ALMOST CONFORMAL TRANSFORMATION IN A CLASS OF RIEMANNIAN MANIFOLDS[1]

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Abstract. *We consider a* 3*-dimensional Riemannian manifold* V *with a metric* g *and an affinor structure* q*. The local coordinates of these tensors are circulant matrices. In* V *we define an almost conformal transformation. Using that definition we construct an infinite series of circulant metrics which are successively almost conformaly related. In this case we get some properties.*

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1. Preliminaries

We consider a 3-dimensional Riemannian manifold M with a metric tensor g and two affine tensors q and S such that: their local coordinates form circulant matrices. So these matrices are as follows:

$$g_{ij} = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}, \quad A > B > 0, \tag{1}$$

where A and B are smooth functions of a point $p\big(x^1, x^2, x^3\big)$ in some $F \subset R^3$,

$$q_i^{.j} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad S_i^{.j} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}. \tag{2}$$

We note by V the class of manifolds like M.

Let M be in V and ∇ be the connection of g. Let us give some results for M in V, obtained in [1].

$$q^3 = E; \quad g(qu, qv) = g(u, v), \quad u,\ v \in \chi M. \tag{3}$$

$$\nabla q = 0 \quad \Leftrightarrow \quad gradA = gradB.S. \tag{4}$$

$$0 < B < A \quad \Rightarrow \quad g\ is\ possitively\ defined. \tag{5}$$

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2. Almost conformal transformation

Let M be in V. We note $f_{ij} = g_{ik}q_j^k + g_{jk}q_i^k$, i.e.

$$f_{ij} = \begin{pmatrix} 2B & A+B & A+B \\ A+B & 2B & A+B \\ A+B & A+B & 2B \end{pmatrix}. \tag{6}$$

We calculate $detf_{ij} = 2(A-B)^2(A+2B) \neq 0$, so we accept f_{ij} for local coordinates of another metric f. Further, we suppose α and β are two smooth functions in $F \subset R^3$ and we construct the metric g_1, as follows:

$$g_1 = \alpha.g + \beta.f. \tag{7}$$

We say that equation (7) define an almost conformal transformation, noting that if $\beta = 0$ then (7) implies the case of the classical conformal transformation in M [2].

From (1), (6) and (7) we get the local coordinates of g_1:

$$g_{1,ij} = \begin{pmatrix} \alpha A + 2\beta B & \beta A + (\alpha+\beta)B & \beta A + (\alpha+\beta)B \\ \beta A + (\alpha+\beta)B & \alpha A + 2\beta B & \beta A + (\alpha+\beta)B \\ \beta A + (\alpha+\beta)B & \beta A + (\alpha+\beta)B & \alpha A + 2\beta B \end{pmatrix}. \tag{8}$$

We see that f_{ij} and $g_{1,ij}$ are both circulant matrices.

Theorem 2.1. Let M be a manifold in V, also g and g_1 be two metrics of M, related by (7). Let ∇ and $\dot{\nabla}$ be the corresponding connections of g and g_1, and $\nabla q = 0$. Then $\dot{\nabla} q = 0$ if and only if, when

$$grad\, \alpha = grad\, \beta.S. \tag{9}$$

Proof. At first we suppose (9) is valid. Using (9) and (4) we can verify that the following identity is true:

$$grad(\alpha A + 2\beta B) = grad\big(\beta A + (\alpha+\beta)B\big).S \tag{10}$$

The identity (10) is analogue to (4), and consequently we conclude $\dot{\nabla} q = 0$.

Inversely, if $\dot{\nabla} q = 0$ then analogously to (4) we have (10). Now (4) and (10) imply (9). So the theorem is proved.

□

Note. We see that (10) is a system of partial differential equations. In this case we know that this system has a solution [3].

Let $w = w\big(x(p), y(p), z(p)\big)$ be an arbitrary vector in T_pM, $p \in M$, $M \subset V$, such that $qw \neq w$. For the metric g of M we suppose $0 < B < A$, i.e. g is positively defined (see (5)).

Let φ be the angle between w and qw with respect to g. Then thank's to (1), (2) and (3) we get $cos\varphi = \dfrac{g(w, qw)}{g(w, w)}$, and we note that $\varphi \in (0, \dfrac{2\pi}{3})$ [1].

Lemma 2.2. Let g_1 be the metric given by (7). If $0 < \beta < \alpha$ and g is positively defined, then g_1 is also positively defined.

