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HIGH-ORDER CONTROL VARIATIONS AND SMALL-TIME LOCAL CONTROLLABILITY*

Mikhail Ivanov Krastanov

Abstract. The importance of "control variations" for obtaining local approximations of the reachable set of nonlinear control systems is well known. Heuristically, if one can construct control variations in all possible directions, then the considered control system is small-time locally controllable (STLC). Two concepts of control variations of higher order are introduced for the case of smooth control systems. The relation between these variations and the small-time local controllability is studied and a new sufficient STLC condition is proved.

1. High-order variations and small-time local controllabiity.

The traditional approach to obtaining local approximations of the reachable set of nonlinear control systems has been to construct "control variations". If one can construct control variations in all possible directions, then the reachable

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set should to be a full neighborhood of the starting point. Let us consider the following control system:

$$(1) \qquad \dot{x}(t) \;=\; f_0(x(t)) \;+\; \sum_{i=1}^{m} u_i(t) f_i(x(t)),\;\; u_i(t) \in [-1,1],$$

where f_i, $i = 0, \ldots, m$, are smooth vector fields defined on a neigbourhood of the point $x_0 \in R^n$ with $f_0(x_0) = 0$. Let $u(\cdot) = (u_1(\cdot), \ldots, u_m(\cdot))$ be an integrable function defined on the interval $[0,T]$ whose components take values from $[-1,1]$. An absolutely continuous function $x(\cdot)$ with $x(0) = x_0$ and satisfying (1) for almost every t from $[0,T]$ is called an admissible trajectory of (1) defined on $[0,T]$, starting from the point x_0 and corresponding to the control $u(\cdot)$. By $\mathcal{R}(x,T)$ we denote all points of R^n reachable from the point x by means of admissible trajectories of (1) defined on $[0,T]$ and starting from the point x.

We introduce two concepts of *high-order control variations* to the reachable set of the control system (1) at the point x_0: the sets of H- and S-control variations. First we denote by $\mathrm{Exp}(Z_t)x_0$ the value of the solution of the equation

$$\dot{x}(\tau) = Z_t(x(\tau)),\;\; x(0) = x_0,$$

at time $\tau = 1$, where $\{Z_t : t \in R^+\}$ is a given family of smooth vector fields, defined on $\mathbf{R}^n$ and depending continuously on $t \geq 0$.

Definition 1. *The smooth vector field g is said to be an H-control variation to the reachable set of the control system* (1) *at the point x_0 if there exist a positive number T, a neighbourhood Ω of x_0, two families of smooth vector fields a_t and b_t parameterized by $t > 0$, and a continuous function $p : R_+ \to R_+$ such that for each $x \in \Omega$ and each $t \in [0,T]$*

$$\mathrm{Exp}\,(tg + a_t + b_t)\,(x) \;\in\; \mathcal{R}(x, p(t)),$$

where

$$\|a_t(x)\| \;\leq\; M\; t^{\theta}\; \|x - x_0\|,\;\; \|b_t(x)\| \;\leq\; N\; t^{\sigma},\;\; p(t) < \nu\; t^{\lambda},$$

for some positive constants M, N, ν, θ, $\sigma > 1$ and λ. We denote by $H^+(x_0)$ the set of all H-control variations to the reachable set of the control system (1) *at x_0.*

Remark 1. To our knowledge, Hermes was the first to realize (cf. [5]) the importance of the control variations for obtaining local approximations of the reachable set of control systems. Similar definitions of high-order control variations can be found in [6], [8], [12], [13] etc. All these definitions use the notion of a Lie bracket. Let f and g be smooth vector fields defined on R^n. Then the Lie bracket $[f, g]$ is defined as

$$[f, g](x) \ := \ \frac{\partial g}{\partial x}(x) f(x) \ - \ \frac{\partial f}{\partial x}(x) g(x).$$

It should be mentioned that the notion of Lie brackets is extended to the non-smooth case in [3].

The next proposition provides constructions of elements of the set $H^+(x_0)$:

Proposition 1. *The following assertions hold true:*

a) *The set $H^+(x_0)$ is a convex cone;*

b) *The vector fields f_i, $\pm[f_i, f_j]$, $[f_i, [f_i, f_0]]$, $i, j = 1, \ldots, m$, are elements of the set $H^+(x_0)$;*

c) *Let $\pm g \in H^+(x_0)$. Then $\pm[g, f_0] \in H^+(x_0)$.*

Slightly different versions of this proposition can be found in [5], [6], [8], [12]. The corresponding proofs are based on the so called Campbell-Baker-Hausdorff formula.

