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ON THE ARITHMETIC OF ERRORS*

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Abstract. An approximate number is an ordered pair consisting of a (real) number and an error bound, briefly *error*, which is a (real) non-negative number. To compute with approximate numbers the arithmetic operations on errors should be well-known. To model computations with errors one should suitably define and study arithmetic operations and order relations over the set of non-negative numbers. In this work we discuss the algebraic properties of non-negative numbers starting from familiar properties of real numbers. We focus on certain operations of errors which seem not to have been sufficiently studied algebraically. In this work we restrict ourselves to arithmetic operations for errors related to addition and multiplication by scalars. We pay special attention to subtractability-like properties of errors and the induced "distance-like" operation. This operation is implicitly used under different names in several contemporary fields of applied mathematics (inner subtraction and inner addition in interval analysis, generalized Hukuhara difference in fuzzy set theory, etc.) Here we present some new results related to algebraic properties of this operation.

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1. Introduction. An approximate number is an ordered pair consisting of a (real) number and an error bound. Error bounds, briefly *errors*, are (real) non-negative numbers. To compute with approximate numbers one should know the arithmetic operations on errors and the properties of these operations. Computations with errors require suitable definitions and study of the arithmetic operations and order relations over the set of non-negative numbers. In this work we discuss the algebraic properties of non-negative numbers starting from familiar properties of real numbers.

Denote $\mathbb{R}$ the set of real numbers. We start by recalling the familiar system $(\mathbb{R}, +, \leq)$ involving the arithmetic operation addition "+" and the order relation preceding "$\leq$".

Real numbers are usually presented by their sign and modulus, e.g.: $2, -2, \pi, -\pi, 3.14, -3.14$, etc. More generally, a real number $a \in \mathbb{R}$ is presented either as $a = +A$ or $a = -A$, where $A = |a| \geq 0$ is the modulus of a. Thus a real number $a \in \mathbb{R}$ is an ordered pair comprising the modulus of a denoted $A = |a| \in \mathbb{R}^+$ and the sign of a denoted $\alpha = \sigma(a)$ where

$$\sigma(a) = \{+, a \geq 0; -, a < 0\} \in \Lambda, \quad \Lambda = \{+, -\}.$$

Here $\mathbb{R}^+ = \{a \in \mathbb{R} \mid a \geq 0\}$ is the set of non-negative real numbers. We write $a = (A; \alpha) \in \mathbb{R}^+ \otimes \Lambda = \{(X; \xi) \mid X \in \mathbb{R}^+, \ \xi \in \Lambda\}$.

Note that the set of pairs $\mathbb{R}^+ \otimes \Lambda$ admits both elements $(0; +)$ and $(0; -)$, which both correspond to the single element $0 \in \mathbb{R}$. Assuming $-0 = +0$, that is $(0; +) = (0; -)$, we can write: $\mathbb{R} \equiv \mathbb{R}^+ \otimes \Lambda$ in the sense that there is a bijection between $\mathbb{R}$ and $\mathbb{R}^+ \otimes \Lambda$.

In the sequel we denote the elements of $\mathbb{R}$ by lower-case letters $a, b, c, \ldots$, and the elements of $\mathbb{R}^+$ by upper-case letters, $A, B, C, \ldots$.

Symbolic formulae for addition and order. We start with finding symbolic presentations for addition and order in terms of non-negative numbers. In other words we wish to describe the arithmetic operation addition "+" and the order relation "$\leq$" in $\mathbb{R}^+ \otimes \Lambda$, so that $(\mathbb{R}, +, \leq) \cong (\mathbb{R}^+ \otimes \Lambda, +, \leq)$.

Let $a = (A; \alpha), \ b = (B; \beta) \in \mathbb{R}$. In the case $\alpha = \beta$ we have

$$a + b = (A; \alpha) + (B; \beta) = (C; \gamma),$$

where $C = A + B$, $\gamma = \alpha = \beta$, that is

$$(A; \alpha) + (B; \alpha) = (A + B; \alpha).$$

Here "$A + B$" is the operation addition in $\mathbb{R}^+$.

To study the case $\alpha \neq \beta$ we need to make use of the following "distance-like" binary operation in $\mathbb{R}^+$:

$$A +^- B = \begin{cases} Y|_{B+Y=A} & \text{if } B \leq A; \\ X|_{A+X=B} & \text{if } A \leq B. \end{cases} \tag{1}$$

In words, $A +^- B$ is either the solution Y of $B + Y = A$ or is the solution X of $A + X = B$ depending on which one exists; note that if both solutions X, Y exist (which only happens when $A = B$), then they coincide: $X = Y = 0$.

