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AN IMPROVEMENT TO THE ACHIEVEMENT OF THE GRIESMER BOUND

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Abstract. We denoted by $n_q(k,d)$, the smallest value of n for which an $[n,k,d]_q$ code exists for given q,k,d. Since $n_q(k,d) = g_q(k,d)$ for all $d \geq d_k + 1$ for $q \geq k \geq 3$, it is a natural question whether the Griesmer bound is attained or not for $d = d_k$, where $g_q(k,d) = \sum_{i=0}^{k-1} \lceil d/q^i \rceil$, $d_k = (k-2)q^{k-1} - (k-1)q^{k-2}$. It was shown by Dodunekov [2] and Maruta [9], [10] that there is no $[g_q(k,d_k),k,d_k]_q$ code for $q \geq k$, $k = 3,4,5$ and for $q \geq 2k-3$, $k \geq 6$. The purpose of this paper is to determine $n_q(k,d)$ for $d = d_k$ as $n_q(k,d) = g_q(k,d) + 1$ for $q \geq k$ with $3 \leq k \leq 8$ except for $(k,q) = (7,7)$, $(8,8)$, $(8,9)$.

1. Introduction. Let $\mathbb{F}_q^n$ denote the vector space of n-tuples over $\mathbb{F}_q$, the field of q elements, where n is an integer ≥ 4 and q is a prime or a prime power. A q-ary linear code $\mathcal{C}$ of length n and dimension k, called an $[n,k]_q$ code, is a

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k-dimensional subspace of $\mathbb{F}_q^n$, where $n > k \geq 3$. An $[n,k]_q$ code $\mathcal{C}$ with minimum Hamming distance d is referred to as an $[n,k,d]_q$ code. Let $G = [\boldsymbol{g}_1^{\mathrm{T}}, \boldsymbol{g}_2^{\mathrm{T}}, \ldots, \boldsymbol{g}_n^{\mathrm{T}}]$ be a $k \times n$ generator matrix of an $[n,k,d]_q$ code $\mathcal{C}$ with $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_n \in \mathbb{F}_q^k$, where $\boldsymbol{g}^{\mathrm{T}}$ denotes the transpose of the vector $\boldsymbol{g}$. If there is no zero vector in $\{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_n\}$, an $[n,k,d]_q$ code $\mathcal{C}$ is called a *nontrivial code*. A fundamental problem in coding theory is to solve the following problem.

Problem 1. *Find the smallest value of n, denoted by $n_q(k,d)$, for which an $[n,k,d]_q$ code exists for given integers q,k,d.*

An $[n,k,d]_q$ code is called *optimal* if $n = n_q(k,d)$. There is a lower bound on $n_q(k,d)$ called the Griesmer bound [3], [11]:

$$n_q(k,d) \geq g_q(k,d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. A $[g_q(k,d),k,d]_q$ code is called a *Griesmer code*. In order to solve Problem 1, we consider the following problem for given integers $k \geq 3$ and $q \geq 3$.

Problem 2. *For given integers k and q, find the value $c(k,q)$ such that*

(a) $n_q(k,d) \geq g_q(k,d) + 1$ *for* $d = c(k,q)$;

(b) $n_q(k,d) = g_q(k,d)$ *for any integer* $d \geq c(k,q) + 1$.

It is known (Theorem 2.12 in [6] or [1]) that the following theorem holds. See [6] for linear codes *of type BV*.

Theorem 1.1. *For given q,k and d, write*

$$d = sq^{k-1} - \sum_{i=1}^{t} q^{u_i - 1}$$

where $s = \lceil d/q^{k-1} \rceil$, $k > u_1 \geq u_2 \geq \cdots \geq u_t \geq 1$, and at most $q-1$ u_i's take any given value. Then there exists a $[g_q(k,d),k,d]_q$ code of type BV if and only if the following condition holds:

$$\sum_{i=1}^{\min\{s+1,t\}} u_i \leq sk.$$

Corollary 1.2. *If q and k are integers with $q \geq k \geq 3$, then*

(1) *there is no* $[g_q(k,d),k,d]_q$ *code of type* BV *for* $d=(k-2)q^{k-1}-(k-1)q^{k-2}$,

(2) $n_q(k,d)=g_q(k,d)$ *for any integer* $d\geq(k-2)q^{k-1}-(k-1)q^{k-2}+1$.

Problem 3. *For given integers* k *and* q*, find the value* $b(k,q)$ *such that*

(*a*) *there is no* $[g_q(k,d),k,d]_q$ *code of type* BV *for* $d=b(k,q)$*;*

(*b*) $n_q(k,d)=g_q(k,d)$ *for any integer* $d\geq b(k,q)+1$.

In the case $q\geq k\geq 3$, Corollary 1.2 shows that if there is no $[g_q(k,d),k,d]_q$ code for $d=(k-2)q^{k-1}-(k-1)q^{k-2}$, then $c(k,q)=(k-2)q^{k-1}-(k-1)q^{k-2}$. Hence we consider the following problem.

Problem 4. *Investigate whether a* $[g_q(k,d),k,d=(k-2)q^{k-1}-(k-1)q^{k-2}]_q$ *code exists or not for given integers* k *and* q *with* $q\geq k\geq 3$.

Hamada conjectured as follows.

Conjecture 1.3. *There is no* $[g_q(k,d),k,d=(k-2)q^{k-1}-(k-1)q^{k-2}]_q$ *code for any integers* k *and* q *with* $q\geq k\geq 3$*. That is,*

$$c(k,q)=(k-2)q^{k-1}-(k-1)q^{k-2}$$

for any integers k *and* q *with* $q\geq k\geq 3$.

Conjecture 1.4. $c(k,q)=b(k,q)$ *for any integers* $k\geq 3$ *and* $q\geq 3$.

As for Conjecture 1.3, the following is known, see Dodunekov [2] and Maruta [9], [10].

Theorem 1.5 ([10])**.** *For* $d=(k-2)q^{k-1}-(k-1)q^{k-2}$*, it holds that* $n_q(k,d)\geq g_q(k,d)+1$ *for* $q\geq k$ *when* $k=3,4,5$ *and for* $q\geq 2k-3$ *when* $k\geq 6$.

Hence Problem 4 is unsolved for any integers k and q with $2k-3>q\geq k\geq 6$. For example, the cases in the next remark are still open.

Remark 1.6. For $6\leq k\leq 13$, Problem 4 is unsolved for the following k and q.

(1)	$k=6$	and	$q=7,8,$	(2)	$k=7$	and	$q=7,8,9,$
(3)	$k=8$	and	$q=8,9,11,$	(4)	$k=9$	and	$q=9,11,13,$
(5)	$k=10$	and	$q=11,13,16,$	(6)	$k=11$	and	$q=11,13,16,17,$
(7)	$k=12$	and	$q=13,16,17,19,$	(8)	$k=13$	and	$q=13,16,17,19.$

In this paper we prove the following two theorems.

Theorem 1.7. *There is no* $[g_q(k,d),k,d=(k-2)q^{k-1}-(k-1)q^{k-2}]_q$ *code for any integers* $k \geq 6$ *and* q *with* $q=2k-2u$ *and* $k>4u-6$ *for* $u=2,3$.

Theorem 1.8. *There is no* $[g_q(k,d),k,d=(k-2)q^{k-1}-(k-1)q^{k-2}]_q$ *code for any integers* $k \geq 6$ *and* q *with* $q=2k-2u-1$ *and* $k>4u-4$ *for* $u=2,3$.

