NATURAL CONNECTIONS WITH TOTALLY SKEW-SYMMETRIC TORSION ON MANIFOLDS WITH NORDEN-TYPE METRICS

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Abstract. *This paper is a survey of results obtained by the authors on the geometry of connections with totally skew-symmetric torsion on the following manifolds: almost complex manifolds with Norden metric, almost contact manifolds with B-metric and almost hypercomplex manifolds with Hermitian and anti-Hermitian metric.*

Keywords: almost complex manifold, almost contact manifold, almost hypercomplex manifold, Norden metric, B-metric, anti-Hermitian metric, skew-symmetric torsion, KT-connection, HKT-connection, Bismut connection

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Introduction

In Hermitian geometry there is a strong interest in the connections preserving the metric and the almost complex structure whose torsion is totally skew-symmetric ([23, 24, 21, 1, 8, 9, 2, 3]). Such connections are called *KT-connections* (or *Bismut connections*). They find widespread application in mathematics as well as in theoretic physics. For instance, it is proved a local index theorem for non-Kähler manifolds by KT-connection in [1] and the same connection is applied in string theory in [21]. According to [8], on any Hermitian manifold, there exists a unique KT-connection. In [3] all almost contact, almost Hermitian and G_2-structures admitting a KT-connection are described.

In this work[1] we provide a survey of our investigations into connections with totally skew-symmetric torsion on almost complex manifolds with Norden metric, almost contact manifolds with B-metric and almost hypercomplex manifolds with Hermitian and anti-Hermitian metric.

In Section 1 we consider an almost complex manifold with Norden metric (i.e. a neutral metric g with respect to which the almost complex structure J is an anti-isometry). On such a manifold we study a natural connection (i.e. a linear connection ∇' preserving J and g) and having totally skew-symmetric torsion. We prove that ∇' exists only when the manifold belongs to the unique basic class with non-integrable structure J. This is the class $\mathcal{W}_3$ of quasi-Kähler manifolds with Norden metric. We establish conditions for the corresponding curvature tensor to be Kählerian as well as conditions ∇' to

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have a parallel torsion. We construct a relevant example on a 4-dimensional Lie group.

In Section 2 we consider an almost contact manifold with B-metric which is the odd-dimensional analogue of an almost complex manifold with Norden metric. On such a manifold we introduce the so-called φKT-connection having totally skew-symmetric torsion and preserving the almost contact structure and the metric. We establish the class of the manifolds where this connection exists. We construct such a connection and study its geometry. We establish conditions for the corresponding curvature tensor to be of φ-Kähler type as well as conditions for the connection to have a parallel torsion. We construct an example on a 5-dimensional Lie group where the φKT-connection has a parallel torsion.

In Section 3 we consider an almost hypercomplex manifold with Hermitian and anti-Hermitian metric. This metric is a neutral metric which is Hermitian with respect to the first almost complex structure and an anti-Hermitian (i.e. a Norden) metric with respect to the other two almost complex structures. On such a manifold we introduce the so-called pHKT-connection having totally skew-symmetric torsion and preserving the almost hypercomplex structure and the metric. We establish the class of the manifolds where this connection exists. We study the unique pHKT-connection D on a nearly Kähler manifold with respect to the first almost complex structure. We establish that this connection coincides with the known KT-connection on nearly Kähler manifolds and therefore it has a parallel torsion. We prove the equivalence of the conditions D be strong, flat and with a parallel torsion with respect to the Levi-Civita connection.

1. Almost complex manifold with Norden metric

Let (M, J, g) be a $2n$-dimensional *almost complex manifold with Norden metric*, i.e. M is a differentiable manifold with an almost complex structure J and a pseudo-Riemannian metric g such that

$$J^2x = -x, \qquad g(Jx, Jy) = -g(x, y)$$

for arbitrary x, y of the algebra $\mathfrak{X}(M)$ on the smooth vector fields on M. Further x, y, z, w will stand for arbitrary elements of $\mathfrak{X}(M)$.

The associated metric $\tilde{g}$ of g on M is defined by $\tilde{g}(x, y) = g(x, Jy)$. Both metrics are necessarily of signature (n, n). The manifold $(M, J, \tilde{g})$ is also an almost complex manifold with Norden metric.