Remark. In familiar terms operation (1) is written as $A +^- B = |A - B|$. However, strictly speaking we have no right to write $|A - B|$ in $\mathbb{R}^+$, as the operation subtraction $A - B$ (in real arithmetic) makes no sense in $\mathbb{R}^+$.

Denote by $\mu(a, b)$ the sign of those of the two real arguments (a or b) that has larger modulus. Symbolically, we define a mapping $\mu : \mathbb{R}^2 \longrightarrow \Lambda$ as follows:

$$\mu(a, b) = \mu((A; \alpha), (B; \beta)) = \begin{cases} \alpha & \text{if } B \leq A, \\ \beta & \text{if } B > A. \end{cases} \tag{2}$$

An equivalent expression for (2) is

$$\mu(a, b) = \begin{cases} \sigma(a) & \text{if } |b| \leq |a|, \\ \sigma(b) & \text{if } |b| > |a|. \end{cases}$$

As we know in the case $\alpha \neq \beta$ the sum $(C; \gamma) = (A; \alpha) + (B; \beta)$ is given by $C = A +^- B$, $\gamma = \mu(a, b)$. Summarizing, we have

$$a + b = (A; \alpha) + (B; \beta) = \begin{cases} (A + B; \alpha) & \text{if } \alpha = \beta; \\ (A +^- B; \mu(a, b)) & \text{if } \alpha \neq \beta, \end{cases}$$

which can be compactly written as

$$(A; \alpha) + (B; \beta) = (A +^{\alpha\beta} B; \mu(a, b)). \tag{3}$$

In (3) we assume that for $\alpha, \beta \in \Lambda$ a binary boolean operation "$\cdot$" is defined by $\alpha \cdot \beta = \alpha\beta = \{+, \alpha = \beta;\ -, \alpha \neq \beta\}$. In addition we assume $+^+ = +$. Formula (3) gives the concise symbolic expression we looked for.

Similarly we can symbolically describe the order relation preceding "$\leq$" in $\mathbb{R}^+ \otimes \Lambda$, so that $(\mathbb{R}, \leq) \cong (\mathbb{R}^+ \otimes \Lambda, \leq)$. We have

$$(4) \qquad (A;\alpha) \leq (B;\beta) \Longleftrightarrow \begin{cases} (\alpha = -) \ \text{ and } \ (\beta = +) \text{ or} \\ (\alpha = \beta = -) \ \text{ and } \ (A \geq B) \text{ or} \\ (\alpha = \beta = +) \ \text{ and } \ (A \leq B). \end{cases}$$

Note that both formulae (3) and (4) make use of the restrictions on the correspondent operation/relation in the set of non-negative numbers; thereby formula (3) makes use of a new operation for non-negative numbers, namely (1).

Algebraic properties of $(\mathbb{R}, +, \leq)$. Recall first the algebraic properties of $(\mathbb{R}, +)$. System $(\mathbb{R}, +)$ is an additive group, that is

i) "+" is a closed (total) operation;

ii) "+" is associative: $(a + b) + c = a + (b + c)$;

iii) there is an identity (neutral, null) element 0, such that $a + 0 = a$ for all a;

iv) for every a there exists an additive inverse (opposite) element $-a$, such that $a + (-a) = 0$.

Property iv) induces operation subtraction $a - b = a + (-b)$ in $\mathbb{R}$.

Using algebraic terminology we can say:

— property i) defines a magma;

— properties i)–ii) define a semigroup;

— properties i)–iii) define a monoid;

— properties i)–iv) define a group.

Every group obeys:

— property v) "cancellation law": $a + x = b + x \Longrightarrow a = b$.

An algebraic system may satisfy also:

— property vi) "commutative law": $a + b = b + a$.

System $(\mathbb{R}, +)$ satisfies all enlisted properties i)– vi) and thus is a commutative (abelian) group. The commutative group $(\mathbb{R}, +)$ satisfies also:

— property vii) subtractability, in the sense that equation $a + x = b$ has an unique solution for all $a, b \in \mathbb{R}$, namely $x = b - a = b + (-a)$.