Theorems 1.7 and 1.8 imply that Conjecture 1.3 is valid for the following k and q:

(1)	$k=6$	and	$q=7,8,$	(2)	$k=7$	and	$q=8,9,$
(3)	$k=8$	and	$q=11,$	(4)	$k=9$	and	$q=11,13,$
(5)	$k=10$	and	$q=13,16,$	(6)	$k=11$	and	$q=16,17,$
(7)	$k=12$	and	$q=17,19,$	(8)	$k=13$	and	$q=19.$

For $d'=(k-2)q^{k-2}-(k-1)q^{k-3}$ with $q \geq k \geq 3$, there exists a $[g_q(k-1,d'),k-1,d']_q$ code, say $\mathcal{C}'$, by Theorem 1.1. Applying Theorem 4.5 of [5] to $\mathcal{C}'$, one can get a $[g_q(k,d)+1,k,d]_q$ code for $d=(k-2)q^{k-1}-(k-1)q^{k-2}$. Hence, the nonexistence of Griesmer codes determines the exact value of $n_q(k,d)$. As a result of the previous theorems, Theorem 1.5 for $k \leq 13$ can be improved to the following.

Theorem 1.9. *For* $d=(k-2)q^{k-1}-(k-1)q^{k-2}$, *it holds that* $n_q(k,d)=g_q(k,d)+1$ *for* $q \geq k$ *with* $3 \leq k \leq 13$ *except for* $(k,q)=(7,7)$, $(8,8)$, $(8,9)$, $(9,9)$, $(10,11)$, $(11,11)$, $(11,13)$, $(12,13)$, $(12,16)$, $(13,13)$, $(13,16)$, $(13,17)$.

Remark 1.10. (1) If $q=2k-2u$ and $k>4u-6$, then $2q-(3k-6)=k-4u+6>0$. If $q=2k-2u-1$ and $k>4u-4$, then $2q-(3k-6)=k-4u+4>0$. Hence it holds that $q>(3k-6)/2$ for both cases. When $q \leq (3k-6)/2$ (e.g. $(k,q)=(7,7)$), the situation is quite complicated, see Section 4.
(2) For the nonexistence of a $[g_q(k,d),k,d]_q$ code for $d=(k-2)q^{k-1}-(k-1)q^{k-2}-\varepsilon$ for some small ε, see Klein [8].

2. A geometric method. To obtain a necessary and sufficient condition for the existence of a $[g_q(k,d),k,d]_q$ code for the case $d \leq q^{k-1}$, the concept of minihyper has been introduced by Hamada [4]. To prove Theorems 1.7 and 1.8, we generalize the concept of minihyper for the case $d>q^{k-1}$ and we give a necessary and sufficient condition for the existence of a nontrivial $[n,k,d]_q$ code for given integers n, k, d, q with $n > k \geq 3$, $q \geq 3$ and $(s-1)q^{k-1} < d \leq sq^{k-1}$ for some positive integer s.

For $k \geq 3$, let $\Sigma = \mathrm{PG}(k-1, q)$ be the finite projective space of dimension $k-1$ over $\mathbb{F}_q$ and let $\mathcal{F}_j$ be the set of all j-flats in Σ, where a j-flat is a projective subspace of dimension j in Σ. 0-flats, 1-flats, 2-flats, 3-flats and $(k-2)$-flats are called *points*, *lines*, *planes*, *solids* and *hyperplanes*, respectively. The number of points in a j-flat is denoted by θ_j, where

$$\theta_j = (q^{j+1} - 1)/(q-1) = q^j + q^{j-1} + \cdots + q + 1$$

for $j = 0, 1, \ldots, k-1$. We set $\theta_{-1} = 0$ for convenience.

A point in Σ is denoted by $(\boldsymbol{h})$ using a nonzero vector $\boldsymbol{h} \in \mathbb{F}_q^k$, where two points $(\boldsymbol{h}_1)$ and $(\boldsymbol{h}_2)$ are same points if and only if there exists a nonzero element $\sigma \in \mathbb{F}_q$ with $\boldsymbol{h}_2 = \sigma \boldsymbol{h}_1$. Each hyperplane of Σ can be expressed as the set of all points $(\boldsymbol{g}) \in \mathcal{F}_0$ such that $(\boldsymbol{g}, \boldsymbol{h}) = 0$ and $\boldsymbol{g} \in \mathbb{F}_q^k \setminus \{\boldsymbol{0}\}$ for some nonzero vector $\boldsymbol{h} \in \mathbb{F}_q^k$, where $(\boldsymbol{g}, \boldsymbol{h})$ denotes the inner product of two vectors $\boldsymbol{g}$ and $\boldsymbol{h}$, i.e., $(\boldsymbol{g}, \boldsymbol{h}) = \boldsymbol{g}\boldsymbol{h}^{\mathrm{T}}$ over $\mathbb{F}_q$. In this case, the hyperplane H is denoted by $H(\boldsymbol{h})$, i.e.,

$$H(\boldsymbol{h}) = \{(\boldsymbol{g}) \mid (\boldsymbol{g}, \boldsymbol{h}) = 0 \text{ and } \boldsymbol{g} \in \mathbb{F}_q^k \setminus \{\boldsymbol{0}\}\}$$

for some nonzero vector $\boldsymbol{h} \in \mathbb{F}_q^k$.

Let $\mathcal{C}$ be a nontrivial $[n, k, d]_q$ code and let $G = [\boldsymbol{g}_1^{\mathrm{T}}, \boldsymbol{g}_2^{\mathrm{T}}, \ldots, \boldsymbol{g}_n^{\mathrm{T}}]$ be a generator matrix of $\mathcal{C}$ with $\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n \in \mathbb{F}_q^k$. Let $\mathbf{M}(G)$ be the multiset of n points of Σ corresponding to the n columns of G, i.e.,

$$\mathbf{M}(G) = \{(\boldsymbol{g}_1), \ldots, (\boldsymbol{g}_n)\}.$$

A point P of Σ is an *i-point* if P has multiplicity i in $\mathbf{M}(G)$. Let γ_0 be the maximum multiplicity of points in Σ and let C_i be the set of i-points in Σ. For any subset K of $\mathcal{F}_0$ we define *the multiplicity of K* as

$$m(K) = \sum_{i=1}^{\gamma_0} i \cdot |K \cap C_i|,$$

where $|T|$ denotes the number of points in a subset T of $\mathcal{F}_0$. Then the multiset $\mathbf{M}(G)$ gives a partition $\bigcup_{i=0}^{\gamma_0} C_i$ of $\mathcal{F}_0$. For a t-flat Π in Σ we define

$$\gamma_j(\Pi) = \max\{m(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\},\ 0 \leq j \leq t.$$

We denote simply by γ_j instead of $\gamma_j(\mathcal{F}_0)$. A line l is called a *w-line* if $m(l) = w$. A *w-plane*, a *w-solid* and so on are defined similarly. We prove Theorems 1.7 and 1.8 using the following theorem.

Theorem 2.1. *For $k \geq 3$, there exists a nontrivial $[n, k, d]_q$ code if and only if there exists a partition $\bigcup_{i=0}^{\gamma_0} C_i$ of $\mathcal{F}_0$ which satisfies the following conditions:*

(a) $m(\mathcal{F}_0) = n$,

(b) $\gamma_{k-2} = n - d$.