A classification of the almost complex manifolds with Norden metric is given in [4]. This classification is made with respect to the tensor F of type (0,3) defined by $F(x, y, z) = g\big((\nabla_x J)\, y, z\big)$, where ∇ is the Levi-Civita connection of g. The tensor F has the following properties

$$\text{(1)} \qquad F(x, y, z) = F(x, z, y) = F(x, Jy, Jz), \quad F(x, Jy, z) = -F(x, y, Jz).$$

The basic classes are $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$. Their intersection is the class $\mathcal{W}_0$ of the Kählerian-type manifolds, determined by $\mathcal{W}_0 : F(x,y,z) = 0 \Leftrightarrow \nabla J = 0$.

The class $\mathcal{W}_3$ of the *quasi-Kähler manifolds with Norden metric* is determined by the condition

$$\mathcal{W}_3 : \quad F(x,y,z) + F(y,z,x) + F(z,x,y) = 0. \tag{2}$$

This is the only class of the basic classes $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$, where each manifold (which is not a Kähler-type manifold) has a non-integrable almost complex structure J, i.e. the Nijenhuis tensor N, determined by $N(x,y) = (\nabla_x J)\,Jy - (\nabla_y J)\,Jx + (\nabla_{Jx} J)\,y - (\nabla_{Jy} J)\,x$ is non-zero.

The components of the inverse matrix of g are denoted by g^{ij} with respect to a basis $\{e_i\}$ of the tangent space T_pM of M at a point $p \in M$.

The *square norm of* ∇J is defined by

$$\|\nabla J\|^2 = g^{ij} g^{ks} g\big((\nabla_{e_i} J)\,e_k, \big(\nabla_{e_j} J\big)\,e_s\big).$$

Definition 1. ([19]). An almost complex manifold with Norden metric and $\|\nabla J\|^2 = 0$ is called an *isotropic-Kähler manifold*.

1.1. KT-connection

Let ∇' be a linear connection on an almost complex manifold with Norden metric (M, J, g). If T is the torsion tensor of ∇', i.e. $T(x,y) = \nabla'_x y - \nabla'_y x - [x,y]$, then the corresponding tensor of type (0,3) is determined by $T(x,y,z) = g(T(x,y),z)$.

Definition 2. ([6]). A linear connection ∇' preserving the almost complex structure J and the Norden metric g, i.e. $\nabla' J = \nabla' g = 0$, is called a *natural connection* on (M, J, g).

By analogy with Hermitian geometry we have given the following

Definition 3. ([16]). A natural connection ∇' on an almost complex manifold with Norden metric is called a *KT-connection* if its torsion tensor T is totally skew-symmetric, i.e. a 3-form.

We have proved the following

Theorem 1. ([18]). *If a KT-connection ∇' exists on an almost complex manifold with Norden metric then the manifold is quasi-Kählerian with Norden metric.*

A partial decomposition of the space $\mathcal{T}$ of the torsion (0,3)-tensors T is valid on an almost complex manifold with Norden metric (M, J, g) according to [6]: $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3 \oplus \mathcal{T}_4$, where $\mathcal{T}_i$ $(i = 1, 2, 3, 4)$ are invariant orthogonal subspaces.

Theorem 2. ([18]). *Let ∇' be a KT-connection with torsion T on a quasi-Kähler manifold with Norden metric $(M, J, g) \notin \mathcal{W}_0$. Then*

1) $T \in \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_4$;
2) T *does not belong to any of the classes* $\mathcal{T}_1 \oplus \mathcal{T}_2$ *and* $\mathcal{T}_1 \oplus \mathcal{T}_4$;
3) $T \in \mathcal{T}_2 \oplus \mathcal{T}_4$ *if and only if* T *is determined by*

$$T(x,y,z) = -\frac{1}{2}\big\{F(x,y,Jz) + F(y,z,Jx) + F(z,x,Jy)\big\}. \tag{3}$$

Bearing in mind that T is a 3-form, the following is valid

$$g\left(\nabla'_x y - \nabla_x y, z\right) = \frac{1}{2}T(x,y,z). \tag{4}$$

Then, by (4), (1) and (2), it follows directly that the tensor T, determined by (3), is the unique torsion tensor of a KT-connection, which is a linear combination of the components of the basic tensor F on (M, J, g) [22].