Order isotonicity. We note that in system $(\mathbb{R}, +, \leq)$ the preceding order "$\leq$" is consistent with addition. Indeed, for $a, b, c \in \mathbb{R}$ we have $a \leq b \implies a+c \leq b + c$. As a consequence we have for $a, b, c, d \in \mathbb{R}$:

$$a \leq b, \ c \leq d \implies a + c \leq b + d.$$

Inverse isotonicity of addition. If $a, b, c \in \mathbb{R}$, then

$$a + c \leq b + c \Longrightarrow a \leq b,$$

in particular $a + c = b + c \Longrightarrow a = b$ (cancellation law).

In what follows we use the terms "error" and "non-negative number" as synonyms. In the next section we are interested in finding out to what extent properties i)–vii), as well as order isotonicity, apply to the system of errors.

2. Properties of errors.

Consider the algebraic properties of the system of errors $(\mathbb{R}^+, +, \leq)$.

The system $(\mathbb{R}^+, +, \leq)$. Properties i)–iii) and vi) are satisfied so $(\mathbb{R}^+, +)$ is a commutative monoid, but property iv) fails; indeed, equation $A + X = 0$ has no solution when $A \neq 0$. However, the cancellation property v) $A + X = B + X \Longrightarrow A = B$ holds true. Note that every group is cancellative, but a cancellative monoid may not be a group, as is the case with $(\mathbb{R}^+, +)$.

As mentioned, due to failure of property iv) there is no additive inverse (opposite) in the cancellative monoid $(\mathbb{R}^+, +)$. Subtractability vii) does not hold either, as $A + X = B$ does not possess a solution in general; in algebraic language this means that $(\mathbb{R}^+, +)$ is neither a group nor a quasigroup.

Order isotonicity properties take place in $\mathbb{R}^+$ as well. Namely, we have for $A, B, C, D \in \mathbb{R}^+$

$$A \leq B \iff A + C \leq B + C,$$

$$A \leq B,\ C \leq D \implies A + C \leq B + D.$$

We shall next focus our attention on operation (1). As we have seen this operation appears whenever we want to present the (group) operation addition by means of non-negative elements, or roughly speaking whenever we want to "project" the group $(\mathbb{R}, +)$ into the monoid system $(\mathbb{R}^+, +)$. Thus the group addition induces two operations in the monoid system: the standard addition "$+$" and the special operation "$+^-$". The operation "$+^-$" has been used implicitly under the name inner (or non-standard) addition (or inner subtraction) in the literature on interval analysis [2], [6], [7], [8] and, since recently under the name of "generalized Hukuhara difference" in fuzzy set analysis [1], [9]. As this operation is defined by means of a conditional formula (1) we shall call it here conditional addition, briefly *c-addition.* (Conditional statements and formulae are typical for monoid systems.)

Remark. As we shall see next (cf. property iv) below), operation (1) possesses certain features that are characteristic to subtraction, thus it can be called *c-subtraction* as well.

We next consider the algebraic properties of the system $(\mathbb{R}^+, +^-, \leq)$ in some detail.

The system $(\mathbb{R}^+, +^-, \leq)$. The following properties of c-addition "$+^-$" follow easily from definition (1):

— i) c-addition "$+^-$" is a closed (total) operation. Hence $(\mathbb{R}^+, +^-)$ is a magma;

— associativity property ii) $(A +^- B) +^- C = A +^- (B +^- C)$ fails, indeed, e.g. $(7 +^- 5) +^- 3 \neq 7 +^- (5 +^- 3)$, as $(7 +^- 5) +^- 3 = 1$ and $7 +^- (5 +^- 3) = 5$;

— property iii) for the existence of an identity (neutral, null) element, such that $A +^- 0 = A$ for all $A \in \mathbb{R}^+$, holds true; hence $(\mathbb{R}^+, +^-)$ is an unital magma;

— property iv) for the existence of an inverse (opposite) element holds true as well. Indeed, the inverse of A is the element A itself, since $A +^- A = 0$ for all A.

— property v) "cancellation law": $A +^- X = B +^- X \Longrightarrow A = B$ fails, e.g. take $A = 1, B = 5, X = 3$. Then $1 +^- 3 = 5 +^- 3$, but $1 \neq 5$;

— property vi) "commutative law": $A +^- B = B +^- A$ holds true.

System $(\mathbb{R}^+, +^-)$ satisfies properties i), iii) and vi) hence is a commutative unital magma.

