex set $V(G_1) \times V(G_2)$, and vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $V(G_1) \times V(G_2)$ are connected by an edge if and only if either ($[u_1 = v_1$ and $u_2v_2 \in E(G_2)]$) or ($[u_2 = v_2$ and $u_1v_1 \in E(G_1)]$). It is a well-known fact that the Cartesian product of graphs is commutative and associative up to isomorphism, see [13] for details. To simplify our argument we write below $deg(u, v)$ as $deg((u, v))$ and $\delta(u, v)$ as $\delta((u, v))$.

By a classical result in metric graph theory [13], the distance between two vertices in $G_1 \Box G_2$ is given by $d_{G_1\Box G_2}(u, v) = d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2)$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$. The degree of a vertex (u_1, u_2) of $G_1 \Box G_2$ is the sum of the degrees of its projections to the respective components, $\delta_{G_1\Box G_2}(u_1, u_2) = \delta_{G_1}(u_1) + \delta_{G_2}(u_2)$. The eccentricity of a vertex (u_1, u_2) of $G_1 \Box G_2$ is obtained in the same way.

Lemma 2. *If $u_i \in G_i, i = 1, 2,$ then*

$$\delta_{G_1\Box G_2}(u_1, u_2) = \delta_{G_1}(u_1) + \delta_{G_2}(u_2) + 2deg_{G_1}(u_1)deg_{G_2}(u_2).$$

P r o o f. Applying an argument similar to Lemma 1, we have:

$$\begin{aligned} \delta_{G_1\Box G_2}(u_1, u_2) &= \sum_{a\in N_{G_1}(u_1)} deg_{G_1\Box G_2}(a, u_2) + \sum_{b\in N_{G_2}(u_2)} deg_{G_1\Box G_2}(u_1, b) \\ &= \sum_{a\in N_{G_1}(u_1)} (deg_{G_1}(a) + deg_{G_2}(u_2)) \\ &\quad + \sum_{b\in N_{G_2}(u_2)} (deg_{G_1}(u_1) + deg_{G_2}(b)) \\ &= \delta_{G_1}(u_1) + \delta_{G_2}(u_2) + 2deg_{G_1}(u_1)deg_{G_2}(u_2). \qquad \Box \end{aligned}$$

The Cartesian product of more than two graphs is defined inductively, $G_1 \Box \dots \Box G_s = (G_1 \Box \dots \Box G_{s-1}) \Box G_s$. We denote $G_1 \Box G_2 \Box \dots \Box G_s$ by $\prod_{i=1}^{s} G_i$. If $G_1 = G_2 = \cdots = G_s = G$, we have the s-th Cartesian power of G and denote it by G^s.

Corollary 3. *If $u_i \in G_i, 1 \le i \le k,$ then*

$$\delta_{\prod_{i=1}^{k} G_i}(u_1, u_2, \cdots, u_k) = \sum_{i=1}^{k} \delta_{G_i}(u_i) + \sum_{i=1}^{k} deg_{G_i}(v_i) \sum_{j=1, j\neq i}^{k} deg_{G_j}(v_j).$$

P r o o f. Induct on k. $\Box$

Lemma 4. *Let* $G = \prod_{i=1}^{s} G_i$ *and* $u_i \in G_i, 1 \leq i \leq s$. *Then*

$$\varepsilon_G((u_1, u_2, \cdots, u_s)) = \sum_{i=1}^{s} \varepsilon_{G_i}(u_i).$$

In the following theorem the relationship between the eccentric and modified eccentric polynomials are investigated.

Theorem 5. *Suppose* $G_i, 1 \leq i \leq k$, *are a connected graph. Then*

$$\xi_C(\prod_{i=1}^{s} G_i; x) = \sum_{i=1}^{k} (\xi_C(G_i, x)) \prod_{j=1, j\neq i}^{k} \Theta(G_j; x) + 2 \sum_{1\leq i<j\leq k}^{s} \Xi(G_i; x)\Xi(G_j; x) \prod_{r=1, r\neq i,j}^{k} \Theta(G_j; x)$$