Proof. Suppose there exists a nontrivial $[n, k, d]_q$ code $\mathcal{C}$ which has a generator matrix $G = [\boldsymbol{g}_1^{\mathrm{T}}, \boldsymbol{g}_2^{\mathrm{T}}, \ldots, \boldsymbol{g}_n^{\mathrm{T}}]$. Then it holds that $m(\mathcal{F}_0) = n$. Since the minimum weight of $\mathcal{C}$ is equal to d, $\mathcal{C}$ must satisfies the following conditions:

$$d = \min\{wt(\boldsymbol{h}G) \mid \boldsymbol{h} \in \mathbb{F}_q^k \setminus \{\boldsymbol{0}\}\} \tag{2.1}$$

where $wt(\boldsymbol{c})$ stands for the number of nonzero entries in the vector $\boldsymbol{c} \in \mathbb{F}_q^n$. Since $wt(\boldsymbol{h}G)$ denotes the number of vectors $\boldsymbol{g}_i$ such that $(\boldsymbol{g}_i, \boldsymbol{h}) \neq 0$ and $m(H(\boldsymbol{h}))$ denotes the number of vectors $\boldsymbol{g}_i$ such that $(\boldsymbol{g}_i, \boldsymbol{h}) = 0$, we have $wt(\boldsymbol{h}G) + m(H(\boldsymbol{h})) = n$. It follows from (2.1) that $\gamma_{k-2} = \max\{m(H(\boldsymbol{h})) \mid \boldsymbol{h} \in \mathbb{F}_q^k \setminus \{\boldsymbol{0}\}\} = n - d$. Hence the part of "only if" holds.

Conversely, suppose there exists a partition in Theorem 2.1 which satisfies the conditions (a) and (b). Let λ_i denote the number of points in C_i. We construct a matrix G consisting of i matrices G_i for $1 \leq i \leq \gamma_0$ as follows.

$$G = [G_1, G_2, G_2, G_3, G_3, G_3, \ldots, G_{\gamma_0}, G_{\gamma_0}, \ldots, G_{\gamma_0}]$$

where G_i denotes a matrix constructed by λ_i colomun vectors $\boldsymbol{g}^{\mathrm{T}}$ with $\boldsymbol{g} \in \mathbb{F}_q^k$ such that $(\boldsymbol{g}) \in C_i$. Then G is a generator matrix of a nontrivial $[n, k, d]_q$ code $\mathcal{C}$. □

For $d = (k-2)q^{k-1} - (k-1)q^{k-2}$, $g_q(k, d)$ can be expressed as follows.

$$g_q(k, d) = (k-2)q^{k-1} - \theta_{k-2}. \tag{2.2}$$

If $n = g_q(k, d)$, then $n - d = (k-1)q^{k-2} - \theta_{k-2} = (k-2)q^{k-2} - \theta_{k-3}$. Hence we have the following corollary from Theorem 2.1.

Corollary 2.2. *For $q \geq k \geq 3$, there exists a $[g_q(k, d), k, d = (k-2)q^{k-1} - (k-1)q^{k-2}]_q$ code if and only if there exists a partition $\bigcup_{i=0}^{k-2} C_i$ of $\mathcal{F}_0$ with $\gamma_0 = k - 2$ in $PG(k-1, q)$ which satisfies the following conditions:*

(a) $m(\mathcal{F}_0) = (k-2)q^{k-1} - \theta_{k-2}$,

(b) $\gamma_{k-2} = (k-2)q^{k-2} - \theta_{k-3}$.

Hence in order to prove Theorems 1.7 and 1.8, it is sufficient to prove the following theorem for integers k and q in the theorems.

Theorem 2.3. *For any integers k and q in Theorems 1.7 and 1.8, there is no partition $\bigcup_{i=0}^{k-2} C_i$ of $\mathcal{F}_0$ with $\gamma_0 = k-2$ in $PG(k-1,q)$ which satisfies the following conditions:*

(a) $m(\mathcal{F}_0) = (k-2)q^{k-1} - \theta_{k-2}$,

(b) $\gamma_{k-2} = (k-2)q^{k-2} - \theta_{k-3}$.

In Sections 3, 4, 5, 6, we shall use repeatedly the following well known result.

Proposition 2.4. *Let k, u, w be integers such that $k \geq 3$, $k-1 \geq w \geq u+2$ and $u \geq 0$. Let $\delta \in \mathcal{F}_u$, $\Pi \in \mathcal{F}_w$.*

(1) *In Π, there are b flats $\Delta_1, \Delta_2, \ldots, \Delta_b \in \mathcal{F}_{u+1}$ containing δ, where $b = \theta_{w-u-1}$.*

(2) *If there exists such a partition of $\mathcal{F}_0$ as Theorem 2.1, then*

$$\sum_{i=1}^{b} m(\Delta_i) = m(\Pi) + (b-1)m(\delta). \tag{2.3}$$

Remark 2.5. In Proposition 2.4 (2), there is a partition of Π as follows.

$$\left(\bigcup_{i=1}^{b} (\Delta_i \setminus \delta)\right) \cup \delta = \Pi. \tag{2.4}$$

Remark 2.6. In the case $d = sq^{k-1}$ for some positive integer s, it is known that there exists a $[g_q(k,d) = s\theta_{k-2}, k, d = sq^{k-1}]_q$ code (take s copies of

Σ as the multiset $\mathbf{M}(G)$). Hence, to solve Problem 1, we only need to consider the case $(s-1)q^{k-1} < d < sq^{k-1}$ for some positive integer s.

3. Preliminary results. Recall from the previous section that γ_j is defined for $1 \le j \le k-1$ as

$$\gamma_j = \max\{m(\Delta) \mid \Delta \in \mathcal{F}_j\}. \tag{3.1}$$

Throughout this section, we assume that there exists a partition $\bigcup_{i=0}^{k-2} C_i$ of $\mathcal{F}_0$ with $\gamma_0 = k-2$ in $\mathrm{PG}(k-1,q)$ which satisfies the conditions (a) and (b) in Corollary 2.2 for $q \ge k \ge 5$. The following lemma due to Maruta [10] plays an important role in proving Theorems 1.7 and 1.8.

Lemmma 3.1 ([10]).

(1) $\gamma_j = (k-2)q^j - \theta_{j-1}$ *for* $0 \le j \le k-1$.

(2) *A j-flat Δ satisfies $m(\Delta) = \gamma_j$ if and only if $\gamma_0(\Delta) = k-2$, for $1 \le j \le k-2$.*

It is already known by Lemma 3.4 of [10] that every line l satisfies $\gamma_0(l) \ge 1$.

Lemma 3.2. *$m(l) \ge tq-1$ for any line l with $\gamma_0(l) = t$.*

Proof. Our assertion follows from the previous lemma for $t = k-2$. Let l be a line with $\gamma_0(l) = t$, $1 \le t \le k-3$. Take a point P of C_{k-2} and let $\delta = \langle l, P\rangle$, where $\langle \chi_1, \chi_2, \dots\rangle$ denotes the smallest flat containing subsets $\chi_1, \chi_2, \dots$ of $\mathcal{F}_0$. Then $m(\delta) = \gamma_2 = (k-2)q^2 - \theta_1$ by Lemma 3.1. Let Q be a t-point on l and let $l_1, \dots, l_q$ be the lines in δ through Q other than l. It follows from (2.3) that

$$m(l) + \sum_{i=1}^{q} m(l_i) = m(\delta) + m(Q)q = \gamma_2 + tq.$$

Since $m(l_i) \le \gamma_1 = (k-2)q-1$ for $1 \le i \le q$, we have

$$m(l) \ge \gamma_2 + tq - q\gamma_1 = tq-1. \qquad \square$$

Lemma 3.3. *Assume that there is no line l with $\gamma_0(l) = k-3$ and $m(l) = (k-3)q+s$, $0 \le s \le k-3$, where $q \ge k \ge 5$. If l_0 is a line with $\gamma_0(l_0) = t \le k-4$, then $m(l_0) = tq-1$.*