Definition 2. *Let $\mathcal{H} = \{h^1, \ldots, h^k\}$ be a finite set of smooth vector fields. $\mathcal{H}$ is said to be a set of S-control variations of the control system* (1) *at the point x_0 if there exist positive reals ν, λ, γ_0 and γ_1 with $\gamma_0 < \gamma_1$, a neighbourhood Ω of x_0, two families of smooth vector fields c_t and d_t parameterized by $t > 0$ such that for each γ from the open interval (γ_0, γ_1) and for each vector $s = (s_1, \ldots, s_k)$ whose components belong to the interval $[0, 1]$ there exists a continuous function $p_{s,\gamma} : R_+ \to R_+$ with $p_{s,\gamma}(\eta) < \nu\, \eta^{-\lambda}$ such that for each $x \in \Omega$ and for each sufficiently large positive integer η*

$$\mathrm{Exp}\left(\eta^{-\gamma} \sum_{j=1}^{k} s_j h^j + c_\eta + d_\eta \right)(x) \ \in \ \mathcal{R}(x, p_{s,\gamma}(\eta)), \tag{2}$$

where

$$\|c_\eta(x)\| \ \le \ Q\, \frac{1}{\eta^\theta}\, \|x - x_0\|, \ \|d_\eta(x)\| \ \le \ R\, \frac{1}{\eta^\sigma},$$

for some positive constants Q, R, θ and $\sigma > \gamma_1$. We denote by $S^+(x_0)$ the set of all sets of S-control variations to the reachable set of the control system (1) *at the point x_0.*

Remark 2. It is shown in [9] that the general sufficient STLC condition obtained by Sussmann in [11] (cf. also [1]) is closely related to the above defined set of high-order control variations. More precisely, the proofs of the sufficient STLC conditions in [1] and [11] can be used to construct elements of the set $S^+(x_0)$. For different applications of the control theory, it is very important to implement them explicitly by using admissible controls.

The following proposition shows how these high-order control variations are related to the small-time local controllability (STLC):

Proposition 2. *Let* $\mathcal{H}_i = \left\{h_i^1, \ldots, h_i^{k_i}\right\}$, $i = 1, \ldots, \alpha$, *be sets of* S*-control variations at* x_0, *let* $g_1, \ldots, g_k \in H^+(x_0)$ *and let the origin belong to the interior of the convex hull of the set*

$$\left\{h_1^1(x_0), \ldots, h_1^{k_1}(x_0), \ldots, h_\alpha^1(x_0), \ldots, h_\alpha^{k_\alpha}(x_0), g_1(x_0), \ldots, g_k(x_0)\right\}. \tag{3}$$

Then the control system (1) *is STLC.*

P r o o f. According to Definition 2, for each index $i \in \{1, \ldots, \alpha\}$ there exist positive reals ν_i, λ_i, γ_0^i and γ_1^i with $\gamma_0^i < \gamma_1^i$, a neighbourhood Ω_i of x_0, families of smooth vector fields c_t^i and d_t^i parameterized by $t > 0$ such that for each γ_i from the open interval $\left(\gamma_0^i, \gamma_1^i\right)$ and for each vector $s_i = (s_1^i, \ldots, s_i^{k_i})$ whose components belong to the interval $[0, 1]$ there exists a continuous function $p_{s_i,\gamma_i} : R_+ \to R_+$ with $p_{s_i,\gamma_i}(\eta) < \nu_i\, \eta^{-\lambda_i}$ such that for each $x \in \Omega_i$ and for each sufficiently large positive integer η

$$\operatorname{Exp}\left(\eta^{-\gamma_i} \sum_{j=1}^{k_i} s_i^j h_i^j + c_\eta^i + d_\eta^i\right)(x) \;\in\; \mathcal{R}(x, p_{s_i,\gamma_i}(\eta)), \tag{4}$$

where

$$\|c_\eta^i(x)\| \;\leq\; Q^i\, \eta^{-\theta_i}\, \|x - x_0\|, \;\|d_\eta^i(x)\| \;\leq\; R^i\, \eta^{-\sigma_i},$$

for some positive constants Q^i, R^i, θ_i and $\sigma_i > \gamma_1^i$, $i = 1, \ldots, \alpha$.