Further, the notion of the KT-connection ∇' on (M, J, g) we refer to the connection with the torsion tensor determined by (3).

1.2. KT-connection with Kähler curvature tensor or parallel torsion

Definition 4. ([5]). A tensor L is called a *Kähler tensor* if it has the following properties:

$$\begin{aligned} &L(x,y,z,w) = -L(y,x,z,w) = -L(x,y,w,z),\\ &L(x,y,z,w) + L(y,z,x,w) + L(z,x,y,w) = 0,\\ &L(x,y,Jz,Jw) = -L(x,y,z,w). \end{aligned}$$

Let R' be the curvature tensor of the KT-connection ∇', i.e. $R'(x,y)z = \nabla'_x\big(\nabla'_y z\big) - \nabla'_y\big(\nabla'_x z\big) - \nabla'_{[x,y]}z$. The corresponding tensor of type $(0,4)$ is determined by $R'(x,y,z,w) = g(R'(x,y)z,w)$.

We have therefore proved the following

Theorem 3. ([16]). *The following conditions are equivalent:*

i) R' *is a Kähler tensor;*
ii) $12R'(x,y,z,w) = 12R(x,y,z,w) + 2g\left(T(x,y),T(z,w)\right)$
$\qquad -g\left(T(y,z),T(x,w)\right) - g\left(T(z,x),T(y,w)\right)$;
iii) $\underset{x,y,z}{\mathfrak{S}}\big\{g\big(\left(\nabla_x J\right)Jy + \left(\nabla_{Jx}J\right)y, \left(\nabla_z J\right)Jw + \left(\nabla_{Jz}J\right)w\big)\big\} = 0$, *where* $\mathfrak{S}$ *denotes the cyclic sum by three arguments.*

Proposition 4. ([16, 17]). *Let* τ *and* τ' *be the scalar curvatures for* R *and* R'*, respectively. Then the following is valid*

i) $3\left\|\nabla J\right\|^2 = 8(\tau' - \tau)$ *if* ∇' *has a Kähler curvature tensor;*
ii) $\left\|\nabla J\right\|^2 = 8(\tau - \tau')$ *if* ∇' *has a parallel torsion.*

Corollary 5. ([17]). *If* ∇' *has a Kähler curvature tensor and a parallel torsion then* (M, J, g) *is an isotropic-Kähler manifold.*

1.3. An example

Let (G, J, g) be a 4-dimensional almost complex manifold with Norden metric, where G is the connected Lie group with an associated Lie algebra $\mathfrak{g}$ determined by a global basis $\{X_i\}$ of left invariant vector fields, and J and g are the almost complex structure and the Norden metric, respectively, determined by

$$JX_1 = X_3, \qquad JX_2 = X_4, \qquad JX_3 = -X_1, \qquad JX_4 = -X_2$$

and

$$g(X_1, X_1) = g(X_2, X_2) = -g(X_3, X_3) = -g(X_4, X_4) = 1,$$
$$g(X_i, X_j) = 0 \quad \text{for} \quad i \neq j.$$

Theorem 6. ([18]).*The manifold (G, J, g) is a quasi-Kählerian with a Killing associated Norden metric $\tilde{g}$, i.e.*

$$g\left([X_i, X_j], JX_k\right) + g\left([X_i, X_k], JX_j\right)=0,$$

if and only if $\mathfrak{g}$ is defined by

$$[X_1, X_2] = \lambda_1 X_1 + \lambda_2 X_2, \qquad [X_1, X_3] = \lambda_3 X_2 - \lambda_1 X_4,$$
$$[X_1, X_4] = -\lambda_3 X_1 - \lambda_2 X_4, \qquad [X_2, X_3] = \lambda_4 X_2 + \lambda_1 X_3,$$
$$[X_2, X_4] = -\lambda_4 X_1 + \lambda_2 X_3, \qquad [X_3, X_4] = \lambda_3 X_3 + \lambda_4 X_4,$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$.