P r o o f. Suppose $\gamma_0(l_0) = t$ and $m(l_0) = tq + t'$, $0 \le t' \le t \le k-4$. Let δ be a plane containing l_0 and a $(k-2)$-point. Then, by Lemma 3.1, we have $m(\delta) = \gamma_2$. Let P be a t-point on l_0 and let l_1 be another line through P in δ. Considering the lines through P in δ, we obtain

$$\gamma_2 = m(\delta) \le m(l_0) + m(l_1) + (q-1)\gamma_1 - qt,$$

whence $m(l_1) \ge (k-2)q - 2 - t' > (k-3)q - 1$, for $t' + 1 \le k-3 < k \le q$. This implies that all lines through P in δ other than l_0 are γ_1-lines from our assumption, and we have $\gamma_2 = \gamma_1 q + t' > \gamma_2$, a contradiction. $\square$

Lemma 3.4. *Let Π be a hyperplane of Σ with $\gamma_0(\Pi) = t$, $1 \le t \le k-3$. Assume that every line l in Π with $\gamma_0(l) = u \le k-3$ satisfies $m(l_0) = uq - 1$. Then*

(1) $c(\Pi) = tq^{k-2} - \theta_{k-3}$.

(2) *For a $(t+1)$-flat π in Π containing a t-point, the partition $\pi = \bigcup_{i=0}^{t}(\pi \cap C_i)$ gives a $[tq^{t+1} - \theta_t, t+2, tq^{t+1} - (t+1)q^t]_q$ code.*

P r o o f. See Lemma 3.5 of [10]. $\square$

Since there exists no $[tq^{t+1} - \theta_t, t+2, tq^{t+1} - (t+1)q^t]_q$ code for $q \ge t+2$ with $1 \le t \le 3$ from Theorem 1.5, we get a contradiction using induction on k for $k \ge 6$. Hence, from Lemmas 3.3 and 3.4, we get the following theorem.

Theorem 3.5. *For $q \ge k \ge 5$, there is no $[g_q(k,d), k, d = (k-2)q^{k-1} - (k-1)q^{k-2}]_q$ code if there is no line l in Σ with $\gamma_0(l) = k-3$ and $m(l) = (k-3)q+s$ for $0 \le s \le k-3$.*

4. A γ_3-solid containing a putative $((k-3)q+s)$-line.

In this section, we assume that there exists a partition $\bigcup_{i=0}^{k-2} C_i$ of $\mathcal{F}_0$ with $\gamma_0 = k-2$ in $\Sigma = \mathrm{PG}(k-1,q)$ which satisfies the conditions (a) and (b) in Corollary 2.2 for given integers q and k with $q > (3k-6)/2$, $k \ge 6$. Since it is known that Theorems 1.7 and 1.8 hold for $q \ge 2k-3$ and $k \ge 6$, it is sufficient to prove the theorems for q and k with

$$2k-4 \ge q > (3k-6)/2 \quad \text{and} \quad k \ge 6. \tag{4.1}$$

Hence, to prove the theorems, it suffices to prove the following three theorems by Theorem 3.5.

Theorem 4.1. *For any integers k and q with* (a) $q = 2k-4$, $k \geq 6$ *or* (b) $q = 2k-5$, $k \geq 6$, *there is no line l in $\Sigma = \mathrm{PG}(k-1, q)$ such that $\gamma_0(l) = k-3$ and $m(l) = (k-3)q + s$ for some integer s with $0 \leq s \leq k-3$.*

Theorem 4.2. *For any integers k and q with $q = 2k-6$, $k \geq 7$, there is no line l in $\Sigma = \mathrm{PG}(k-1, q)$ such that $\gamma_0(l) = k-3$ and $m(l) = (k-3)q + s$ for some integer s with $0 \leq s \leq k-3$.*

Theorem 4.3. *For any integers k and q with $q = 2k-7$, $k \geq 9$, there is no line l in $\Sigma = \mathrm{PG}(k-1, q)$ such that $\gamma_0(l) = k-3$ and $m(l) = (k-3)q + s$ for some integer s with $0 \leq s \leq k-3$.*

The proofs of Theorems 4.2 and 4.3 are given in Sections 5 and 6, respectively. In order to prove these theorems, we shall prepare four lemmas in this section. Theorem 4.1 is a corollary of one of these lemmas. Suppose for some integers k and q satisfying the condition (4.1) that there exists a line l in Σ such that $\gamma_0(l) = k-3$ and

$$m(l) = (k-3)q + s \tag{4.2}$$

for some integer s with $0 \leq s \leq k-3$. Let Δ be a solid in Σ containing l and a $(k-2)$-point. Then $m(\Delta) = \gamma_3 = (k-2)q^3 - \theta_2$ by Lemma 3.1. Let $\delta_0, \delta_1, \ldots, \delta_q$ be the planes in Δ containing l. Without loss of generality, we may assume that $m(\delta_0) \leq m(\delta_1) \leq \cdots \leq m(\delta_q)$. It follows from (2.3) and (4.2) that

$$\sum_{i=0}^{q} m(\delta_i) = m(\Delta) + m(l)q = (k-2)q^3 + (k-4)q^2 + (s-1)q - 1. \tag{4.3}$$

If $\gamma_0(\delta_i) = k-2$ for all i with $0 \leq i \leq q$, it follows from Lemma 3.1 that the left hand side of (4.3) is equal to

$$\begin{aligned}(q+1)((k-2)q^2 - \theta_1) &= (k-2)q^3 + (k-4)q^2 + (q-2)q - 1 \\ &> (k-2)q^3 + (k-4)q^2 + (s-1)q - 1,\end{aligned}$$

a contradiction, since

$$(q-2) - (s-1) = q - s - 1 > (3k-6)/2 - (k-3) - 1 = (k-2)/2 > 0$$

by (4.1). Hence $\gamma_0(\delta_0) = k-3$. Since $m(\delta_i) \leq \gamma_2$, it follows from (4.3) and Lemma 3.1 that

$$m(\delta_0) + m(\delta_1) + m(\delta_2) \geq (3k-7)q^2 + (s-2)q - 3. \tag{4.4}$$

Lemma 4.4. *If $2q > 3k-6$, then $\gamma_0(\delta_0) = k-3$ and $\gamma_0(\delta_i) = k-2$ for $2 \le i \le q$.*

Proof. It suffices to prove $\gamma_0(\delta_2) = k-2$. Suppose $\gamma_0(\delta_2) = k-3$. Then it holds that

$$m(\delta_0) + m(\delta_1) + m(\delta_2) \le 3(k-3)\theta_2.$$

If $2q > 3k-6$, then

$$((3k-7)q^2 + (s-2)q - 3) - 3(k-3)\theta_2 = (2q-3k+6)(q+1) + (s-1)q \ge sq+1 > 0,$$

which implies that

$$m(\delta_0) + m(\delta_1) + m(\delta_2) < (3k-7)q^2 + (s-2)q - 3.$$

This is contradictory to (4.4). Hence $\gamma_0(\delta_2) = k-2$. $\square$

Let P be a $(k-3)$-point in l and let $l_1, \ldots, l_q$ be the lines in δ_q through P other than l. Without loss of generality, we may assume that $m(l_1) \le \cdots \le m(l_q)$. It follows from (2.3) and $m(P) = k-3$ that

$$\sum_{i=1}^{q} m(l_i) + m(l) = m(\delta_q) + m(P)q = (k-2)q^2 + (k-4)q - 1. \tag{4.5}$$