Similarly, according to Definition 1, for each index $j \in \{1, \ldots, k\}$, there exist a positive real T, a neighbourhood Ω_j of x_0, two families of smooth vector fields a_t^j and b_t^j parameterized by $t > 0$, and a continuous function $p^j : R_+ \to R_+$ such that for each $x \in \Omega_j$ and each $t \in [0, T]$

$$\operatorname{Exp}\left(tg_j + a_t^j + b_t^j\right)(x) \;\in\; \mathcal{R}(x, p^j(t)), \tag{5}$$

where

$$\|a_t^j(x)\| \;\leq\; M_j\, t^{\theta_j}\, \|x - x_0\|, \;\|b_t^j(x)\| \;\leq\; N_j\, t^{\sigma_j}, \; p^j(t) < \tilde{\nu}_j\, t^{\tilde{\lambda}_j},$$

for some positive constants M_j, N_j, $\tilde{\nu}_j$, θ_j, $\sigma_j > 1$ and $\tilde{\lambda}_j$.

Without loss of generality, we may think that

$$\gamma_0^1 = \max\{\gamma_0^i : \ i = 1, \dots, \alpha\}.$$

Let us fix an arbitrary γ from (γ_0^1, γ_1^1) and an arbitrary vector

$$\tilde{s} := (\tilde{s}_1, \dots, \tilde{s}_k, s_1, \dots, s_k) \quad \text{with } \tilde{s}_i := (\tilde{s}_i^1, \dots, \tilde{s}_i^{k_i}),$$

where each $\tilde{s}_i^j \in [0,1]$, $j = 1, \dots, k_i$, $i = 1, \dots, \alpha$ and each $s_j \in [0,1]$, $j = 1, \dots, k$.

For each index $i \in \{1, \dots, \alpha\}$ two cases are possible: a) $\gamma < \gamma_1^i$; b) $\gamma \geq \gamma_1^i$. In the case a) we have that $\gamma_0^i \leq \gamma_0^1 < \gamma < \gamma_1^i$. By setting $\gamma_i := \gamma$ and $s_i^j := \tilde{s}_i^j$, $j = 1, \dots, k_i$, we obtain that (4) is fulfilled for each sufficiently large positive integer $\eta > 0$. Let us assume that the case b) holds true, i.e., $\gamma \geq \gamma_1^i$. We fix a positive number γ_i from (γ_0^i, γ_1^i). Then $\gamma_i < \gamma$ and for each positive integer η there exists a positive integer $\mu_i = \mu_i(\eta) \geq \eta$ such that the following inequalities hold true:

$$\left(\frac{1}{\mu_i(\eta)}\right)^{\gamma_i} \geq \left(\frac{1}{\eta}\right)^{\gamma} > \left(\frac{1}{\mu_i(\eta)+1}\right)^{\gamma_i}.$$

We set $\bar{s}_i := (s_i^1, \dots, s_i^{k_i})$, where

$$s_i^j := \frac{\tilde{s}_i^j(\mu_i(\eta))^{\gamma_i}}{\eta^{\gamma}}, \ j = 1, \dots, k_i.$$

Clearly, $0 \leq s_i^j \leq 1$. For this choice of $\bar{s}_i$ and γ_i and replacing η by $\mu_i(\eta)$, the inclusion (4) can be written as follows

$$\text{(6)} \qquad \operatorname{Exp}\left(\eta^{-\gamma}\sum_{j=1}^{k_i}\tilde{s}_i^j h_i^j + c_{\mu_i(\eta)}^i + d_{\mu_i(\eta)}^i\right)(x) \ \in \ \mathcal{R}(x, p_{\bar{s}_i,\gamma_i}^i(\mu_i(\eta))),$$

where

$$\|c_{\mu_i(\eta)}^i(x)\| \ \leq \ Q \ \mu_i(\eta)^{-\theta_i} \ \|x - x_0\|, \ \|d_{\mu_i(\eta)}^i(x)\| \ \leq \ R_i \ \mu_i(\eta)^{\sigma_i},$$