As mentioned, associativity fails in $(\mathbb{R}^+, +^-)$. Associativity means that every three elements can be "summed up" (in the sense of c-addition) in any order and produce the same result. Notice that associativity holds true under the requirement that the element participating in both brackets is the largest one. We call this property "conditional associativity" (briefly "c-associativity").

C-associativity. Let $A, B, C \in \mathbb{R}^+$ be such that $B \geq A, B \geq C$. Then $(A +^- B) +^- C = A +^- (B +^- C)$.

C-associativity is practically important, since "summing up" three elements (in the sense of c-addition) does not depend on the order of summation unless we start summation from the largest element. In practice this restriction is often not too hard.

We mentioned that the cancellation law $A +^- X = B +^- X \Longrightarrow A = B$ fails in $(\mathbb{R}^+, +^-)$. However, cancellation is valid under certain conditions. Indeed, consider relation $A +^- X = B +^- X$ as an equation for X, then it has a unique solution. Namely X is the midpoint between A and B, that is the arithmetic mean $X = (A + B)/2$. It is easy to see that cancellation of X holds true unless X is not equal to the arithmetic mean of A and B. We formulate this property as follows:

C-cancellation. Let $A, B \in \mathbb{R}^+$. Equation $A +^- X = B +^- X$ is satisfied for $X = (A + B)/2$. If $X \neq (A + B)/2$, then cancellation of X holds true: $A +^- X = B +^- X \Longrightarrow A = B$.

Summarizing we obtain:

Proposition 1. *The set of errors with c-addition* $(\mathbb{R}^+, +^-)$ *is a c-associative and c-cancellative commutative unital magma.*

The next proposition links the operation c-addition and the order relation preceding "$\leq$".

Proposition 2. (Conditional inclusion isotonicity w. r. t. c-addition). *Let* $A, B, C, D \in \mathbb{R}^+$. *Assuming* $A \geq B, C \leq D$, *we have*

$$\begin{aligned} &\textit{if}\, A \leq C, \quad \textit{then}\ A +^- C \leq B +^- D, \\ &\textit{if} \quad B \geq D, \quad \textit{then}\ A +^- C \geq B +^- D. \end{aligned}$$

In the special case $D = C$ *we have: For* $A, B, C \in \mathbb{R}^+$ $A \geq B$ *implies*

$$\begin{aligned} &\textit{if}\, A \leq C, \quad \textit{then}\ A +^- C \leq B +^- C, \\ &\textit{if} \quad B \geq C, \quad \textit{then}\ A +^- C \geq B +^- C. \end{aligned}$$

The extended monoidal system $(\mathbb{R}^+, +, 0, +^-, \leq)$. As mentioned above, subtractability does not hold in the cancellative monoid $(\mathbb{R}^+, +)$, it does not hold in $(\mathbb{R}^+, +^-)$ either. This means that we cannot solve directly equations $A+X = B$ and $A+^- X = B$. However, the operation c-addition "$+^-$" defined by (1) allows solving equations of the form $A+X = B$ in certain cases. Namely, using "$+^-$" we can solve equation $A + X = B$ when $A \leq B$ and we can solve equation

$B + X = A$ when $A \geq B$. Thus c-addition plays a role in $(\mathbb{R}^+, +)$ analogous to the role of subtraction in the group $(\mathbb{R}, +)$. (This is one more "subtraction-like" feature of c-addition.)

Up to now we have seen that the algebraic system $(\mathbb{R}^+, +, \leq)$ possesses null and c-addition; to remind this we shall sometimes denote fully this system as $(\mathbb{R}^+, +, 0, +^-, \leq)$ and to emphasize that the system includes c-addition we shall call it *extended (monoidal) system*.

Clearly, the enlisted algebraic properties of $(\mathbb{R}^+, +, \leq)$ induce certain manipulation rules, such as the rule based on formula (1), saying that the solution X of equation $A + X = B$ when $A \leq B$ is $X = A +^- B$. We may call the set of algebraic properties and arithmetic rules in $\mathbb{R}^+$ "error arithmetic". Hence error arithmetic is a set of rules needed to compute with errors (error bounds of approximate numbers).

Definition (1) shows that the operation addition "+" induces c-addition "$+^-$" in the monoid $(\mathbb{R}^+, +)$, in a way analogous to the way addition "+" induces negation/subtraction "−" in the group $(\mathbb{R}, +)$. The next Proposition shows that operations "+" and "$+^-$" are closely related.