Let (G, J, g) be the manifold determined by the conditions in the last theorem.

The non-trivial components $T_{ijk} = T(X_i, X_j, X_k)$ of the torsion T of the KT-connection ∇' on (G, J, g) are $T_{134} = \lambda_1$, $T_{234} = \lambda_2$, $T_{123} = -\lambda_3$, $T_{124} = -\lambda_4$.

Moreover it is proved the following

Theorem 7. ([18]). *The following propositions are equivalent:*

i) *The manifold (G, J, g) is isotropic-Kählerian;*
ii) *The manifold (G, J, g) is scalar flat;*
iii) *The KT-connection ∇' has a Kähler curvature tensor;*
iv) *The equality $\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = 0$ is valid.*

2. Almost contact manifolds with B-metric

Let $(M, \varphi, \xi, \eta, g)$ be an *almost contact manifold with B-metric* (an *almost contact B-metric manifold*), i.e. M is a $(2n+1)$-dimensional differentiable manifold with an almost contact structure (φ, ξ, η) which consists of an endomorphism φ of the tangent bundle, a vector field ξ, its dual 1-form η as well as M is equipped with a pseudo-Riemannian metric g of signature $(n, n+1)$, such that the following algebraic relations are satisfied

$$\varphi\xi = 0, \qquad \varphi^2 = -I + \eta \otimes \xi, \qquad \eta \circ \varphi = 0, \qquad \eta(\xi) = 1,$$
$$g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y),$$

where I denotes the identity.

Let us remark that the so-called B-metric g one can say a *metric of Norden type* in the odd-dimensional case, because the restriction of g on the contact distribution $\ker\eta$ is a Norden metric with respect to the almost complex structure derived by φ.

The associated metric $\tilde{g}$ of g on M is defined by $\tilde{g}(x,y) = g(x,\varphi y) + \eta(x)\eta(y)$. Both metrics are necessarily of signature $(n, n+1)$. The manifold $(M, \varphi, \xi, \eta, \tilde{g})$ is also an almost contact B-metric manifold.

A classification of the almost contact manifolds with B-metric is given in [7]. This classification is made with respect to the tensor F of type (0,3) defined by $F(x,y,z) = g\big((\nabla_x\varphi)\, y, z\big)$, where ∇ is the Levi-Civita connection of g. The tensor F has the following properties

$$F(x,y,z) = F(x,z,y) = F(x,\varphi y,\varphi z) + \eta(y)F(x,\xi,z) + \eta(z)F(x,y,\xi).$$

This classification includes eleven basic classes $\mathcal{F}_1$, $\mathcal{F}_2$, $\dots$, $\mathcal{F}_{11}$. The special class $\mathcal{F}_0$, belonging to any other class $\mathcal{F}_i$ $(i = 1, 2, \dots , 11)$, is determined by the condition $F(x,y,z) = 0$. Hence $\mathcal{F}_0$ is the class of almost contact B-metric manifolds with ∇-parallel structures, i.e. $\nabla\varphi = \nabla\xi = \nabla\eta = \nabla g = 0$.

In the present work we pay attention to $\mathcal{F}_3$ and $\mathcal{F}_7$, where each manifold (which is not a $\mathcal{F}_0$-manifold) has a non-integrable almost contact structure, i.e. the Nijenhuis tensor N, determined by $N(x,y) = [\varphi,\varphi](x,y) + d\eta(x,y)\xi$, is non-zero. These basic classes are characterized by the conditions

$$\mathcal{F}_3: \quad \underset{x,y,z}{\mathfrak{S}} F(x,y,z) = 0, \quad F(\xi,y,z) = F(x,y,\xi) = 0,$$

$$\mathcal{F}_7: \quad \underset{x,y,z}{\mathfrak{S}} F(x,y,z) = 0, \quad F(x,y,z) = -F(\varphi x,\varphi y,z) - F(\varphi x,y,\varphi z).$$

Let us consider the linear projectors h and v over T_pM which split (orthogonally and invariantly with respect to the structural group) any vector x into a horizontal component $h(x) = -\varphi^2 x$ and a vertical component $v(x) = \eta(x)\xi$.