P r o o f. Let $G = \prod_{i=1}^{k} G_i$. By definition, we have:

$$\begin{aligned} \xi_C(G; x) &= \sum_{(u_1,u_2,\cdots,u_k)\in G} \delta_G(u_1, u_2, \cdots, u_k) x^{\varepsilon_G(u_1,u_2,\cdots,u_k)} \\ &= \sum_{(u_1,u_2,\cdots,u_k)\in G} \left(\sum_{i=1}^{k} \delta_{G_i}(u_i) + \sum_{i=1}^{k} deg_{G_i}(v_i) \sum_{j=1, j\neq i}^{k} deg_{G_j}(v_j) \right) x^{\sum_{i=1}^{k} \varepsilon_{G_i}(u_i)} \\ &= \sum_{i=1}^{k} (\xi_C(G_i, x)) \prod_{j=1, j\neq i}^{k} \Theta(G_j; x) + 2 \sum_{1\leq i<j\leq k}^{s} \Xi(G_i; x)\Xi(G_j; x) \prod_{r=1, r\neq i,j}^{k} \Theta(G_j; x). \end{aligned}$$

□

Corollary 6. *Let* G *and* H *be connected graphs, then*

$$\xi_C(G\Box H; x) = \xi_C(G; x)\Theta(H; x) + \xi_C(H; x)\Theta(G; x) + 2\Xi(G; x)\Xi(H; x).$$

Corollary 7.

$$\xi_C(G^k;x) = k\Theta(G;x)^{k-2}\left(\xi_C(G;x)\Theta(G;x) + (k-1)\Xi(G;x)^2\right).$$

Example 7. A **Hamming graph** $H_{n_1,\cdots,n_s}$ is defined as $H_{n_1,\cdots,n_s} = \prod_{i=1}^{s} K_{n_i}$, where $n_1,\cdots,n_s$ are positive integers. It can be easily seen that $\Xi(K_{n_i};x) = n_i(n_i-1)x$, $\xi_C(K_{n_i};x) = n_i(n_i-1)^2x$ and $\Theta(K_{n_i};x) = n_ix$. Then,

$$\xi_C(H_{n_1,\cdots,n_k};x) = (\prod_{i=1}^{k} n_i \left(2\sum_{1\le i<j\le k}(n_i-1)(n_j-1) + \sum_{i=1}^{k}(n_i-1)\right) x^k.$$

For $n_1 = n_2 = \cdots = n_s = 2$ we obtain the s-dimensional hypercube Q_s; its eccentric connectivity polynomial is given by $\xi_C(Q_s;x) = s^2 2^s x^s$.

Example 8. A cylinder covered by squares is called a C_4**-nanotube** and a torus covered by squares is called a C_4**-nanotorus.** The names are taken from physics literature, because these are the molecular graphs of nanotubes and nanotori, respectively. Here, a molecular graph is a graph in which atoms are vertices and bonds determine the set of edges of the graph. By a well-known fact in chemistry, the maximum degree in such a graph is four which is related to carbon atoms. So, a molecular graph is a graph in which the degree of each vertex is at most four.

The C_4-nanotubes and nanotori arise as Cartesian products of paths and cycles and of two cycles, respectively. By using the results of Examples 2 and 6 and combining them with Theorem 5 we obtain the following explicit formulas for C_4-nanotubes and C_4-nanotori. We denote $R = P_n\square C_m$ and $S = C_k\square C_m$ and assume $k,m \ge 3$ and $n \ge 4$. Then,

$$\xi_C(R,x) = \begin{cases} 2m\left(2(3x^{\lceil m/2\rceil} + 2x^{\lceil n/2\rceil})\right)x^{n-1} + \left(7x^{\lceil m/2\rceil} + 2x^{\lceil n/2\rceil}\right)x^{n-2} & 2\mid n \\ +2\left(4x^{\lceil m/2\rceil} + x^{\lceil n/2\rceil}\right)\sum\limits_{i=\lfloor n/2\rfloor}^{n-3} x^i & \\ & \\ 2m\left(5x^{\lceil m/2\rceil} + 4x^{\lceil n/2\rceil}\right)x^{n-1} + \left(7x^{\lceil m/2\rceil} + 8x^{\lceil n/2\rceil}\right)x^{n-2} & 2\nmid n \\ +8\left(x^{\lceil m/2\rceil} + x^{\lceil n/2\rceil}\right)\left(\sum\limits_{i=\lfloor n/2\rfloor}^{n-3} x^i + \frac{1}{2}x^{\lceil n/2\rceil}\right) & \end{cases}$$

and $\xi_C(S,x) = 16kmx^{\lfloor m/2\rfloor+\lfloor k/2\rfloor}$.