If $\gamma_0(l_i) = k-2$ for $2 \le i \le q$, it follows from Lemma 3.1 and (4.5) that $m(l_i) = (k-2)q - 1$ for $2 \le i \le q$ and

$$m(l) + m(l_1) = (2k-5)q - 2. \tag{4.6}$$

Since $m(l_i) \le \gamma_1$, it follows from (4.5) that

$$m(l) + m(l_1) + m(l_2) \ge (3k-7)q - 3. \tag{4.7}$$

Lemma 4.5. *If $2q > 3k-6$, then $\gamma_0(l_i) = k-2$ for $2 \le i \le q$, $\gamma_0(l_1) = k-3$ and $m(l_1) = (k-3)q + q - s - 2$.*

Proof. Suppose $\gamma_0(l_2) = k-3$. Then, from our assumptions $\gamma_0(l) = k-3$ and $m(l_1) \le m(l_2)$, we have $m(l) + m(l_1) + m(l_2) \le 3(k-3)\theta_1$. If $2q > 3k-6$, then $(3k-7)q - 3 - 3(k-3)\theta_1 = 2q - 3k + 6 > 0$. This implies that

$$m(l) + m(l_1) + m(l_2) < (3k-7)q - 3,$$

contradicting (4.7). Hence $\gamma_0(l_i) = k-2$ for $2 \le i \le q$, and $m(l_1) = (k-3)q + q - s - 2$ by (4.6). It holds that $\gamma_0(l_1) = k-3$ by Lemma 3.1 since $(k-3)q + q - s - 2 < \gamma_1$. □

Let $s_1 = q - 2 - s$. When $2q > 3k-6$, we have $m(l_1) = (k-3)q + s_1$ by Lemma 4.5. Since $s + s_1 = q - 2$, we may assume without loss of generality that

$$s \ge s_1, \quad (q-2)/2 \le s \le k-3. \tag{4.8}$$

Thus, if $\gamma_0(\delta_i) = k-2$, there always exists a pair of lines l and l_{i1} in δ_i such that

$$m(l) = (k-3)q + s, \quad m(l_{i1}) = (k-3)q + s_1,$$

where $s + s_1 = q - 2$. Hence, to prove Theorem 4.1, it is sufficient to show that there is no line l in Σ such that $\gamma_0(l) = k-3$ and $m(l) = (k-3) + s$ for any integer s satisfying the condition (4.8).

Assume $2q > 3k-6$, $k \ge 6$. Let l be a $((k-3)q+s)$-line with $0 \le s \le k-3$ and let Δ be a γ_3-solid containing l and a $(k-2)$-point in Σ. Let $\delta_0, \delta_1, \ldots, \delta_q$ be the planes through l in Δ with $m(\delta_0) \le m(\delta_1) \le \cdots \le m(\delta_q)$. Then $\gamma_0(\delta_0) = k-3$, $\gamma_0(\delta_1) = k-3$ or $k-2$ and $\gamma_0(\delta_i) = k-2$ for $2 \le i \le q$ by Lemma 4.4. Let P be a $(k-3)$-point on l and let $l_{i1}, l_{i2}, \ldots, l_{iq}$ be the lines in δ_i through P other than l with $m(l_{i1}) \le m(l_{i2}) \le \cdots \le m(l_{iq})$ for $1 \le i \le q$. When $\gamma_0(\delta_i) = k-2$, it follows from Lemma 4.5 that $\gamma_0(l_{ij}) = k-2$ for $2 \le j \le q$ and that $\gamma_0(l_{i1}) = k-3$, $m(l_{i1}) = (k-3)q + s_1$, where $s_1 = q - s - 2$. Note that $s_1 \ge 0$ since $q > 3(k-2)/2 \ge 3(s+1)/2$.

Lemma 4.6. *If $2q > 3k-6$, $k \ge 6$, then*

(1) $\gamma_0(\delta_0) = k-3$, $\gamma_0(\delta_i) = k-2$ *for* $1 \le i \le q$ *and* $m(\delta_0) = (k-3)q^2 + sq - 1$,

(2) *there are* q $((k-3)q+s)$*-lines and one* $((k-3)q-1)$*-line through* P *in* δ_0,

(3) *there is a* $((k-3)q^2+s_1q-1)$*-plane* $\tilde{\delta_1}$ *through* P *meeting* δ_0 *in a* $((k-3)q-1)$*-line,*

(4) *for any* $(k-3)$*-point* P' *in* δ_0 *there are* q $((k-3)q+s)$*-lines and one* $((k-3)q-1)$*-line through* P' *in* δ_0,

(5) $s \le k-4$, $s_1 \le k-4$ *and* $q \le 2k-6$.

P r o o f. (1) To prove (1), it suffices to determine $\gamma_0(\delta_1)$ and $m(\delta_0)$ by Lemma 4.4. Recall that in a γ_2-plane containing l, the lines through P consist of

l and a $((k-3)q+s_1)$-line and $q-1$ γ_1-lines. So, the γ_2-plane $\langle l_{q1}, l_{q-1,j}\rangle$ meets δ_0 in a $((k-3)q+s)$-line, say l_{0j}, for $2 \le j \le q$. Hence $\langle l_{q1}, l_{q-1,j}\rangle$ with $2 \le j \le q$ meets δ_u in a γ_1-line for $1 \le u \le q-2$. Thus $\gamma_0(\delta_1) = k-2$. By Lemma 3.1 we get

$$\begin{aligned} m(\delta_0) &= m(\Delta) - \sum_{i=1}^{q} m(\delta_i) + m(l)q \\ &= \gamma_3 - \gamma_2 q + ((k-3)q+s)q = (k-3)q^2 + sq - 1. \end{aligned}$$

(2) From the proof of (1), there are q $((k-3)q+s)$-lines $l, l_{02}, l_{03}, \dots, l_{0q}$ through P in δ_0. Let l_{01} be the other line through P in δ_0. Then it follows from (1) that

$$\begin{aligned} m(l_{01}) &= m(\delta_0) - \sum_{i=2}^{q} m(l_{0i}) - m(l) + m(P)q \\ &= (k-3)q^2 + sq - 1 - ((k-3)q+s)q + (k-3)q = (k-3)q - 1. \end{aligned}$$

(3) Put $\tilde{\delta_1} = \langle l_{q1}, l_{q-1,1}\rangle$. Then $\tilde{\delta_1}$ meets δ_u in a $((k-3)q+s_1)$-line for $1 \le u \le q$. Hence $\gamma_0(\tilde{\delta_1}) = k-3$. Since $m(\delta_0 \cap \tilde{\delta_1}) = m(l_{01}) = (k-3)q-1$, it holds that

$$\begin{aligned} m(\tilde{\delta_1}) &= \sum_{i=0}^{q} m(\delta_i \cap \tilde{\delta_1}) - m(P)q = (k-3)q - 1 + ((k-3)q+s_1)q - (k-3)q \\ &= (k-3)q^2 + s_1 q - 1. \end{aligned}$$

(4) Note from (1) that for any $((k-3)q+s)$-line l with $0 \le s \le k-3$, there is only one plane through l in Δ which has no $(k-2)$-point. If all the lines through P' in δ_0 are $((k-3)q-1)$-lines, then

$$m(\delta_0) = ((k-3)q-1)\theta_1 - (k-3)q = (k-3)q^2 - \theta_1,$$

a contradiction. Hence there is a $((k-3)q+s')$-line l' in δ_0 through P' for some $0 \le s' \le k-3$. In Δ there is only one plane, say δ', through l' which has no $(k-2)$-point. From (1) we have $m(\delta') = (k-3)q^2 + s'q - 1$. Since δ_0 is also a plane containing l' which has no $(k-2)$-point, we obtain $\delta' = \delta_0$ and $s' = s$. Hence our assertion follows from (1) and (2).