Proof. For g_1 we have that $\alpha A + 2\beta B - (\beta A + (\alpha+\beta)B = (\alpha-\beta)(A-B) > 0$. Analogously to (6) we state that g_1 is positively defined.

□

Lemma 2.3. Let $w = w\big(x(p),\, y(p),\, z(p)\big)$ be in T_pM, $p \in M$, $M \subset V$, $qw \neq w$. Let g and g_1 be the metrics of M, related by (7). Then we have

$$\begin{aligned} g_1(w,w) &= \alpha g(w,w) + 2\beta g(w,qw) \\ g_1(w,qw) &= \beta g(w,w) + (\alpha+\beta) g(w,qw). \end{aligned} \tag{11}$$

Proof. Using (1) and (2) we find

$$\begin{aligned} g(w,w) &= A(x^2+y^2+z^2) + 2B(xy+yz+zx) \\ g(w,qw) &= B(x^2+y^2+z^2) + (A+B)(xy+yz+zx). \end{aligned} \tag{12}$$

Now, we use (8) and (12) after some computations we get (11).

□

Theorem 2.4. Let $w = w\big(x(p),\, y(p),\, z(p)\big)$ be a vector in T_pM, $p \in M$, $M \subset V$, $qw \neq w$. Let g and g_1 be two positively defined metrics of M, related by (7). If φ and φ_1 are the angles between w and qw, with respect to g and g_1 respectively, then the following equation is true

$$\cos\varphi_1 = \frac{\beta + (\alpha+\beta) cos\varphi}{\alpha + 2\beta cos\varphi}\,. \tag{13}$$

Proof. Since g and g_1 are both positively defined metrics we can calculate $\cos\varphi$ and $\cos\varphi_1$, respectively [2]. Then by using (11) from Lemma 2.2 and Lemma 2.3 we get (13).

□

We note $\varphi \in (0, \dfrac{2\pi}{3})$. Theorem 2.4 implies immediately the assertions:

Corollary 2.5. If φ_1 is the angle between w and qw with respect to g_1 then $\varphi_1 \in (0, \dfrac{2\pi}{3})$.

Corollary 2.6. Let φ and φ_1 be the angles between w and qw with respect to g and g_1. Then

1) $\varphi = \dfrac{\pi}{2}$ if and only if when $\varphi_1 = \arccos\dfrac{\beta}{\alpha}$;

2) $\varphi_1 = \dfrac{\pi}{2}$ if and only if when $\varphi = \arccos\Big(-\dfrac{\beta}{\alpha+\beta}\Big)$.

Further, we consider an infinite series of the metrics of M in V as follows:

$$g_0,\ g_1,\ g_2, \dots,\ g_n, \dots$$

where

$$g_0 = g,\ \ g_n = \alpha g_{n-1} + \beta f_{n-1},\ \ f_{n-1,is} = g_{n-1,ia} q_s^a + g_{n-1,sa} q_i^a,\ 0 < \beta < \alpha. \tag{14}$$

By the method of the mathematical induction we can see that the matrix of every g_n is circulant one and every g_n is positively defined.

Theorem 2.7. Let $w = w\big(x(p),\, y(p),\, z(p)\big)$ be in T_pM, $p \in M$, $M \subset V$, $qw \neq w$. Let φ_n be the angle between w and qw with respect to metric g_n from (14). Then the infinite series:

$$\varphi_0,\ \varphi_1,\ \varphi_2,\ \dots,\ \varphi_n,\ \dots$$

is converge and $\lim \varphi_n = 0$.

Proof. Using the method of the mathematical induction and Theorem 2.4 we obtain

$$\cos\varphi_n = \frac{\beta + (\alpha+\beta)\cos\varphi_{n-1}}{\alpha + 2\beta\cos\varphi_{n-1}} \tag{15}$$

as well as $\varphi_n \in (0, \frac{2\pi}{3})$. From (15) we get

$$\cos\varphi_n - \cos\varphi_{n-1} = \frac{\beta(1-\cos\varphi_{n-1})(1+2\cos\varphi_{n-1})}{\alpha + 2\beta\cos\varphi_{n-1}}\,. \tag{16}$$

The equation (16) implies $\cos\varphi_n > \cos\varphi_{n-1}$, so the series $\{\cos\varphi_n\}$ is increasing one and since $\cos\varphi_n < 1$ then it is converge. From (15) we have $\lim\cos\varphi_n = 1$, so $\lim\varphi_n = 0$.

□

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