The decomposition $T_pM = h(T_pM) \oplus v(T_pM)$ generates the corresponding distribution of basic tensors F, which gives the horizontal component $\mathcal{F}_3$ and the vertical component $\mathcal{F}_7$ of the class $\mathcal{F}_3 \oplus \mathcal{F}_7$.

The *square norm of* $\nabla\varphi$ is defined by

$$\|\nabla\varphi\|^2 = g^{ij}g^{ks}g\big((\nabla_{e_i}\varphi)\, e_k, (\nabla_{e_j}\varphi)\, e_s\big).$$

Definition 5. ([14]). An almost contact B-metric manifold with $\|\nabla\varphi\|^2=0$ is called an *isotropic-$\mathcal{F}_0$-manifold.*

2.1. φKT-connection

Definition 6. ([14]). A linear connection D preserving the almost contact B-metric structure (φ, ξ, η, g), i.e. $D\varphi = D\xi = D\eta = Dg = 0$, is called a *natural connection* on $(M, \varphi, \xi, \eta, g)$.

Definition 7. ([14]). A natural connection D on an almost contact B-metric manifold is called a *φKT-connection* if its torsion tensor T is totally skew-symmetric, i.e. a 3-form.

The following theorem is proved.

Theorem 8. ([14]). *If a φKT-connection D exists on an almost contact B-metric manifold $(M, \varphi, \xi, \eta, g)$ then ξ is a Killing vector field and $\mathfrak{S}F = 0$, i.e. $(M, \varphi, \xi, \eta, g)$ belongs to the class $\mathcal{F}_3 \oplus \mathcal{F}_7$.*

The existence of a φKT-connection D on a manifold in $\mathcal{F}_3 \oplus \mathcal{F}_7$ is given by the following

Proposition 9. ([14]). *Let $(M, \varphi, \xi, \eta, g)$ be in the class $\mathcal{F}_3 \oplus \mathcal{F}_7$. Then the connection D with a torsion tensor T, determined by*

$$T(x, y, z) = -\frac{1}{2} \mathop{\mathfrak{S}}_{x,y,z} \big\{F(x, y, \varphi z) - 3\eta(x)F(y, \varphi z, \xi)\big\}, \tag{5}$$

is a φKT-connection on $(M, \varphi, \xi, \eta, g)$.

Further, the notion of the φKT-connection D on $(M, \varphi, \xi, \eta, g)$ we refer to the connection with the torsion tensor determined by (5). For this connection we have

$$D_x y = \nabla_x y + \frac{1}{4}\big\{2\,(\nabla_x\varphi)\,\varphi y - (\nabla_y\varphi)\,\varphi x + (\nabla_{\varphi y}\varphi)\,x + 3\eta(x)\nabla_y\xi - 4\eta(y)\nabla_x\xi + 2\,(\nabla_x\eta)\,y.\xi\big\}.$$

2.2. The φKT-connection on the horizontal component

Let us consider a manifold from the class $\mathcal{F}_3$ – the horizontal component of $\mathcal{F}_3 \oplus \mathcal{F}_7$. Since the restriction on the contact distribution of any $\mathcal{F}_3$-manifold is an almost complex manifold with Norden metric belonging to the class $\mathcal{W}_3$ (known as a quasi-Kähler manifold with Norden metric), then the curvature properties are obtained in a way analogous to that in Section 1.

2.3. The φKT-connection on the vertical component

Let $(M, \varphi, \xi, \eta, g)$ belong to the class $\mathcal{F}_7$ – the vertical component of $\mathcal{F}_3 \oplus \mathcal{F}_7$. For such a manifold the torsion of the φKT-connection D has the form

$$T(x, y) = 2\left\{\eta(x)\nabla_y\xi - \eta(y)\nabla_x\xi + (\nabla_x\eta)\,y.\xi\right\}.$$

A *tensor of φ-Kähler type* we call a tensor with the properties from Definition 4 with respect to the structure φ.