3.2. Symmetric difference and disjunction. The **symmetric difference** $G_1 \oplus G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ in which (u_1, u_2) is adjacent with (v_1, v_2) whenever u_1 is adjacent with v_1 in G_1 or u_2 is adjacent with v_2 in G_2, but not both. It follows from the definition that the degree of a vertex (u_1, u_2) of $G_1 \oplus G_2$ is given by

$$deg_{G_1 \oplus G_2}((u_1, u_2))$$
$$= |V(G_2)|deg_{G_1}(u_1) + |V(G_1)|deg_{G_2}(u_2) - 2deg_{G_1}(u_1)deg_{G_2}(u_2).$$

The **disjunction** $G_1 \vee G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \times V(G_2)$ in which (u_1, u_2) is adjacent with (v_1, v_2) whenever u_1 is adjacent with v_1 in G_1 or u_2 is adjacent with v_2 in G_2. Obviously, the degree of a vertex (u_1, u_2) of $G_1 \vee G_2$ is given by the following equations:

$$deg_{G_1 \vee G_2}((u_1, u_2)) = |V(G_2)|deg_{G_1}(u_1) + |V(G_1)|deg_{G_2}(u_2) - deg_{G_1}(u_1)deg_{G_2}(u_2).$$

The distance between any two vertices of a disjunction or a symmetric difference cannot exceed 2. If none of the components contains well-connected vertices, the eccentricity of all vertices is constant and equal to 2.

Lemma 8. *If $u_i \in G_i (i = 1, 2)$, then,*

$$\begin{aligned}
\delta_{G_1 \vee G_2}(u_1, u_2) &= \left(|V(G_2)|^2 - 2|E(G_2)|\right)\delta_{G_1} + \left(|V(G_1)|^2 - 2|E(G_1)|\right)\delta_{G_2} \\
&+ 2\left(|V(G_1)||E(G_2)|deg_{G_1}(u_1) + |V(G_2)||E(G_2)|deg_{G_2}(u_2)\right. \\
&- |V(G_2)|\delta_{G_1}(u_1)deg_{G_2}(u_2) - |V(G_1)|\delta_{G_2}(u_2)deg_{G_1}(u_1) \\
&+ \left.\delta_{G_1}(u_1)\delta_{G_2}(u_2)\right).
\end{aligned}$$

$$\begin{aligned}
\delta_{G_1 \oplus G_2}(u_1, u_2) &= \left(|V(G_2)|^2 - 2|E(G_2)|\right)\delta_{G_1} + \left(|V(G_1)|^2 - 2|E(G_1)|\right)\delta_{G_2} \\
&\quad +2\left(|V(G_1)||E(G_2)|deg_{G_1}(u_1) + |V(G_2)||E(G_2)|deg_{G_2}(u_2)\right. \\
&\quad -2|V(G_2)|\delta_{G_1}(u_1)deg_{G_2}(u_2) - 2|V(G_1)|\delta_{G_2}(u_2)deg_{G_1}(u_1) \\
&\quad + \left.2\delta_{G_1}(u_1)\delta_{G_2}(u_2)\right) + \left(|V(G_1)|^2 - 2|E(G_1)|\right)\delta_{G_2}.
\end{aligned}$$

Proof. By the formula given above, we have:

$$\begin{aligned}
\delta_{G_1\vee G_2}(u_1,u_2) &= \sum_{a\in N_{G_1}(u_1)}\sum_{v\in V(G_2)} deg_{G_1\vee G_2}((a,v)) \\
&\quad + \sum_{b\in N_{G_2}(u_2)}\sum_{u\in V(G_1)} deg_{G_1\vee G_2}((u,b)) \\
&\quad - \sum_{a\in N(G_1)(u_1)}\sum_{b\in N(G_2)(u_2)} deg_{G_1\vee G_2}((a,b)) \\
&= \sum_{a\in N_{G_1}(u_1)}\sum_{v\in V(G_2)} [|V(G_2)|deg_{G_1}(a) \\
&\qquad\qquad + |V(G_1)|deg_{G_2}(v) - deg_{G_1}(a)deg_{G_2}(v)] \\
&\quad + \sum_{u\in V(G_1)}\sum_{b\in N(G_2)(u_2)} [|V(G_2)|deg_{G_1}(u) + |V(G_1)|deg_{G_2}(b)] \\
&\quad - deg_{G_1}(u)deg_{G_2}(b) \\
&\quad - \sum_{a\in N_{G_1}(u_1)}\sum_{b\in N(G_2)(u_2)} |V(G_2)|deg_{G_1}(a) \\
&\quad + |V(G_1)|deg_{G_2}(b) - deg_{G_1}(a)deg_{G_2}(b) \\
&= \left(|V(G_2)|^2 - 2|E(G_2)|\right)\delta_{G_1} + \left(|V(G_1)|^2 - 2|E(G_1)|\right)\delta_{G_2} \\
&\quad + 2\left(|V(G_1)||E(G_2)|deg_{G_1}(u_1) + (|V(G_2)||E(G_1)|deg_{G_2}(u_2)\right) \\
&\quad - |V(G_2)|\delta_{G_1}(u_1)deg_{G_2}(u_2) - (|V(G_1)|\delta_{G_2}(u_2) \\
&\quad - \delta_{G_1}(u_1)\delta_{G_2}(u_2),
\end{aligned}$$