(5) Suppose $s = k-3$. Then $l \subset C_{k-3}$, and every line in δ_0 contains a $(k-3)$-point. So, from (4), every line in δ_0 is a $((k-3)\theta_1)$-line or a $((k-3)q-1)$-line. Let R be a t-point on a $((k-3)q-1)$-line in δ_0 with $t \le k-4$. Since all the

lines in δ_0 through R are $((k-3)q-1)$-lines, we get $m(\delta_0) = ((k-3)q-1)\theta_1 - tq$, whence $k-4-t = s = k-3$, i.e., $t = -1$, a contradiction. Hence $s \neq k-3$. Since $s \geq s_1$ from (4.8), we have $s_1 \leq k-4$. From $s \leq k-4$ and $s_1 = q-s-2 \leq k-4$, we have $q-k+2 \leq s \leq k-4$, so $q \leq 2k-6$. $\square$

Remark 4.7. (1) In the proof of Lemma 4.6(3), it is easily checked that the $q-1$ planes through l_{01} other than δ_0, $\tilde{\delta_1}$ are γ_2-planes.

(2) It follows from Lemma 4.6(4) that every $((k-3)q+s')$-line with $0 \leq s' \leq k-3$ in δ_0 satisfies $s' = s$ since $(k-3)q+s' > (k-4)\theta_1$.

(3) We obtain Theorem 4.1 as a consequence of Lemma 4.6(5).

Lemma 4.8. *Assume that δ_0 contains an s-point S and that l_{01} contains a* 0*-point R and a $(k-4)$-point Q. Then*

(1) $l_R = \langle R, S\rangle$ *is an $((s+1)q-1)$-line containing $q-1$ $(s+1)$-points and $l_Q = \langle Q, S\rangle$ is a $((k-4)q+s)$-line with $l_Q \setminus \{S\} \subset C_{k-4}$, and any point of $\delta_0 \setminus (l_Q \cup l_R)$ is a $(k-3)$-point.*

(2) *Every line through R in δ_0 other than l_R is a $((k-3)q-1)$-line.*

(3) *Every line through Q in δ_0 other than l_{01}, l_Q is a $((k-3)q+s)$-line.*

Proof. Since $m(l_{01}) = (k-3)q-1$, l_{01} contains $q-1$ $(k-3)$-points, say $P_1, P_2, \ldots, P_{q-1}$. It follows from Lemma 4.6(4) that each line $\langle S, P_i\rangle$ is a $((k-3)q+s)$-line containing q $(k-3)$-points for $1 \leq i \leq q-1$. Hence any line l' through R in δ_0 other than l_R, l_{01} contains $q-1$ $(k-3)$-points. Then we have $m(l') = (k-3)q-1$ by Lemma 4.6(4) again, and l' meets l_Q in a $(k-4)$-point. Thus $m(l_Q) = (k-4)q+s$ and l_Q contains q $(k-4)$-points except the s-point S. Hence

$$m(l_R) = m(\delta_0) - ((k-3)q-1)q = (s+1)q-1.$$

If $\gamma_0(l_R) \geq s+2$, we have $m(l_R) \geq (s+2)q-1$ by Lemma 3.4, a contradiction. It follows from $(s+1)q-1 > s\theta_1$ that $\gamma_0(l_R) = s+1$ and that l_R contains $q-1$ $(s+1)$-points. Hence our assertions follow. $\square$

5. Proof of Theorem 4.2. Throughout this section, we assume that $2k-6 \geq q > (3k-6)/2$, $k \geq 6$, $s = k-4$ and that l, P, Δ, δ_0, $l_{01}, l_{02}, \ldots, l_{0q}$, $\tilde{\delta_1}$, s_1 are as in the proof of Lemma 4.6. We also use the following notations:

$$\eta_1 = (k-3)q+k-4,\ \eta_j = \eta_{j-1}q - 1 \text{ for } 2 \leq j \leq k-2,$$

$$\mu_1 = (k-3)q - 1,\ \mu_j = \mu_{j-1}q - 1 \text{ for } 2 \le j \le k-2.$$

Note that $\gamma_1 = (k-2)q - 1$ and $\gamma_j = \gamma_{j-1}q - 1$ for $2 \le j \le k-2$ by Lemma 3.1.

Lemma 5.1. *Assume $2k-6 \ge q > (3k-6)/2$, $k \ge 6$ and $s = k-4$.*

(1) *The η_2-plane δ_0 consists of one 0-point R, θ_1 collinear $(k-4)$-points and q^2-1 $(k-3)$-points.*

(2) *The lines in δ_0 are the $((k-4)\theta_1)$-line $L(\subset C_{k-4})$, θ_1 μ_1-lines through R and q^2-1 η_1-lines.*

Proof. We first note that each of η_1-lines $l, l_{02}, \dots, l_{0q}$ through a $(k-3)$-point P in the η_2-plane δ_0 contains exactly q $(k-3)$-points and one $(k-4)$-point. Let $Q_0, Q_2, \dots, Q_q$ be the $(k-4)$-points in $l, l_{02}, \dots, l_{0q}$, respectively and let $P_1, P_2, \dots, P_{q-1}$ be the $(k-3)$-points in l_{0q} other than P.

Suppose that l_{01} contains no t-point for $t \le k-5$. Then the number of $(k-4)$-points in the μ_1-line l_{01} in δ_0 through P is $(k-3)\theta_1 - \mu_1 = k-2$. Since $k \ge 6$, there are at least four $(k-4)$-points in l_{01}. Since P_i is a $(k-3)$-point in δ_0 for $1 \le i \le q-1$, it follows from Lemma 4.6 and $m(Q_0) = k-4$ that $\langle Q_0, P_i\rangle$ must be an η_1-line for $1 \le i \le q-1$. That is, $\langle Q_0, P_i\rangle$ contains q $(k-3)$-points and one $(k-4)$-point Q_0 for $1 \le i \le q-1$. This implies that the q points $Q_0, Q_2, \dots, Q_q$ must be on the line $\langle Q_0, Q_q\rangle$ and that there are q $(k-3)$-points and at most one $(k-4)$-point in l_{01}, a contradiction. Hence there is a t-point R in l_{01} with $t \le k-5$.

Next, we show that every line in δ_0 through R is a μ_1-line. Actually, such a line other than $\langle Q_0, R\rangle$ is a μ_1-line since it meets l in a $(k-3)$-point. Hence we have

$$m(\langle Q_0, R\rangle) = m(\delta_0) - \mu_1 q + tq = (k-3+t)q - 1.$$

Since $\gamma_0(\delta_0) = k-3$, it follows from Lemma 3.1 that $t = 0$. Hence the line $\langle Q_0, R\rangle$ is also a μ_1-line, and l_{01} contains exactly one $(k-4)$-point, say Q_1. The points of l_{01} other than R, Q_1 are $(k-3)$-points. Note that each of other lines in δ_0 through R also contains only one $(k-4)$-point. Put $L = \delta_0 \cap C_{k-4} = \{Q_0, Q_1, Q_2, \dots, Q_q\}$. Then L forms a line by Lemma 4.8. Hence our assertions follow. □

Since $m(\Delta) = m(\delta_0) + \gamma_2 q - m(L)q - q^2$ and $\gamma_2 - q^2 = (k-3)q^2 - \theta_1 > (k-4)\theta_2$, it holds that $m(\Delta) > m(\delta_0) + \gamma_2(q-1) + (k-4)\theta_2 - m(L)q$. Hence we get the following.