Proposition 3. *i) For $A, B \in \mathbb{R}^+$, such that $A \leq B$, the unique solution of $A + X = B$ is $X = B +^- A$. ii) Equation $A +^- X = B$ has a solution $X = A + B$ for $A, B \in \mathbb{R}^+$. If $A, B \in \mathbb{R}^+$ are such that $A \geq B > 0$, then equation $A +^- X = B$ has one more solution $X = A +^- B$.*

We thus see that: i) the solution of $A + X = B$ (when existing) can be expressed in terms of c-addition, and ii) the solution of $A +^- X = B$ can be expressed in terms of usual addition. Thus, in light of Proposition 3 solutions of both $A + X = B$ and $A +^- X = B$ become possible under certain conditions. This property is further called "conditional subtractability", briefly "c-subtractability".

Proposition 3 shows that standard addition and c-addition complement each other. Other examples of complementary rules are the associative-like properties to be discussed in the next section.

3. C-associativity in the extended system. The operations addition and c-addition tightly complement each other in the extended monoid system $(\mathbb{R}^+, +, +^-, \leq)$ regarding associative-like properties.

Associative-like rules can be obtained using the simple method that we used in the introduction to obtain formulae (1), (3), namely by writing the real variables as signed non-negative ones (in [3] we use this method in the more general setting of so-called Kaucher intervals).

In the familiar associative law $(a + b) + c = a + (b + c)$, $a, b, c \in \mathbb{R}$, we substitute $a = (A; \alpha)$, $b = (B; \beta)$, $c = (C; \gamma) \in \mathbb{R}$ to obtain:

$$((A; \alpha) + (B; \beta)) + (C; \gamma) = (A; \alpha) + ((B; \beta) + (C; \gamma)).$$

Using (3) we can write:

$$((A +^{\alpha\beta} B; \mu(a, b))) + (C; \gamma) = (A; \alpha) + ((B +^{\beta\gamma} C; \mu(b, c))). \tag{5}$$

Now substituting specific values for α, β, γ in (5) we obtain a number of associative-like rules, some of which are given below.

Define the mapping $\phi : \mathbb{R}^{+2} \longrightarrow \Lambda$ by

$$\phi(A, B) = \begin{cases} +, & \text{if } A \geq B; \\ -, & \text{otherwise.} \end{cases}$$

Remark. The mapping ϕ is analogous to mapping μ given in (2) but defined on $\mathbb{R}^{+2}$ instead on $\mathbb{R}^2$.

Proposition 4. *Let $A, B, C \in \mathbb{R}^+$. Then*

$$(A + B) +^- C = A +^{\phi(B,C)} (B +^- C). \tag{6}$$

P r o o f. Formula (6) can be obtained from the general formula (5) by substituting $\alpha = \beta = -\gamma$. Alternatively, it can be directly checked that both sides of (6) are equal to $A + B - C$ (in real arithmetic). $\square$

For some applications the two operations for addition $+, +^-$ can be considered as one operation in two modes (directions). We shall use below the notation "$+^\theta$", wherein $\theta \in \Lambda$, and refer to "$+^\theta$" as "directed addition". For $\theta = +$ the operation "$+^\theta$" is the positively directed (standard) addition, "$+^+ = +$", whereas for $\theta = -$, "$+^\theta$" is the negatively directed c-addition, "$+^-$".

Associative-like rules for algebraic transformations. Directed addition is conditionally associative in the following sense:

Proposition 5. *For each triple $A, B, C \in \mathbb{R}^+$ and each pair $\theta_1, \theta_2 \in \Lambda$, there exists a pair $\theta_3, \theta_4 \in \Lambda$, such that*

$$(A +^{\theta_1} B) +^{\theta_2} C = A +^{\theta_3} (B +^{\theta_4} C). \tag{7}$$

Proof. Formula (7) generalizes (6); indeed (6) is a special case of (7) when $\theta_1 = +, \theta_2 = -$. Assume now $\theta_1 = -$. It can be directly checked (or derived from (5)) that for $A, B, C \in \mathbb{R}^+$ we have

$$(A +^- B) + C = \begin{cases} A +^{-\phi(B,C)} (B +^- C), & A \geq B, \\ A +^- (B + C), & A < B; \end{cases} \tag{8}$$

$$(A +^- B) +^- C = \begin{cases} A +^{-\phi(B,C)} (B +^- C), & A < B, \\ A +^- (B + C), & A \geq B. \end{cases} \tag{9}$$

From formulae (6), (8) and (9) we see that θ_3, θ_4 are simple functions of the errors $A, B, C \in \mathbb{R}^+$ and $\theta_1, \theta_2 \in \Lambda$ and can be effectively computed. This proves (7). $\square$

Example. For $A, B \in \mathbb{R}^+$, $(A + B) +^- A = B$. Indeed, using (6) we obtain: $(A+B)+^- A = (B+A)+^- A = B +^{\phi(A,A)} (A +^- A) = B +^{\phi(A,A)} 0 = B$.