We have proved the following

Theorem 10. ([14]). *The curvature tensor K of D on a $\mathcal{F}_7$-manifold is of φ-Kähler type if and only if it has the form*

$$
\begin{aligned}
K(x,y,z,w) = R(x,y,z,w)& \\
&+ \tfrac{1}{3}\big\{2\,(\nabla_x\eta)\,y\,(\nabla_z\eta)\,w - (\nabla_y\eta)\,z\,(\nabla_x\eta)\,w - (\nabla_z\eta)\,x\,(\nabla_y\eta)\,w\big\} \\
&+ \eta(x)\eta(z)g\,(\nabla_y\xi,\nabla_w\xi) - \eta(x)\eta(w)g\,(\nabla_y\xi,\nabla_z\xi) \\
&- \eta(y)\eta(z)g\,(\nabla_x\xi,\nabla_w\xi) + \eta(y)\eta(w)g\,(\nabla_x\xi,\nabla_z\xi)\,.
\end{aligned}
$$

Theorem 11. ([14]). *If D has a curvature tensor K of φ-Kähler type and a parallel torsion T on a $\mathcal{F}_7$-manifold then*

$$
\begin{aligned}
K(x,y,z,w) = R(x,y,z,w)& \\
&+ \tfrac{1}{3}\big\{2\,(\nabla_x\eta)\,y\,(\nabla_z\eta)\,w + (\nabla_x\eta)\,z\,(\nabla_y\eta)\,w - (\nabla_x\eta)\,w\,(\nabla_y\eta)\,z\big\},
\end{aligned}
$$

$$
\rho(K)(y,z) = \rho(y,z), \qquad \tau(K) = \tau,
$$

where $\rho(K)$ and ρ are the Ricci tensors for K and R, respectively, and $\tau(K)$ and τ are the their corresponding scalar curvatures.

2.4. An example

Let $(G,\ \varphi,\ \xi,\ \eta,\ g)$ be a 5-dimensional almost contact manifold with B-metric, where G is the connected Lie group with an associated Lie algebra $\mathfrak{g}$ determined by a global basis $\{X_i\}$ of left invariant vector fields, and (φ,ξ,η) and g are the almost contact structure and the B-metric, respectively, determined by

$$
\begin{gathered}
\varphi X_1 = X_3, \quad \varphi X_2 = X_4, \quad \varphi X_3 = -X_1, \quad \varphi X_4 = -X_2, \quad \varphi X_5 = 0; \\
\xi = X_5; \qquad \eta(X_i) = 0\ (i = 1,\,2,\,3,\,4), \quad \eta(X_5) = 1;
\end{gathered}
$$

$$
\begin{gathered}
g(X_1,X_1) = g(X_2,X_2) = -g(X_3,X_3) = -g(X_4,X_4) = g(X_5,X_5) = 1, \\
g(X_i,X_j) = 0,\ i \neq j, \quad i,\,j \in \{1,\,2,\,3,\,4,\,5\}.
\end{gathered}
$$

Theorem 12. ([14]). *The manifold $(G,\ \varphi,\ \xi,\ \eta,\ g)$ is a $\mathcal{F}_7$-manifold if and only if $\mathfrak{g}$ is determined by the following non-zero commutators:*

$$
\begin{aligned}
[X_1,X_2] = -[X_3,X_4] &= -\lambda_1X_1 - \lambda_2X_2 + \lambda_3X_3 + \lambda_4X_4 + 2\mu_1X_5, \\
[X_1,X_4] = -[X_2,X_3] &= -\lambda_3X_1 - \lambda_4X_2 - \lambda_1X_3 - \lambda_2X_4 + 2\mu_2X_5,
\end{aligned}
$$

where $\lambda_i,\ \mu_j \in \mathbb{R}$ $(i = 1,\,2,\,3,\,4;\ j = 1,\,2)$.

Let (G,φ,ξ,η,g) be the manifold determined by the conditions in the last theorem.

The non-trivial components $T_{ijk} = T(X_i, X_j, X_k)$ of the torsion T of the φKT-connection D on $(G, \varphi, \xi, \eta, g)$ are $T_{125} = T_{345} = 2\mu_1$, $T_{235} = T_{415} = 2\mu_2$.