Lemma 5.2. *Every plane δ' in Δ through L with $m(\delta') < \gamma_2$ satisfies $\gamma_0(\delta') = k-3$.*

From now on, we assume that $q = 2k-6$ in this section. Then, $s_1 = q-s-2 = k-4 = s$ and $k \geq 7$ from our assumption $q > (3k-6)/2$. Hence, $\tilde{\delta_1}$ in Lemma 4.6 is an η_2-plane meeting δ_0 in the μ_1-line l_{01}. By Lemma 5.1, $\tilde{\delta_1}$ contains a $((k-4)\theta_1)$-line ($\subset C_{k-4}$), say $\tilde{L}$. Put $\delta_L = \langle L, \tilde{L} \rangle$. Suppose $\gamma_0(\delta_L) = k-2$. Considering the lines in δ_L through the $(k-4)$-point $L \cap \tilde{L}$, we get

$$\gamma_2 \leq 2(k-4)\theta_1 + \gamma_1(q-1) - (k-4)q = \gamma_2 - q < \gamma_2,$$

a contradiction. Hence we have $\gamma_0(\delta_L) = k-3$ by Lemma 5.2. Next, we determine $m(\delta_L)$. Suppose there is another plane $\delta'(\neq \delta_L)$ in Δ through L with $\gamma_0(\delta') = k-3$. Then, by Lemma 5.1, δ' meets $\tilde{\delta_1}$ in an η_1-line, which contradicts to the fact that there is only one plane in Δ containing no $(k-2)$-point through a fixed η_1-line by Lemma 4.6(1). Thus, all planes through L other than δ_L and δ_0 are γ_2-planes, and we have

$$m(\delta_L) = m(\Delta) - \gamma_2(q-1) - m(\delta_0) + m(L)q = \mu_2.$$

It follows from

$$\begin{aligned} \mu_2 &= \mu_1\theta_1 - (k-3)q \\ &= \mu_1(q-1) + 2(k-4)\theta_1 - (k-4)q \end{aligned}$$

that every line in δ_L through a $(k-3)$-point is a μ_1-line and that every line in δ_L through the $(k-4)$-point $L \cap \tilde{L}$ other than $L, \tilde{L}$ is a μ_1-line. Recall from Lemma 4.6 that for any $(k-3)$-point P on the η_1-line l, there is another η_2-plane through P meeting the η_2-plane δ_0 in a μ_1-line. Hence, for any μ_1-line l_1' in δ_0 through R, one can find an η_2-plane meeting δ_0 in l_1'. Since there is only one plane through L (other than δ_0) containing no $(k-2)$-point, each $(k-4)$-point of L is on exactly two $((k-4)\theta_1)$-lines in δ_L. Thus there are exactly $q+2$ $((k-4)\theta_1)$-lines in δ_L, say $L, L_0, L_1, \ldots, L_q$. Put $\mathcal{L} = \{L, L_0, L_1, \ldots, L_q\}$. Let $L \cap L_i = \{Q_i\}$ and let ℓ_i be any line in δ_L through the $(k-4)$-point Q_i other than L, L_i, $0 \leq i \leq q$. Since ℓ_i is a μ_1-line, ℓ_i must contain $q/2$ $(k-4)$-points and $q/2$ $(k-3)$-points except for Q_i. Since $|\ell_i \cap L_j| = 1$ for $0 \leq i \leq q$, $0 \leq j \leq q$ with $i \neq j$, this implies that no three lines of $\mathcal{L}$ are concurrent. Thus $\mathcal{L}$ forms a $(q+2)$-arc of lines in δ_L (see [7] for arcs). Hence $|\delta_L \cap C_{k-4}| = |L \cup L_0 \cup L_1 \cup \cdots \cup L_q| = \binom{q+2}{2}$ and any point of δ_L out of the $((k-4)\theta_1)$-lines is a $(k-3)$-point. Just like δ_0 or $\tilde{\delta_1}$, the plane $\langle R, L_i \rangle$ is an η_2-plane for $1 \leq i \leq q$. Any line l^* in δ_L containing a $(k-3)$-point is a μ_1-line and l^* contains exactly $(q+2)/2$ $(k-4)$-points and $q/2$ $(k-3)$-points, since $\mathcal{L}$ forms a $(q+2)$-arc of lines. It follows from $m(\Delta) = \gamma_2 q + m(\delta_L) - \mu_1 q$ that every plane through l^* other than δ_L is a γ_2-plane. Hence, $\langle R, l^* \rangle$ is a γ_2-plane. Since every line containing R and a $(k-4)$-point of l^* is a μ_1-line, the other $q/2$ lines through R and a $(k-3)$-point of l^* are γ_1-lines containing exactly $q-1$ $(k-2)$-points. Therefore we get the following.

Lemma 5.3. *Assume $q = 2k-6$, $k \geq 7$ and that a γ_3-solid Δ contains an η_1-line. Then*

(1) *Δ has one 0-point R and one μ_2-plane δ_L.*

(2) *δ_L contains a $(q+2)$-arc of lines $\mathcal{L}$. Each line of $\mathcal{L}$ consists of $(k-4)$-points. And any point of δ_L out of the lines in $\mathcal{L}$ is a $(k-3)$-point.*

(3) *The plane $\langle R, L \rangle$ is an η_2-plane for any $L \in \mathcal{L}$.*

(4) *The line $\langle P, R \rangle$ contains $q-1$ $(k-2)$-points for any $P \in \delta_L \cap C_{k-3}$, and the line $\langle Q, R \rangle$ contains $q-1$ $(k-3)$-points for any $Q \in \delta_L \cap C_{k-4}$.*

(5) *Any plane in Δ other than δ_L and $q+2$ η_2-planes in (3) is a γ_2-plane.*

Now, let Π be a 4-flat with $m(\Pi) = \gamma_4$ containing the γ_3-solid Δ. Let $\Delta_1, \Delta_2, \ldots, \Delta_q$ be the solids in Π other than Δ containing the η_2-plane δ_0 with $m(\Delta_1) \leq m(\Delta_2) \leq \cdots \leq m(\Delta_q) \leq m(\Delta) = \gamma_3$. It can be proved similarly to Lemma 4.4 that $\gamma_0(\Delta_1) = k-3$ and $\gamma_0(\Delta_q) = k-2$. Let l_0 be any line in δ_0 through the 0-point R. Then l_0 is a μ_1-line, and there is only one η_2-plane, say δ_1, in Δ through l_0 other than δ_0. Let $\delta_{i1}, \delta_{i2}, \ldots, \delta_{iq}$ be the planes in Δ_i through l_0 other than δ_0 with $m(\delta_{i1}) \leq \cdots \leq m(\delta_{iq})$ for $1 \leq i \leq q$. When $\gamma_0(\Delta_i) = k-2$, we have

$$m(\delta_{i1}) = \eta_2, \ m(\delta_{ij}) = \gamma_2 \ \text{ for } \ 2 \leq j \leq q \tag{5.1}$$

by Lemma 5.3. Put $\Delta_{1j} = \langle \delta_1, \delta_{qj} \rangle$ for $1 \leq j \leq q$. Then, from (5.1), we have $\gamma_0(\Delta_{1j}) = k-2$ for $2 \leq j \leq q$. For $2 \leq j \leq q$, Δ_{1j} contains only one η_2-plane, say δ'_j, through l_0 other than δ_0 so that $\Delta_{1j} \cap \Delta_1 = \delta'_j$. Hence the $q-1$ γ_2-planes through l_0 in Δ_{1j} other than δ_0, δ'_j are the planes $\Delta_{1j} \cap \Delta_2, \ldots, \Delta_{1j} \cap \Delta_q$. Hence, $m(\Delta_j) = \gamma_3$ for $2 \leq j \leq q$, and we get

$$m(\Delta_1) = m(\Pi) - \sum_{j=2}^{q} m(\Delta_j) - m(\Delta) + m(\delta_0)q = \gamma_4 - \gamma_3 q + \eta_2 q = \eta_3.$$