We note in passing that c-associativity of c-addition, namely $(A +^- B) +^- C = A +^- (B +^- C)$,when $B \geq A, B \geq C$, is an easy consequence of relation (9).

The *c-associative rules* are useful as they give specific conditions under which "replacement of brackets" can be performed; thereby these conditions are easily programmable. Standard addition is commutative and associative but has no inverse, whereas c-addition is commutative, not associative and has an inverse. Considering both additions together as a "directed" operation in two modes, we can say that this directed operation is c-associative. Thus both modes complement each other.

In the calculus for interval functions [8] an important role is played by c-associative rules involving four elements.

Associative-like rules involving four elements. For $A, B, C, D \in \mathbb{R}^+$ define $\varphi : \mathbb{R}^{+4} \to \{+, -\}$ as

$$\varphi(A, B, C, D) = \phi(A, B)\phi(C, D).$$

Proposition 6. *For $A, B, C, D \in \mathbb{R}^+$ denote*

$$\gamma = \varphi(A, C, B, D) = \phi(A, C)\phi(B, D),$$

$$\delta = \varphi(A, B, C, D) = \phi(A, B)\phi(C, D),$$

then we have

$$\begin{aligned}
(A + B) +^- (C + D) &= (A +^- C) +^\gamma (B +^- D); \\
(A +^- B) + (C +^- D) &= \begin{cases} (A +^- C) +^{-\gamma} (B +^- D), & \textit{if } \delta < 0; \\ (A + C) +^- (B + D), & \textit{if } \delta \geq 0; \end{cases} \\
(A +^- B) +^- (C +^- D) &= \begin{cases} (A +^- C) +^{-\gamma} (B +^- D), & \textit{if } \delta \geq 0; \\ (A + C) +^- (B + D), & \textit{if } \delta < 0. \end{cases}
\end{aligned}$$

Example. The above relations may be specified in particular cases, e.g., when we know ranges for the arguments. Take for example $A \in [3, 4], B \in [1, 2], C \in [4, 5], D \in [6, 7]$. Since $\gamma = -, \delta = +$, Proposition 6 obtains the form:

$$\begin{aligned}
(A + B) +^- (C + D) &= (A +^- C) +^- (B +^- D), \\
(A +^- B) + (C +^- D) &= (A + C) +^- (B + D), \\
(A +^- B) +^- (C +^- D) &= (A +^- C) + (B +^- D).
\end{aligned}$$

Note in the last relation in Proposition 6

$$(A +^- B) +^- (C +^- D) = (A +^- C) +^{-\gamma} (B +^- D) \ \text{ if } \ \delta \geq 0,$$

the condition $\delta \geq 0$ is not as restrictive as it seems, due to commutativity of "$+^-$", allowing us to replace (if necessary) $(B +^- A)$ with $(A +^- B)$ and/or $(D +^- C)$ with $(C +^- D)$.

The main algebraic properties of systems $(\mathbb{R}, +)$, $(\mathbb{R}^+, +)$, $(\mathbb{R}^+, +^-)$ and the extended system $(\mathbb{R}^+, +, +^-)$ are summarized in Table 1.

Table 1

Axiom/System	$(\mathbb{R}, +)$	$(\mathbb{R}^+, +)$	$(\mathbb{R}^+, +^-)$	$(\mathbb{R}^+, +, +^-)$
Closure	Yes	Yes	Yes	Yes & Yes
Associativity	Yes	Yes	C	Yes & C
Identity	Yes	Yes	Yes	Yes & Yes
Inverse	Yes	No	Yes	No & Yes
Cancellation	Yes	Yes	C	Yes & C
Commutativity	Yes	Yes	Yes	Yes & Yes
Subtractability	Yes	No	No	C

Table 1. Summary of the algebraic properties of the group $(\mathbb{R}, +)$, the monoid $(\mathbb{R}^+, +)$, the magma $(\mathbb{R}^+, +^-)$ and the extended monoid $(\mathbb{R}^+, +, +^-)$. The letter "C" stands for "conditional".