Hence, using the components of D, we calculate that the corresponding components of the covariant derivative of T with respect to D are zero. Thus, we have proved the following

Theorem 13. ([14]). *The φKT-connection D on $(G, \varphi, \xi, \eta, g)$ has a parallel torsion T.*

Theorem 14. ([14]). *The manifold $(G, \varphi, \xi, \eta, g)$ is an isotropic-$\mathcal{F}_0$-manifold if and only if $\mu_1 = \pm\,\mu_2$.*

3. Almost hypercomplex manifolds with Hermitian and Norden metric

Let (M, H) be an *almost hypercomplex manifold*, i.e. M is a $4n$-dimensional differentiable manifold and $H = (J_1, J_2, J_3)$ is a triple of almost complex structures with the properties:

$$J_\alpha = J_\beta \circ J_\gamma = -J_\gamma \circ J_\beta, \qquad J_\alpha^2 = -I$$

for all cyclic permutations (α, β, γ) of $(1, 2, 3)$.

The standard structure of H on a $4n$-dimensional vector space with a basis

$$\{X_{4k+1}, X_{4k+2}, X_{4k+3}, X_{4k+4}\}_{k=0,1,\dots,n-1}$$

has the form:

$$\begin{array}{lll}
J_1 X_{4k+1} = X_{4k+2}, & J_2 X_{4k+1} = X_{4k+3}, & J_3 X_{4k+1} = -X_{4k+4}, \\
J_1 X_{4k+2} = -X_{4k+1}, & J_2 X_{4k+2} = X_{4k+4}, & J_3 X_{4k+2} = X_{4k+3}, \\
J_1 X_{4k+3} = -X_{4k+4}, & J_2 X_{4k+3} = -X_{4k+1}, & J_3 X_{4k+3} = -X_{4k+2}, \\
J_1 X_{4k+4} = X_{4k+3}, & J_2 X_{4k+4} = -X_{4k+2}, & J_3 X_{4k+4} = X_{4k+1}.
\end{array}$$

Let g be a pseudo-Riemannian metric on (M, H) with the properties

$$g(x, y) = \varepsilon_\alpha g(J_\alpha x, J_\alpha y), \qquad \varepsilon_\alpha = \begin{cases} 1, & \alpha = 1; \\ -1, & \alpha = 2;\ 3. \end{cases}$$

In other words, for $\alpha = 1$, the metric g is Hermitian with respect to J_1, whereas in the cases $\alpha = 2$ and $\alpha = 3$ the metric g is an anti-Hermitian (i.e. Norden) metric with respect to J_2 and J_3, respectively. Moreover, the associated bilinear forms g_1, g_2, g_3 are determined by

$$g_\alpha(x, y) = g(J_\alpha x, y) = -\varepsilon_\alpha g(x, J_\alpha y), \qquad \alpha = 1, 2, 3.$$

Then, we call a manifold with such a structure briefly an *almost (H, G)-manifold* [12, 13].

The structural tensors of an almost (H, G)-manifold are the three $(0, 3)$-tensors determined by

$$F_\alpha(x, y, z) = g\big((\nabla_x J_\alpha)\, y, z\big) = \big(\nabla_x g_\alpha\big)(y, z), \qquad \alpha = 1, 2, 3,$$

where ∇ is the Levi-Civita connection generated by g.

In the classification of Gray-Hervella [11] for almost Hermitian manifolds the class $\mathcal{G}_1 = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ is determined by the condition $F_1(x,x,z) = F_1(J_1x, J_1x, z)$.

Theorem 15. ([15]). *If M is an almost (H,G)-manifold which is a quasi-Kähler manifold with Norden metric regarding J_2 and J_3, then it belongs to the class $\mathcal{G}_1$ with respect to J_1.*

3.1. pHKT-connection

Definition 8. ([15]). A linear connection D preserving the almost hypercomplex structure H and the metric g, i.e. $DJ_1 = DJ_2 = DJ_3 = Dg = 0$, is called a *natural connection* on (M,H,G).

Definition 9. ([15]). A natural connection D on an almost (H,G)-manifold is called a *pseudo-HKT-connection* (briefly, a *pHKT-connection*) if its torsion tensor T is totally skew-symmetric, i.e. a 3-form.