$$
\begin{aligned}
\delta_{G_1\oplus G_2}(u_1,u_2) &= \sum_{a\in N_{G_1}(u_1)}\sum_{v\in V(G_2)} deg_{G_1\vee G_2}((u,b)) \\
&\quad -2\sum_{a\in N_{G_1}(u_1)}\sum_{b\in N_{G_2}(u_2)} deg_{G_1\vee G_2}((a,b)) \\
&= \sum_{a\in N_{G_1}(u_1)}\sum_{v\in V(G_2)} [|V(G_2)|deg_{G_1}(a) \\
&\qquad\qquad +|V(G_1)|deg_{G_2}(v) - deg_{G_1}(a)deg_{G_2}(v)] \\
&\quad +\sum_{u\in V(G_1)}\sum_{b\in N_{G_2}(u_2)} [|V(G_2)|deg_{G_1}(u) + |V(G_1)|deg_{G_2}(b) \\
&\quad -deg_{G_1}(u)deg_{G_2}(b)] \\
&\quad -2\sum_{a\in N_{G_1}(u_1)}\sum_{b\in N_{G_2}(u_2)} [|V(G_2)|deg_{G_1}(a) \\
&\quad +|V(G_1)|deg_{G_2}(b) - deg_{G_1}(a)deg_{G_2}(b)] \\
&= \left(|V(G_2)|^2 - 2|E(G_2)|\right)\delta_{G_1} + \left(|V(G_2)|^2 - 2|E(G_1)|\right)\delta_{G_2} \\
&\quad +2\left(|V(G_1)|E(G_2)deg_{G_1}(u_1) + |V(G_2)||E(G_1)|deg_{G_2}(u_2)\right) \\
&\quad -2|V(G_2)|\delta_{G_1}(u_1)deg_{G_2}(u_2) - 2|V(G_1)|\delta_{G_2}(u_2)deg_{G_1}(u_1) \\
&\quad +2\delta_{G_1}(u_1)\delta_{G_2}(u_2). \qquad \square
\end{aligned}
$$

Lemma 9. *Let G_1 and G_2 be two graphs without well-connected vertices. Then*

$$\varepsilon_{G_1\vee G_2}((u,v)) = \varepsilon_{G_1\oplus G_2}((u,v)) = 2.$$

Proof. Let (u,v) be an arbitrary vertex of $V(G_1 \vee G_2)$. We know that its eccentricity cannot exceed two. It remains to show that it cannot be equal to one. But this would mean that the vertex (u,v) is well-connected in $V(G_1 \vee G_2)$, and this is impossible, since a vertex of $G_1 \vee G_2$ is well-connected if and only if both of its projections are well-connected in their respective components. Hence, $\varepsilon_{G_1\vee G_2}((u,v)) = 2$ for all vertices (u,v). The same argument can be applied for $G_1 \oplus G_2$. $\square$