From Table 1 we see that (standard) addition and c-addition complement each other. For example, addition "+" has no inverse in $\mathbb{R}^+$, whereas c-addition "$+^-$" is invertible. Similar complementarity is observed especially with respect to associativity, cancellation and subtractability.

4. Linear and quasilinear spaces. All said above can be generalized component-wise for n-vectors: $(\mathbb{R}^n, +) = (\mathbb{R}^n, +, 0, -, \leq)$, resp. $(\mathbb{R}^{+n}, +) = (\mathbb{R}^{+n}, +, 0, +^-, \leq)$, noticing that then the order relation "$\leq$" is not total but partial.

We introduce multiplication by scalars from the real ordered field $\mathbb{R} = (\mathbb{R}, +, \cdot, \leq)$, arriving thus at the familiar vector space $(\mathbb{R}^n, +, \mathbb{R}, \cdot, \leq)$, where now $\mathbb{R}^n$ is the set of real vectors $a = (a_1, a_2, \ldots, a_n)$.

Component-wise generalizations of previous definitions like $a = (A; \alpha) \in \mathbb{R}^n$ with $A = (A_1, A_2, \ldots, A_n) \in \mathbb{R}^{+n}, \alpha \in \Lambda^n$, etc. are obvious.

Multiplication of the vector $a = (A; \alpha) \in \mathbb{R}^n$ by a scalar $c \in \mathbb{R}$ is given by

$$c \cdot a = c \cdot (A; \alpha) = (|c| \cdot A; \sigma(c)\alpha).$$

The above shows that multiplication by scalars induces a "quasivector" multiplication by scalars "$*$" in the "error space" $(\mathbb{R}^{+n}, +, +^-, \mathbb{R}, *, \leq)$ given by

$$c * A = |c| \cdot A, \; c \in \mathbb{R}, \; A \in \mathbb{R}^{+n}.$$

Proposition 7 (Quasi-distributive law)**.** *For $C \in \mathbb{R}^{+n}$, $\alpha, \beta \in \mathbb{R}$ and "$*$" multiplication by scalars we have*

$$(\alpha + \beta) * C = \alpha * C +^{\sigma(\alpha)\sigma(\beta)} \beta * C. \tag{10}$$

P r o o f. For $\sigma(\alpha) = \sigma(\beta)$ (10) holds true. Let $\sigma(\alpha) \neq \sigma(\beta)$, e. g. $\sigma(\alpha) = +, \sigma(\beta) = -$, then we have $\alpha * C +^- \beta * C = |\alpha C - |\beta|C| = |\alpha C + \beta C| = |\alpha + \beta|C = (\alpha + \beta) * C$. $\square$

Proposition 8. *Multiplication "$*$" of error vectors by scalars is order isotone. Namely, for $A, B \in \mathbb{R}^{+n}$, $\gamma \in \mathbb{R}$: $A \leq B$ implies $\gamma * A \leq \gamma * B$, $\gamma \in \mathbb{R}$.*

Remark. Note that for $a, b \in \mathbb{R}^n$: $a \le b$ does not imply $\gamma \cdot a \le \gamma \cdot b$, $\gamma \in \mathbb{R}$, showing that multiplication by scalars "$\cdot$" is not order isotone in real arithmetic. The order relation "$\le$" in the space of errors is interpreted as inclusion (of symmetric intervals) and is also denoted "$\subseteq$".

In what follows we give an axiomatic definition of quasilinear space, which is an abstract generalisation of system $(\mathbb{R}^{+n}, +, \mathbb{R}, *)$. To do this we start with the space of real vectors $(\mathbb{R}^n, +, \mathbb{R}, \cdot)$.

The space $(\mathbb{R}^n, +, \mathbb{R}, \cdot)$ is the well-known n-dimensional vector space, which is a finite linear space. In textbooks a linear space is usually abstractly defined as a commutative group w. r. t. addition that also satisfies four axioms w. r. t. multiplication by scalars. The following is a review of such a definition, with a small modification for the purposes of the discussion to follow.