Since $\Delta_{1j} \cap \Delta_1$ is an η_2-plane through l_0 for $2 \le j \le q$, we have

$$m(\Delta_{11} \cap \Delta_1) = m(\Delta_1) - \eta_2 q + m(l_0)q = \eta_3 - \eta_2 q + \mu_1 q = \mu_2.$$

Thus it holds that $m(\delta_{11}) = \mu_2$ and $m(\delta_{1j}) = \eta_2$ for $2 \le j \le q$.

Let Q_0 be the $(k-4)$-point on l_0. Take a $(k-4)$-point $Q_1(\neq Q_0)$ in δ_0 and put $l_1 = \langle Q_1, R\rangle$. Then, like as for l_0, the planes in Δ_1 through l_1 are η_2-planes except for one plane (which is a μ_2-plane). These q η_2-planes meet δ_{11} in a μ_1-line through R. Hence the remaining line, say $\tilde{l}$, through R in δ_{11} satisfies

$$m(\tilde{l}) = m(\delta_{11}) - \mu_1 q = \mu_2 - \mu_1 q = -1,$$

a contradiction. This completes the proof of Theorem 4.2.

6. Proof of Theorem 4.3. In this section, we assume that $q = 2k-7$, $k \ge 9$ so that the condition $2q > 3k-6$ holds, and let l, P, Δ, δ_0, l_{01}, $\tilde{\delta_1}$, s, s_1 be as in the proof of Lemma 4.6. We also use the notations $\eta_1 = (k-3)q + k - 4$, $\eta_2 = \eta_1 q - 1$, $\mu_1 = (k-3)q - 1$ and $\mu_2 = \mu_1 q - 1$ as in the previous section and

$$\eta_1' = (k-3)q + k - 5,\ \eta_2' = \eta_1' q - 1.$$

Since $0 \le s \le k-4$ and $0 \le s_1 \le k-4$ with $s + s_1 = q - 2 = 2k - 9$ by Lemma 4.6(5), we may assume that $s = k-4, s_1 = k-5$. Hence we have

$$m(\delta_0) = \eta_2,\ m(\tilde{\delta_1}) = \eta_2', \quad m(\delta_0 \cap \tilde{\delta_1}) = m(l_{01}) = \mu_1$$

by Lemma 4.6. Since $s = k-4$, the η_2-plane δ_0 consists of one 0-point R, θ_1 collinear $(k-4)$-points and $q^2 - 1$ $(k-3)$-points by Lemma 5.1. Note that an η_1'-line contains either one $(k-5)$-point or two $(k-4)$-points.

Lemma 6.1. $\tilde{\delta_1}$ *contains no* $(k-5)$*-point.*

P r o o f. Recall from the proof of Lemma 5.1 that the μ_1-line l_{01} contains the 0-point R, a $(k-3)$-point P and the $(k-4)$-point Q_1. Suppose $\tilde{\delta_1}$ contains a $(k-5)$-point S. Then, by Lemma 4.8, $l_{Q_1} = \langle Q_1, S\rangle$ is a $((k-4)q + k - 5)$-line containing q $(k-4)$-points and every line through R in $\tilde{\delta_1}$ other than $l_R = \langle R, S\rangle$ is a $((k-3)q-1)$-line. If there exists a plane through L in Δ whose multiplicity is less than γ_2 except for δ_0 and $\delta_L = \langle L, l_{Q_1}\rangle$, it meets $\tilde{\delta_1}$ in an η_1'-line, contradicting to Lemma 4.6(1). Hence we have

$$m(\delta_L) = m(\Delta) - \gamma_2(q-1) - m(\delta_0) + m(L)q = \mu_2$$

and $\gamma_0(\delta_L) = k-3$ by Lemma 5.2. It can be proved similarly that every plane through l_{Q_1} other than $\tilde{\delta_1}, \delta_L$ is a γ_2-plane.

Take a $(k-4)$-point $Q'(\neq Q_1)$ on l_{Q_1} and put $P' = \langle R, Q'\rangle \cap \langle S, P\rangle$. Since P' is a $(k-3)$-point on the η_1'-line $\langle S, P\rangle$, one can find another η_2-plane δ_0' through P' meeting $\tilde{\delta_1}$ in the $((k-3)q-1)$-line $\langle R, P'\rangle$. Let L' be the $((k-4)\theta_1)$-line in δ_0'. It turns out similarly to δ_L that the plane $\delta_{L'} = \langle L', l_{Q_1}\rangle$ is a μ_2-plane with $\gamma_0(\delta_{L'}) = k-3$. Since $\delta_{L'}$ contains l_{Q_1}, we have $\delta_{L'} = \delta_L$, and L' is on δ_L. It follows from the multiplicity of δ_L and Lemma 4.6(1) that every line l' in δ_L with $\gamma_0(l') = k-3$ is a μ_1-line. Considering the lines in δ_L through $L \cap L'$, we have

$$m(\delta_L) = m(L) + m(L') + \mu_1(q-1) - m(L\cap L')q - 1,$$

giving the existence of a $(\mu_1 - 1)$-line in δ_L. This is a contradiction, for $\mu_1 - 1 > (k-4)\theta_1$. □

It follows from Lemma 6.1 that every line through P in $\tilde{\delta_1}$ other than l_{01} contains exactly two $(k-4)$-points and that the points of $\tilde{\delta_1}$ out of l_{01} are the $2q$ $(k-4)$-points and q^2-2q $(k-3)$-points. Let $m_1, m_2, \ldots, m_q$ be the lines through R in $\tilde{\delta_1}$ other than l_{01} with $m(m_1) \le m(m_2) \le \cdots \le m(m_q)$. If $\gamma_0(m_1) = k-3$, we have

$$\eta_2' = m(\tilde{\delta_1}) = m(l_{01}) + \sum_{i=1}^{q} m(m_i) \ge \mu_1\theta_1 = (k-3)q^2 + (k-4)q - 1 > \eta_2',$$

a contradiction. Hence $\gamma_0(m_1) = k-4$ and m_1 contains q $(k-4)$-points. If m_q contains no $(k-4)$-point, then we have $m(m_q) = (k-3)q$, which is contradictory to Lemma 4.6(4). Hence each of $m_2, \ldots, m_q$ contains a $(k-4)$-point. Since the number of $(k-4)$-points in $\tilde{\delta_1}$ out of $l_{01} \cup m_1$ is equal to $(k-3)(q^2+q) - \eta_2' - (q+1) = q$, m_2 contains two $(k-4)$-points. Hence $m(m_2) = \mu_1 - 1$, a contradiction again. This completes the proof of Theorem 4.3.

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