Theorem 10. *Let G_1 and G_2 be two graphs without well-connected vertices. Then*

$$\begin{aligned} \xi_C(G_1 \vee G_2, x) &= \Big(\big(\big(|V(G_2)|^2 - 4|E(G_2)\big)\,|V(G_2)|\big)\, M_1(G_1) \\ &+ \big(\big(|V(G_1)|^2 - 4|E(G_1)\big)\,|V(G_1)|\big)\, M_1(G_2) + M_1(G_1)M_1(G_2) \\ &+ 8|V(G-1)||V(G_2)||E(G_1)||E(G_2)|\,\Big)x^2\,, \end{aligned}$$

$$\begin{aligned} \xi_C(G_1 \oplus G_2, x) &= \Big(\big(\big(|V(G_2)|^2 - 6|E(G_2)\big)\,|V(G_2)|\big)\, M_1(G_1) \\ &+ \big(\big(|V(G_1)|^2 - 6|E(G_1)\big)\,|V(G_1)|\big)\, M_1(G_2) + 2M_1(G_1)M_1(G_2) \\ &+ 8|V(G-1)||V(G_2)||E(G_1)||E(G_2)|\,\Big)x^2. \end{aligned}$$

P r o o f. Applying Lemma 9, we have:

$$\begin{aligned} \xi_C(G_1 \vee G_2, x) &= \sum_{(u_1,u_2)} \delta_{G_1\vee G_2}((u_1,u_2))x^{\varepsilon_{G_1\vee G_2}((u_1,u_2))} \\ &\quad + \sum_{(u_1,u_2)} (2(|V(G_1)||E(G_2)|deg_{G_1}(u_1) \\ &\qquad\qquad + |V(G_2)||E(G_1)|deg_{G_2}(u_2)))x^2 \\ &\quad + \sum_{(u_1,u_2)} (-|V(G_2)|\delta_{G_1}(u_1)deg_{G_2}(u_2) \\ &\qquad\qquad - |V(G_1)|\delta_{G_2}(u_2)deg_{G_1}(u_1) + \delta_{G_1}(u_1)\delta_{G_2}(u_2))x^2 \\ &= \Big(\big(\big(|V(G_2)|^2 - 6|E(G_2)\big)\,|V(G_2)|\big)\, M_1(G_1) \\ &\quad + \big(\big(|V(G_1)|^2 - 6|E(G_1)\big)\,|V(G_1)|\big)\, M_1(G_2) + 2M_1(G_1)M_1(G_2) \\ &\quad + 8|V(G-1)||V(G_2)||E(G_1)||E(G_2)|\,\Big)x^2. \end{aligned}$$

$$
\begin{aligned}
\xi_C(G_1 \oplus G_2, x) &= \sum_{(u_1,u_2)} \delta_{G_1 \oplus G_2}((u_1,u_2)) x^{\varepsilon_{G_1 \oplus G_2}((u_1,u_2))} \\
&= \sum_{(u_1,u_2)} \Big(\left(|V(G_2)|^2 - 2|E(G_2)| \right) \delta_{G_1}(u_1) \\
&\qquad\qquad + \left(|V(G_1)|^2 - 2|E(G_1)| \delta_{G_2}(u_2) \right) x^2 \\
&\quad + \sum_{(u_1,u_2)} (2\,(|V(G_1)||E(G_2)| deg_{G_1}(u_1) \\
&\qquad\qquad + |V(G_2)||E(G_1)| deg_{G_2}(u_2)))\, x^2 \\
&\quad + \sum_{(u_1,u_2)} (-2|V(G_2)| \delta_{G_1}(u_1) deg_{G_2}(u_2) \\
&\qquad\qquad + |V(G_1)| \delta_{G_2}(u_2) deg_{G_1}(u_1) + \delta_{G_1}(u_1)\delta_{G_2}(u_2))\, x^2 \\
&= \big[\left(\left(|V(G_2)|^2 - 6|E(G_2)\right) |V(G_2)| \right) M_1(G_1) \\
&\quad + \left(\left(|V(G_1)|^2 - 6|E(G_1)\right) |V(G_1)| \right) M_1(G_2) \\
&\quad + 2M_1(G_1)M_1(G_2) + 8|V(G-1)||V(G_2)||E(G_1)||E(G_2)| \big] x^2.
\end{aligned}
$$

□

3.3. Join. The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{xy \mid x \in V(G_1) \text{ \& } y \in V(G_2)\}$. The definition can be generalized to the case of $s \geq 3$ graphs in a straightforward manner.

The following result is a direct consequence of the definition of join.