Definition. *An algebraic system $(L, +, \mathbb{R}, \cdot)$ is a linear space over the field $\mathbb{R}$ if there exists $0 \in L$ such that for all $a, b, c \in L$ and all $\alpha, \beta, \gamma \in \mathbb{R}$:*

$$(a + b) + c = a + (b + c), \tag{11}$$

$$a + 0 = a, \tag{12}$$

$$a + b = b + a, \tag{13}$$

$$a + c = b + c \Longrightarrow a = b, \tag{14}$$

$$\alpha \cdot (\beta \cdot c) = (\alpha\beta) \cdot c, \tag{15}$$

$$1 \cdot a = a, \tag{16}$$

$$\gamma \cdot (a + b) = \gamma \cdot a + \gamma \cdot b, \tag{17}$$

$$(\alpha + \beta) \cdot c = \alpha \cdot c + \beta \cdot c. \tag{18}$$

The above definition differs from the traditional one by the weaker cancellation axiom (14), that replaces the axiom for the existence of an additive inverse (opposite) element, which implies that L is a group under addition.

To check the equivalence between Definition (11)–(18) and the usual textbook one, note that substituting $\alpha = 1$ and $\beta = -1$ in (18) implies $0 = c + (-1) \cdot c$, i. e., the element $(-1) \cdot c$ is opposite to c.

Condition (12) for existence of a neutral (null) element for addition is redundant and can be omitted. Indeed, the element $0 \cdot c$ is such a neutral element. To show this, substitute in (18) the values $\alpha = 1$ and $\beta = 0$ to obtain:

$$c + 0 \cdot c = 1 \cdot c + 0 \cdot c = (1 + 0) \cdot c = 1 \cdot c = c,$$

implying $0 \cdot c = 0$.

Error spaces can be introduced in an axiomatic way. We briefly sketch this approach in what follows. Definition (11)–(18) is useful for the comparison between linear and quasilinear spaces to be considered next.

A quasilinear space of monoid structure is defined by relaxing the distributive law (18) from the set of linear space axioms (11)–(18) as follows.

Definition. *An algebraic system* $(\mathbb{Q}, +, \mathbb{R}, *)$ *is a* quasilinear space (of monoid structure, over $\mathbb{R}$)*, if there exists* $0 \in L$ *such that for all* $A, B, C \in \mathbb{Q}$*, and for all* $\alpha, \beta, \gamma \in \mathbb{R}$*:*

$$(A + B) + C = A + (B + C), \tag{19}$$

$$A + 0 = A, \tag{20}$$

$$A + B = B + A, \tag{21}$$

$$A + C = B + C \implies A = B, \tag{22}$$

$$\alpha * (\beta * C) = (\alpha\beta) * C, \tag{23}$$

$$1 * A = A, \tag{24}$$

$$\gamma * (A + B) = \gamma * A + \gamma * B, \tag{25}$$

$$(\alpha + \beta) * C = \alpha * C + \beta * C, \quad \text{if } \alpha\beta \geq 0. \tag{26}$$

Condition (20) for existence of a neutral element for addition is redundant, as it was in the definition of a linear space. The limitation $\alpha\beta \geq 0$ in axiom (26) allows us to substitute in (26) the values $\alpha = 1$ and $\beta = 0$ to obtain:

$$C + 0 * C = 1 \cdot C + 0 * C = (1 + 0) * C = 1 * C = C,$$

implying $0 * C = 0$.

It is easy to check the following

Proposition 9. *The error system* $(\mathbb{R}^{+n}, +, \mathbb{R}, *)$ *is a quasilinear space in the sense of definition* (19)*–*(26)*.*

5. Conclusions. In the present work we show that:

i) the study of approximate numbers imposes the study of the algebraic properties of errors, that is non-negative numbers;

ii) addition of real numbers naturally induces the operation c-addition of non-negative numbers (distance, modulus of the difference);

iii) the operation c-addition of non-negative numbers enriches the additive monoidal system of non-negative numbers up to a structure close to a group (where many typically group operations can be performed under somewhat sophisticated conditions);

iv) the operation c-addition of non-negative numbers is fundamental in introducing inner operations in interval analysis, resp. generalized Hukuhara difference in fuzzy-set analysis and other fields of applied and computational mathematics;

v) error arithmetic involves naturally an operation "multiplication by scalars" which leads to a special algebraic structure "quasilinear space", close but yet different from linear spaces.

The idea of the present paper appeared while the authors worked on a motion paper [4] presented at the IEEE P1788 Working Group on Standardardization of interval arithmetic [5]. Some of the results were reported at the SCAN-2010 Symposium, `http://scan2010.ens-lyon.fr/`

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