for the positive constants Q_i, R_i, θ_i and $\sigma_i > \gamma_1^i$, $i = 1, \dots, \alpha$. Our choice of $\mu_i(\eta)$ and the inequalities $\sigma_i > \gamma_1^i > \gamma_i$ imply that

$$\frac{\eta^{\gamma}}{\mu_i(\eta)^{\sigma_i}} < \frac{(\mu_i(\eta)+1)^{\gamma_i}}{\mu_i(\eta)^{\sigma_i}} = \frac{\left(1 + \dfrac{1}{\mu_i(\eta)}\right)^{\gamma_i}}{\mu_i(\eta)^{\sigma_i - \gamma_i}} \underset{\eta \to \infty}{\longrightarrow} 0,$$

and

$$\left(\frac{1}{\eta}\right)^{\gamma} > \left(\frac{1}{\mu_i(\eta)}\right)^{\sigma_i}$$

for all sufficiently large positive integers η. Thus we have shown that

$$\|d^i_{\mu(\eta)}(x)\| \; = o\left(1/\eta^{\gamma}\right),$$

where by definition $\dfrac{o(t)}{t} \to 0$ as $t \downarrow 0$.

Hence, for each index i the inclusion (4) or the inclusion (6) is fulfilled (depending on the cases a) and b)). Then there exists a neighborhood Ω of x_0 such that for each point $x \in \Omega$ and for all sufficiently large positive integers η

$$\mathcal{E}(\tilde{s},\eta,\gamma)(x) := \mathrm{Exp}\left(\eta^{-\gamma}\sum_{j=1}^{k_1}\tilde{s}_1^j h_1^j + c_\eta^1 + \tilde{d}_\eta^1\right)\cdots$$

$$\mathrm{Exp}\left(\eta^{-\gamma}\sum_{j=1}^{k_\alpha}\tilde{s}_\alpha^j h_\alpha^j + c_\eta^\alpha + \tilde{d}_\eta^\alpha\right)\ \mathrm{Exp}\left(s_1\eta^{-\gamma}g^1 + a^1_{s_1\eta^{-\gamma}} + b^1_{s_1\eta^{-\gamma}}\right)\cdots$$

$$\mathrm{Exp}\left(s_k\eta^{-\gamma}g^k + a^k_{s_k\eta^{-\gamma}} + b^k_{s_k\eta^{-\gamma}}\right)(x) \in\ \mathcal{R}^N_{\vec{X}}\left(x,\tilde{p}_{\tilde{s},\tilde{\gamma}}(\eta)\right),$$

where by $\tilde{d}^i_\eta$, ,$i = 1,\ldots,\alpha$, we have denoted the corresponding d^i_η (in the case a)) or $d^i_{\mu_i(\eta)}$ (in the case b)). Also, here we have set

$$\tilde{p}_{\tilde{s},\tilde{\gamma}}(\eta) = \tilde{p}^1_{\hat{s}_1,\gamma_1}(\eta) + \cdots + \tilde{p}^\alpha_{\hat{s}_\alpha,\gamma_\alpha}(\eta) + p^1\left(s_1\eta^{-\gamma}\right) + \cdots + p^k\left(s_k\eta^{-\gamma}\right),$$

where $\tilde{p}^i_{\hat{s}_i,\gamma_i}(\eta)$ is equal to $p^i_{\tilde{s}_i,\gamma_i}(\eta)$ (in the case a)) or to $p^i_{\tilde{s}_i,\gamma_i}(\mu_i(\eta))$ (in the case b)). Taking into account the estimations for $p^i_{s_i,\gamma_i}(\cdot)$, $i = 1,\ldots,\alpha$, for $p^j(\cdot)$, $j = 1,\ldots,k$, and the inequalities $\mu_i(\eta) \geq \eta$ for all indices i for which the case b) holds true, we obtain that

$$\tilde{p}_{\tilde{s},\tilde{\gamma}}(\eta) < \sum_{i=1}^{\alpha}\nu_i\,\eta^{-\lambda_i} + \sum_{j=1}^{k}\tilde{\nu}_j\,\left(s_j\eta^{-\gamma}\right)^{\tilde{\lambda}_j} < \nu\,\eta^{-\overline{\lambda}}, \tag{7}$$