For an almost complex manifold with Hermitian metric (M,J,g), in [3] it is proved that there exists a unique KT-connection if and only if the Nijenhuis tensor $N_J(x,y,z) := g(N_J(x,y),z)$ is a 3-form, i.e. the manifold belongs to the class of cocalibrated structures $\mathcal{G}_1$.

3.2. The class $\mathcal{W}_{133}$

Next, we restrict the class $\mathcal{G}_1(J_1)$ to its subclass $\mathcal{W}_1(J_1)$ of *nearly Kähler manifolds with neutral metric* regarding J_1 defined by $F_1(x,y,z) = -F_1(y,x,z)$. In this case (M,H,G) belongs to the class $\mathcal{W}_{133} = \mathcal{W}_1(J_1) \cap \mathcal{W}_3(J_2) \cap \mathcal{W}_3(J_3)$ and $\dim M \geq 8$.

We have proved the following

Theorem 16. ([15]). *The curvature tensor R of ∇ on $(M,H,G) \in \mathcal{W}_{133}$ has the following property with respect to the almost hypercomplex structure H:*

$$R(x,y,z,w) + \sum_{\alpha=1}^{3} R(x,y,J_\alpha z, J_\alpha w) = \sum_{\alpha=1}^{3} \big\{ A_\alpha(x,z,y,w) - A_\alpha(y,z,x,w) \big\},$$

where $A_\alpha(x,y,z,w) = g\big((\nabla_x J_\alpha)\, y, (\nabla_z J_\alpha)\, w\big)$, $\alpha = 1, 2, 3$.

3.3. The pHKT-connection on a $\mathcal{W}_{133}$-manifold

KT-connections on nearly Kähler manifolds are investigated for instance in [20]. The unique KT-connection D^1 for the nearly Kähler manifold (M, J_1, g) on the considered almost (H,G)-manifold has the form

$$g\left(D^1_x y, z\right) = g\left(\nabla_x y, z\right) + \frac{1}{2} F_1(x,y,J_1 z).$$

Moreover, there exists a unique KT-connection D^α ($\alpha = 2, 3$) for the quasi-Kähler manifold with Norden metric (M, J_α, g) on the considered almost (H, G)-manifold such that

$$g\left(D^\alpha_x y, z\right) = g\left(\nabla_x y, z\right) - \frac{1}{4} \underset{x,y,z}{\mathfrak{S}} F_\alpha(x, y, J_\alpha z).$$

In [15] we have constructed a connection D, using the KT-connections D^1, D^2 and D^3, on an almost (H, G)-manifold from the class $\mathcal{W}_{133}$ and we have proved the following

Theorem 17. ([15]). *The connection D defined by*

$$g\left(D_x y, z\right) = g\left(\nabla_x y, z\right) + \frac{1}{2} F_1(x, y, J_1 z).$$

is the unique pHKT-connection on an almost (H, G)-manifold from the class $\mathcal{W}_{133}$.

Let us remark that the pHKT-connection D on an almost (H, G)-manifold coincides with the known KT-connection D^1 on the corresponding nearly Kähler manifold. Then the torsion of the pHKT-connection D is parallel and henceforth T is coclosed, i.e. $\delta T = 0$ [3]. Moreover, the curvature tensors K of D and R of ∇ has the following relation [10]

$$K(x, y, z, w) = R(x, y, z, w) + \frac{1}{4} A_1(x, y, z, w) + \frac{1}{4} \underset{x,y,z}{\mathfrak{S}} A_1(x, y, z, w).$$

We have proved the following

Theorem 18. ([15]).*Let (M, H, G) be an almost (H, G)-manifold from $\mathcal{W}_{133}$ and D be the pHKT-connection. Then the following characteristics of this connection are equivalent:*

(i) *D is strong ($dT = 0$);*
(ii) *D has a ∇-parallel torsion;*
(iii) *D is flat.*

Theorem 19. ([15]). *Let (M, H, G) be an almost (H, G)-manifold from $\mathcal{W}_{133}$ and D be the pHKT-connection. If D is flat or strong then (M, H, G) is ∇-flat, isotropic-hyper-Kählerian (i.e. $\|\nabla J_\alpha\|^2 = 0$, $\alpha = 1, 2, 3$) and the torsion of D is isotropic (i.e. $\|T\|^2 = 0$).*

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