Lemma 11. *If none of G_i, $i = 1,\ 2$, contains well-connected vertices, then for every $u \in V(G_1 + G_2)$ we have $\varepsilon_{G_1+G_2}(u) = 2$.*

Lemma 12. *Let G_i, $i = 1, 2$ be graphs. Then*

$$
\delta_{G_1+G_2} = \begin{cases} \delta_{G_1}(u) + deg_{G_1}(u)|V(G_2)| + |V(G_1)||V(G_2)| + 2|E(G_2)| & u \in V(G_1) \\ \delta_{G_2}(u) + deg_{G_2}(u)|V(G_1)| + |V(G_1)||V(G_2)| + 2|E(G_1)| & u \in V(G_2) \end{cases}.
$$

P r o o f. If $u \in V(G_1)$, then we have

$$\begin{aligned} \delta_{G_1+G_2}(u) &= \sum_{a \in N_{G_1}(u)} deg_{G_1+G_2}(a) + \sum_{b \in V_{G_2}} deg_{G_1+G_2}(b) \\ &= \sum_{a \in N_{G_1}(u)} (deg_{G_1}(a) + |V(G_2)|) + \sum_{b \in V_{G_2}} (deg_{G_2}(b) + |V(G_1)|) \\ &= \delta_{G_1}(u) + deg_{G_1}(u)|V(G_2)| + |V(G_1)||V(G_2)| + 2|E(G_2)|. \end{aligned}$$

Similary if $u \in V(G_2)$, then

$$\begin{aligned} \delta_{G_1+G_2}(u) &= \sum_{a \in N_{G_2}(u)} deg_{G_1+G_2}(a) + \sum_{b \in V_{G_1}} deg_{G_1+G_2}(b) \\ &= \sum_{a \in N_{G_2}(u)} (deg_{G_2}(a) + |V(G_1)|) + \sum_{b \in V_{G_1}} (deg_{G_1}(b) + |V(G_2)|) \\ &= \delta_{G_2}(u) + deg_{G_2}(u)|V(G_1)| + |V(G_1)||V(G_2)| + 2|E(G_1)|. \end{aligned}$$

□

Theorem 13. *Let G_1 and G_2 be two graphs without well-connected vertices. Then*

$$\begin{aligned} \xi_C(G_1 + G_2, x) &= (M_1(G_1) + M_1(G_2) + 4(|V(G_1)||E(G_2)| + |V(G_2)||E(G_1)|) \\ &+ |V(G_1)||V(G_2)|(V(G_1)| + |V(G_2)|)\, x^2. \end{aligned}$$

P r o o f. By Lemma 12, we have:

$$\begin{aligned} \xi_C(G_1 + G_2, x) &= \sum_{v \in V(G_1)} \delta_{G_1+G_2}(v) x^{\varepsilon_{G_1+G_2}(v)} + \sum_{v \in V(G_2)} \delta_{G_1+G_2}(v) x^{\varepsilon_{G_1+G_2}(v)} \\ &= \sum_{v \in V(G_1)} (\delta_{G_1}(v) + deg_{G_1}(v)|V(G_2)| \\ &\qquad + |V(G_1)||V(G_2)| + 2|E(G_2)|)\, x^2 \\ &\quad + \sum_{v \in V(G_2)} (\delta_{G_2}(v) + deg_{G_2}(v)|V(G_1)| \\ &\qquad + |V(G_1)||V(G_2)| + 2|E(G_1)|)\, x^2 \\ &= (M_1(G_1) + M_1(G_2) + 4(|V(G_1)||E(G_2)| + |V(G_2)||E(G_1)|) \\ &\quad + |V(G_1)||V(G_2)|(V(G_1)| + |V(G_2)|)\, x^2, \end{aligned}$$

which proves the theorem. □

4. Conclusions. In this paper the eccentric and modified eccentric connectivity polynomials of graphs are presented. Our definitions are similar to those are given in Sagan et al. [15]. The exact formulas for some graph operations are obtained. These polynomials are computed for well-known graphs to clarify our formulas. Finally, the polynomial presented by Sagan et al. for the Wiener index can be defined in a unique way, but it is possible to define even more than two different polynomials with the property that their derivative evaluated at $x = 1$ give the eccentric connectivity index.

For given graphs G_1 and G_2 their **tensor product** $G_1 \times G_2$ is defined as the graph on the vertex set $V(G_1) \times V(G_2)$, and vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ of $V(G_1) \times V(G_2)$ are connected by an edge if and only if either $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$. It remains an open question to find exact formulas for the eccentric and modified eccentric connectivity polynomials of graphs under tensor product operation.

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