where $\nu := \left(\displaystyle\sum_{i=1}^{\alpha}\nu_i + \sum_{j=1}^{k}\tilde{\nu}_j s_j^{\tilde{\lambda}_j}\right)$ and $\overline{\lambda} := \min\left(\lambda_1,\ldots,\lambda_\alpha,\gamma\tilde{\lambda}_1,\ldots,\gamma\tilde{\lambda}_k\right)$. Applying the C-B-H formula, we obtain that $\mathcal{E}(\tilde{s},\eta,\gamma)(x) =$

$$= \mathrm{Exp}\left(\eta^{-\gamma}\left(\sum_{i=1}^{\alpha}\sum_{j=1}^{k_i}\tilde{s}_i^j h_i^j + \sum_{j=1}^{k}s_j g^j\right) + \Theta_{\eta,\gamma} + \Delta_{\eta,\gamma,\tilde{s}}\right)(x) \in\ \mathcal{R}\left(x,\tilde{p}_{\tilde{s},\tilde{\gamma}}(\eta)\right), \tag{8}$$

where $\tilde{p}_{\tilde{s},\gamma}(\eta) < \nu\ \eta^{-\overline{\lambda}}$,

$$\|\Theta_{\eta,\gamma}\| \leq Q \left(\frac{1}{\eta}\right)^{\theta} \|x - x_0\| \ \text{ and } \ \|\Delta_{\eta,\gamma,\tilde{s}}\| = o\,(1/\eta^{\gamma})\,. \tag{9}$$

If we expand the right-hand side of (8), it turns out that

$$\mathcal{E}(\tilde{s},\eta,\gamma)(x) = 1+\eta^{-\gamma}\left(\sum_{i=1}^{\alpha}\sum_{\alpha=1}^{k_i}\tilde{s}_i^j h_i^j + \sum_{j=1}^{k} s_j g^j\right)(x) + Y_2(\eta,\gamma)(x) + Y_3(\eta,\tilde{s},\gamma)(x),$$

where $Y_2(\eta,\gamma)$ is a sum of powers of $\Theta_{\eta,\gamma}$ and $Y_3(\eta,\tilde{s},\gamma)$ is a sum of products of the factors $\eta^{-\gamma}\left(\sum_{i=1}^{\alpha}\sum_{\alpha=1}^{k_i}\tilde{s}_i^j h_i^j + \sum_{j=1}^{k} s_j g^j\right)$, $\Theta_{\eta,\gamma}$ and $\Delta_{\eta,\gamma,\tilde{s}}$ and at least one factor is $\Delta_{\eta,\gamma,\tilde{s}}$ or $\eta^{-\gamma}\left(\sum_{i=1}^{\alpha}\sum_{\alpha=1}^{k_i}\tilde{s}_i^j h_i^j + \sum_{j=1}^{k} s_j g^j\right)$. The inequalities (9) imply that $\Theta_{\eta,\gamma}(x_0) = 0$, and hence every power $\Theta_{\eta,\gamma}(x_0) = 0$ vanishes as well. Also, every term of the sum $Y_3(\eta,\tilde{s},\gamma)$ is a product containing at last one factor $o(1/\eta^{-\gamma})$. So, we have that $\mathcal{E}(\tilde{s},\eta,\gamma)(x_0) =$

$$= 1+\eta^{-\gamma}\left(\sum_{i=1}^{\alpha}\sum_{\alpha=1}^{k_i}\tilde{s}_i^j h_i^j + \sum_{j=1}^{k} s_j g^j\right)(x) + Y_3(\eta,\tilde{s},\gamma)(x) \in \ \mathcal{R}\left(x, \tilde{p}_{\tilde{s},\tilde{\gamma}}(\eta)\right),$$

where $\tilde{p}_{\tilde{s},\tilde{\gamma}}(\eta) < \nu\ \eta^{-\overline{\lambda}}$ and $Y_3(\eta,\tilde{s},\tilde{\gamma})/\eta^{\gamma}$ tends to zero as $\eta \to \infty$ (the convergence being uniform with respect to the vector $\tilde{s}$ whose components vary in the interval $[0,1]$). From here (cf. for example, [10]), it follows that $\mathcal{R}(x_0,t)$ contains a neighbourhood of the point x_0 for each $t > 0$. This completes the proof. □

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Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria,
e-mail: `krast@math.bas.bg`

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